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PROGRESS OF THEORETICAL PHYSICS

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Volume 4

JANUARY—DECEMBER

1 9 4 9

Published by

The Physical Society

of Japan

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Progress of Theoretical Physics Vol. IV, No. 1, Jan.~Mar., 1949.

On Dirac's General Transformation Function. II—b.

Satosi WATANABE.

The Scientific Research Institute, Tokyo.

(Received May 22, 1948)

§ 3. Existence of Classical Transformation Function.

In order that the classical general transformation function may be constructed, there must exist a solution which, (i) satisfies the wave equation in the considered domain of the space-time, and (ii) has the given field strength u on the boundary of the domain. If the wave equation derived from the variation principle is a partial differential equation of the elliptic type, our problem becomes a Dirichlet problem which is always soluble. The existence of a Green function will manifestly assure the possibility of solution. Statical fields correspond to this case. Particular attention must be paid in the case of a hyperbolic type of wave equation.

Although the results in the case of two dimensions cannot automatically be generalized to the cases of more than two dimensions because the symmetry between time and space is the characteristic feature of the two dimensions, it will yet be well to start with this case, for its particular simplicity will help to understand certain of the difficulties pertaining to the solution of boundary problems of the hyperbolic type. This problem has been treated in some detail by E. Picard.⁽¹⁾ We want to summarize the results and apply them to our problem.

The equation treated is of the type :

$$a \frac{\partial^2 u}{\partial x^2} + 2b \frac{\partial^2 u}{\partial x \partial t} + c \frac{\partial^2 u}{\partial t^2} + e \frac{\partial u}{\partial x} + f \frac{\partial u}{\partial t} + gu + h = 0. \quad (3.1)$$

with

$$b^2 - ac > 0, \quad (3.2)$$

where a, b, c, e, f, g and h are real functions of x and t . In our application the characteristics of the equation bear the meaning of the light cone. We are, of course, placed in the general case of non-analytic functions.

Cauchy's problem consists in determining the solution whose values, as well as the values of the first order derivatives, are given on a non-characteristic curve. The problem we are facing consists in determining the solution whose values are given on a certain boundary, but not those of the derivatives. The possibility of such a solution naturally depends on the nature of the boundary. We want first to mention the three basic cases where the solution exists:

- (a) u is given on two intersecting characteristics.
- (b) u is given on a characteristic and an everywhere-spatial (or an everywhere-temporal) non-characteristic curve which intersect.
- (c) u is given on two everywhere-spatial (or two everywhere-temporal) non-characteristic curves which intersect at one point.

Figure 1 (a, b, c) shows these three cases, the thick lines indicating the boundaries on which u is given, and the shaded area the domain in which the solution is determined by the boundary condition. The straight lines which are inclined at 45° from the vertical line represent the light cone (characteristics).

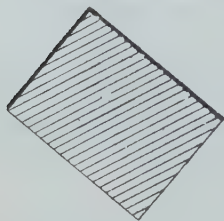


Fig. 1 (a)

by computing, with the help of a simple integral equation, the values of the derivative on a non-characteristic from the field values on the other curve, the latter being a characteristic in case (b) and a non-characteristic in case (c). A similar method could be applied to the corresponding cases of more than two dimensions if the non-characteristic boundary is everywhere-spatial. If, on the contrary, the non-characteristic boundary is everywhere-

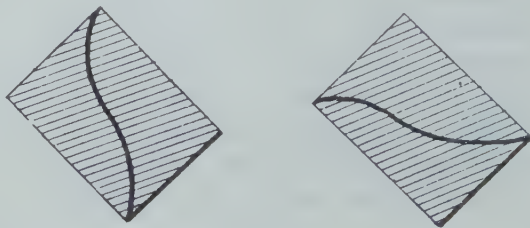


Fig. 1 (b)

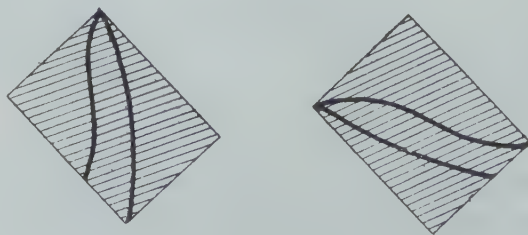


Fig. 1 (c)

temporal, the method will not work because the Cauchy problem on a temporal boundary is in general impossible, except in the case of two dimensional space-time.⁽²⁾ The reduction of the problem to a Cauchy problem consists essentially in reversing the roles of a known and an unknown in the formula of Riemann's method of integration in the case of two-dimensional hyperbolic differential equations. For equations of more than two dimensions, Volterra's generalization of Riemann's method may be applied.⁽³⁾

Let us now determine the possibility of combining these three types and constructing a closed domain in which the field is to be determined by the field values given on the boundary. First, if the field values are given on four characteristics enclosing a rectangular domain, there will be too many given conditions, since two neighbouring sides are enough to determine the field inside. (See class (a) above). Second, let us consider two spatial curves AB and AC, BC being a characteristic, and assume that the field values are given on the whole contour ABC. (See Fig. 2). In view of class (c) above, it is evident that the condition on BC is one too many. However, there is one point of view that

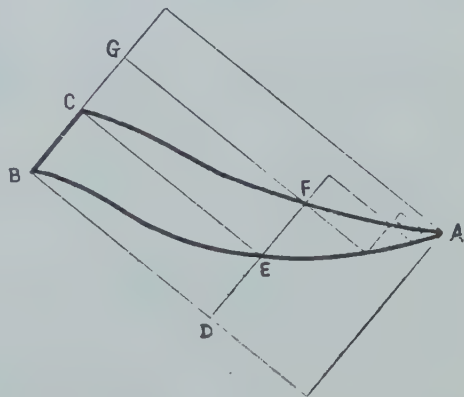


Fig. 2

may obscure this impossibility. Construct a rectangle BCED, E lying on AB. The field in the rectangle is then determined by the given condition on BC and BE. (class b). The field in the rectangle CEFG (F lying on AC) is then determined by CE and CF. We can repeat this operation indefinitely and finally obtain the field in A BC. This procedure, however, shifts the difficulty only to the infinitesimal vicinity of the limiting point A, at which the continuity of

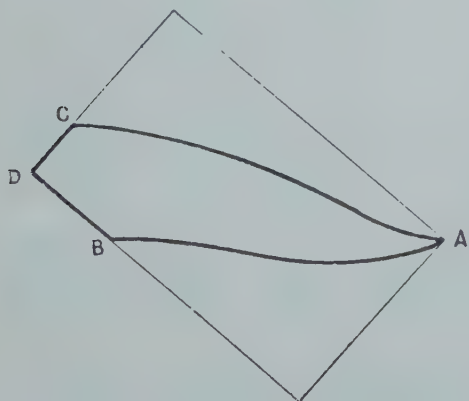


Fig. 3

the derivative will be destroyed and the field equation will not be observed.⁽⁴⁾ A similar argument is valid for a domain, having as its boundary two characteristics and two non-characteristics (Fig. 3), where AB and AC are spatial, and DB and DC are characteristics.

The most hopeful type is the domain bounded by two everywhere-spatial curves or two everywhere-temporal curves which intersect at two points. At first sight this seems to be a special case of class (c) but it will turn out to be quite different in nature. Let ACIB and ADB in Fig.

4 be two spatial curves, CD lying on one characteristic and ID on another. The field values on AC and AD will completely determine the field in the rectangle AGDE. It would seem that we could move CD toward the end-point B and thus solve the problem satisfactorily, but such is not the case, since we must fulfil too many conditions for the rectangular domain BH

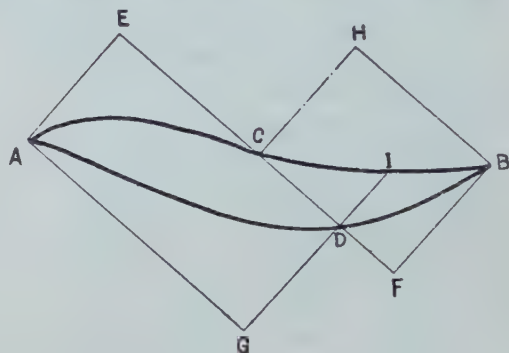


Fig. 4

CF. We have indeed, other than conditions on BC and BD, the condition on CD. From the standpoint of the reduction to a Cauchy problem, we can construct such a problem for the curve AD with the help of the given values on AC and another for the curve DB with the help of the given values on IB. That will be enough for determination of the field in the rectangular domain having A and B as its diametrically opposite corners. Thus the field values given on CI may contradict the rest of the given conditions.

However, if the two curves are infinitely near and if the conditions on them are infinitely proximate, the problem is possible because it then becomes practically a Cauchy problem.

When a closed domain is bounded by two curves, one everywhere-spatial and the other partly spatial and partly temporal, the difficulties are even more pronounced than in the case just considered.

Thus, we are obliged to give up the boundary-value problems for a finite closed domain in the case of a hyperbolic field equation. In general, we demand too many conditions by giving the field values for the entire

periphery of the contour.

As regards the infinite boundary in the case of open domain, in class (c) we can of course legitimately prolong the two spatial (or two temporal) curves indefinitely in both directions, in so far as they do not lose their spatial (or temporal) nature. However, if the field values are given on two non-intersecting, infinite, spatial (or temporal) curves which are separated from each other by a finite distance, we cannot regard the problem as a limiting case of class (c) in which the point of intersection is displaced to infinity. There appears a particular indeterminacy in the solution of such a problem.

Let us take the simplest example :

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, \quad (3.3)$$

and require, as boundary conditions,

$$\left. \begin{aligned} u(x, 0) &= \varphi(x) \\ u(x, \tau) &= \lambda(x) \end{aligned} \right\} \quad (3.4)$$

Then if $u_1(x, t)$ is a solution satisfying the boundary condition,

$$u = u_1 + \frac{1}{2} \int_{x-t}^{x+t} \phi(\xi) d\xi \quad (3.5)$$

is also a solution satisfying the boundary condition, where

$$\phi(\xi) = \sum_{n=-\infty}^{\infty} \left(A_n \sin \frac{n\pi\xi}{\tau} + B_n \cos \frac{n\pi\xi}{\tau} \right) \quad (\text{Note : } n \neq 0) \quad (3.6)$$

is an arbitrary function defined in a domain of length 2τ , the additive constant being so adjusted that the integration of the function over any domain of length 2τ may vanish.

A similar consideration can be applied to the case of more than one spatial dimension. Whenever the boundaries are two infinite, non-intersecting, spatial hypersurfaces (temporal ones are excluded here) separated by a finite temporal distance, there appears an indeterminacy of the same nature.

In summation it can be stated that the boundary-value problem is conceivable if the domain is limited by two spatial surfaces. However, when

the surfaces do not intersect each other at all the problem is interminate (too few conditions), and when they enclose a closed domain the problem is impossible (too many conditions). Even in those cases, indeterminacy or impossibility can be eliminated if the two surfaces are infinitely near.

§ 4. Existence of Quantum-Mechanical General Transformation Function.

i) *Hamilton Scheme.*

As already stated, the classical general transformation function surely exists when the domain of definition is bounded by two spatial hypersurfaces which are separated from each other by an infinitesimal (temporal) distance. In this case the Lagrange scheme is reduced to the Hamilton scheme (§ 5-ii, Part I, and § 1, Part II). If we limit ourselves to such a case, there is no serious question as to the possibility of representation $[u\{s\}]$, and no question as to the relativistically invariant formulation of commutation rules.

For a domain limited by two spatial surfaces which are separated by a finite distance, we can gather up infinitesimal transformation operators and obtain as the resultant general transformation function

$$\prod_{s_0}^s (1 - i\mathbf{H}' ds). \quad (4.1)$$

This operator exists for the given domain and does not necessarily vanish for the "impossible" boundary conditions for which the classical general transformation function does not exist.

Apart from the argumentations which led us to limit the boundary surfaces to spatial ones, there is another reason for which the present formalism is only applicable to the spatial s -surfaces. Indeed if each s -surface was not everywhere-spatial, the definition of the operator (4.1) would be ambiguous, since the q -numbers such as u and π contained in \mathbf{H}' are, in general, not commutable, except those which refer to two points which are separated by a spatial distance. The total operator (4.1) will then depend on the order in which those q -numbers appear.

When all the s -surfaces are spatial and the order of multiplications under \prod is defined by the one-dimensional order of s , such ambiguity will disappear. However, if (4.1) is transcribed in the form:

$$H(1-iH'(d\xi)^4) \quad (4.2)$$

as in (1.21), the factors must be so arranged that two volume-elements, $(d\xi)^4$ lying in the light-cone of each other, appear in a definite order of s .

All these considerations lead us to the conclusion that the Hamilton scheme, to which the current field quantum-mechanics belongs, can be generalized to the curved space-like s -surfaces, in lieu of the time-constant planes, but no further since it would then fundamentally interfere with our current notions of relativity and quantum mechanics.

ii) Lagrange Scheme.

The quantum-mechanical general transformation function in the Lagrange scheme is given by the right-hand side of equation (1.11) Part II. Its existence therefore depends solely on that of the classical general transformation function. (See §3 Part II.)

But from the quantum-mechanical point of view, the left-hand side of (1.11) must have the meaning of a certain probability amplitude. It would therefore be necessary to have a definite value even for classically impossible boundary conditions. A modification of the proposed scheme might be necessary to cope with such difficulties.

At any rate, we can point out one advantage of the Lagrange scheme viz., that here the essential limitations of the current relativistic quantum field theory reveal themselves in a compact form. (A potential usefulness of the Lagrange scheme will be discussed in §5).

§ 5. Physical Interpretation and Conclusion.

Our knowledge of the state of a field on the s -surface is expressed by a Hermitian operator G with real eigen-values: w_1, w_2, w_3, \dots , which are not negative and obey $\sum w_i = 1$. In the case of a pure state, all the w 's except one will vanish. The expectation value \overline{Q} of a physical quantity Q will be given by

$$\begin{aligned} \overline{Q} &= \text{trace } (QG) \\ &= \iint [u' | Q | u''] [u'' | G | u'] \end{aligned} \quad (5.1)$$

where the integration, in general, involves a mathematically awkward manipulation as in (1.5) Part II. This expectation value is dependent on

the parameter s , and its dependence is expressed, with the help of a unitary transformation operator S , by

$$\overline{Q} = \text{trace } (QSGS^{-1}) \quad (5.2)$$

$$= \text{trace } (S^{-1}QSG) \quad (5.3)$$

To give a physical meaning to the foregoing theory, operator (1.11)' or (1.19) Part II is to be substituted in this equation.

The first line (5.2) can be interpreted as expressing the change of state according to

$$G \rightarrow SGS^{-1} \quad (5.4)$$

while the physical quantity Q is assumed to remain unchanged. This interpretation naturally corresponds to the Schrödinger picture. The second line (5.3), which is mathematically equivalent to the first, can be interpreted in terms of the Heisenberg picture in which the state remains unchanged while the physical quantity changes according to

$$Q \rightarrow S^{-1}QS. \quad (5.5)$$

If the operator S consists of two parts

$$S = S_0 S', \quad (5.6)$$

the expectation value may be written

$$\overline{Q} = \text{trace } (S_0^{-1}QS_0 S' G S'^{-1}), \quad (5.7)$$

which is readily interpreted in the "mixed" picture.

In the case of a pure state, in which the field is found to have definite values $u''' \{s_0\}$ on the boundary $s=s_0$, the matrix corresponding to (5.4) will be written

$$[u' \{s\} | u''' \{s_0\}] [u''' \{s_0\} | u'' \{s'\}], \quad (\text{no integration over } u''' \{s_0\}) \quad (5.8)$$

which shows that S itself, in a sense, has the property of a state-representative.

Now if we pass to the "unified" point of view, introduced in connexion with (4.16) Part I and (1.12) Part II, by rendering the portion of the boundary occupied by $s=s_0$ negligibly small, we shall have in the place

of (5.8),

$$[u'\{s\}] [\overline{u''\{s\}}]. \quad (5.9)$$

Expression (5.2) will then become

$$\overline{Q} = \iint [\overline{u'\{s\}}] [u'\{s\} | Q | u''\{s\}] [u''\{s\}], \quad (5.10)$$

which could also be derived directly from (5.1). The causal inference of the state on s from that on s_0 is thus abolished here. This last expression (5.10), as interpreted by Dirac,⁽⁵⁾ may give the *a priori* expectation values of the quantity Q , defined all around the contour.

The essential, physical statement of the current quantum-mechanics is summarized in the equation (5.2) or (5.3), since it is this very relation which binds theory with observation. There are many reasons to believe that a principle of the type (5.2) will, *mutatis mutandis*, survive the future reframing of the quantum field theory even if other accepted laws such as the Schrödinger equation, the Poisson bracket expression etc., are to be discarded. In fact, if the operator S is obtainable, we can give the necessary physical predictions without any knowledge about the Hamiltonian and so on. The future problem might well be concentrated on how to give an adequate form to the S .⁽⁶⁾

The current quantum mechanics takes the Hamiltonian operator as its starting point and derives the operator S from it by the infinite multiplication of the type (1.19) or (1.21) Part II, of course taking the time variable as s . This procedure, however, is hardly justifiable in view of the well-known divergence difficulties of the current theory. Indeed, if we are to avoid these difficulties, we are obliged to make some artifice which, in effect, amounts to giving up division of the space-time into portions smaller than r_0^4 , r_0 being some universal constant of the dimension of length ranging around the classical electron-radius.⁽⁷⁾ The Hamilton scheme will then lose its footing, as is clear from the deduction in § 1 Part II. Although the division of the space-time into cells of finite dimension cannot be formulated in a relativistically invariant way, this situation leads us to believe that a fundamental rôle in the future theory will be played by the S for a finite space-time domain rather than by the familiar Hamiltonian of an infinitesimal space-time domain. The non-commutability of quantities at two points separated by a distance less than r_0 would in fact be inconceivable, at least

under its present-day implications for we must consider the theater of an elementary interaction between two particles to cover a finite domain of linear dimension r_0 , inside which we can scarcely delve, in a physical sense, into the internal mechanism.

It is for these reasons that the properties of the classical and quantum-mechanical expressions of the operator \mathcal{S} have been examined at some length and that even the Lagrange scheme which deviates violently from the current Hamilton scheme has been introduced.

Apart from these basic considerations concerning the future revision of the field theory, our discussion has shown that the current method of the relativistic field quantum-mechanics can be generalized to the case where the field quantities are measured on a curved space-like surface instead of a flat time-constant plane. The important information obtained through this analysis pertains to the reasons why this generalization cannot be carried beyond this limitation (space-like s -surface) without violating the basic postulates of the present-day field theory — a vital point which the other studies have not clarified satisfactorily till now.

The writer would like to acknowledge the value of the opinions given him by Professors T. Yamanouchi, T. Inui, S. Tomonaga, T. Miyazima, K. Husimi, R. Utiyama, H. Yukawa.

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The first five lines of Subsection (iii), page 86, should be replaced by: "If the Lagrangian L depends on ξ^μ only through the Jacobian $\partial(x)/\partial(\xi)$ in (3-6), and if we choose coordinates such that".

Progress of Theoretical Physics Vol. IV, No. 1, Jan.~Mar., 1949.

On the Thermal Reactions in Gas Phase. I.

FIZÔ KANAI.

Institute of Chemistry, Kyoto University.

(Received Aug. 1, 1948)

§ 1. Introduction.

The elementary processes underlying the chemical reactions are, of course, the collisions which take place between the atoms or the molecules. However, the present experiments about the chemical reactions do not measure the microscopic probabilities of the elementary processes, but the macroscopic rate constants of the whole processes. And the theory of the chemical reactions made up by Eyring, Wigner and others⁽¹⁾ seems to be powerful to give the approximate values of the rate constants, while it is not very interested in the microscopic behaviors of the elementary processes. This circumstance is contrasted with the one in the other cases, such as, the collision problems between the electron and the atomic nucleus and between the nucleon and the nucleus, in which cases the elementary reactions are mainly of importance.

Here we shall try to take up the elementary reactions between the molecules and to investigate their characters from the standpoint of the elementary collisions and then to relate them with the macroscopic rates. For this purpose it will be necessary to know the theoretical standpoint suitable for the investigation of the chemical reactions. As is well known, the present theory cited above is based on the following three assumptions:

1. the adiabatic treatment of the electronic motions,
2. the classical treatment of the nuclear motions,
3. the existence of the thermal equilibrium between the activated state and the initial state of the whole system.

We shall briefly criticize these assumptions in succession and ask for the starting point for the subsequent considerations.

Adiabatic assumption seems to be satisfied approximately in the ordinary reactions. That reason essentially consists in the slow motion of the atomic nuclei compared with the rapid one of the valence electrons. In

fact, unless the separations of the electronic potentials are smaller than the thermal energy KT , above condition will hold because of the small mass of the electron compared with the nucleus and the small kinetic energy of the nuclei compared with the electrons. But there are a few examples which seem to proceed non-adiabatically such as the thermal decomposition of N_2O . Hence, it will be desirable to treat the both cases (adiabatic and non-adiabatic) on the unified standpoint.

When the reactions proceed non-adiabatically, it is inevitable to use the quantum mechanics for the computation of the reaction rates. On the other hand, in the case of the adiabatic reactions the nuclear motion has been treated classically. Indeed it was justified when we had considered the chemical reactions as the collision problem between the rigid spheres representing atoms or molecules. For their radii were larger than the nuclear wave lengths and we could trace the nuclear motion in the particle aspect. This type of collisions is known as the "surface reaction" which takes place in a very short time without penetrating into the mutual interiors. However, it will be unnecessary to see the failure of the historical "collision theory" for us to understand the fact that the chemical reactions are never the surface reactions.

Now the progresses of the theory about the molecular structures enable us to visualize the reaction processes as the movement of the representative points on the potential energy surfaces, when the adiabatic assumption is justified. Of course, the quantitative knowledges about the construction of the potential surfaces are never complete on account of the weakness of our mathematical faculties. But our present knowledges are probably powerful as the guide of the qualitative discussions in some cases.

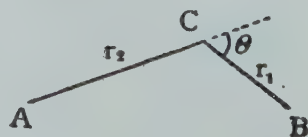
For an example, we shall consider the exchange reaction:



where A , B and C are all atoms and each has one s -electron. In this simplest case, the potential V of the system is the function of r_1 , r_2 and $|\theta|$ (c.f. Fig. 1). When we write the values of V in the three-dimensional space of r_1 , r_2 and θ , we see the high potential region separating the initial state

$$r_2 \approx b, \quad r_1 \gg a$$

and the final state



1.Fig.

$$r_1 \approx a, \quad r_2 \gg b$$

where a and b are the atomic distances of the stable molecules BC and AC respectively. Looking for the position of the pass on the potential mountain, we obtain the activated state on the reaction path. The difference of the values of V in the activated state and in the bottom of the initial valley is the so-called (classical) activation energy W . When the initial state is in the thermal equilibrium, such a group of the elementary systems contributes most effectively to the rate constant, whose energies are given by

$$E \simeq W + KT$$

where K is the Boltzmann's constant and T is the experimental temperature. So we shall investigate the classical motion of the system having the energy $E = W + KT$ for the time being.

First we pick up the potential region satisfying the condition

$$V \leq E$$

which is the space where the representative point with energy E can move classically. As is seen from Fig. 2, this region has the form of an open tube, which we shall call the "reaction tube". One end of this reaction tube is in the initial state and the other is in the final state and the middle part makes a fairly long and narrow bending portion. The accomplishment of the reaction process needs the passage of the representative point through this bending middle portion. In order to test the validity of the classical mechanics at this stage, we have only to compare the diameter of the narrow middle part with the

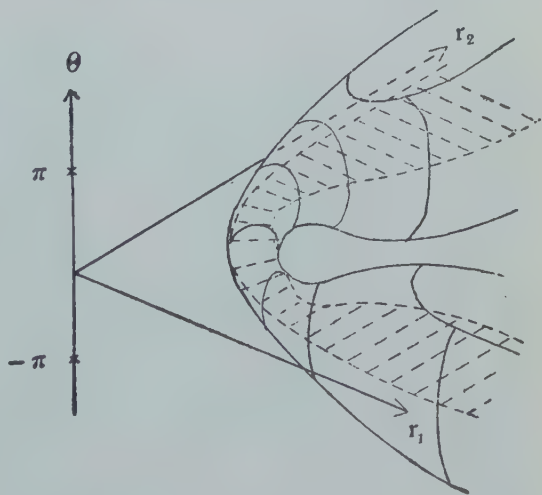


Fig. 2

wave length of the moving representative point in this position. As is well known, we cannot apply the geometrical optics for the light passing the slit of the dimension comparable to its wave length. Borrowing the numerical values from the paper of Hirschfelder, Eyring and Topley,⁽²⁾ we obtain the following results in the reaction, $H_2 + H \rightarrow H + H_2$:

$$\lambda \text{ (wave length)} \sim 1.7 \cdot 10^{-8} \text{ cm}$$

$$l \text{ (diameter)} \sim 10^{-8} \text{ cm}$$

where we put $E = W + 1 \text{ K cal mol}^{-1}$ (c.f. at 29°C , $KT = 0.6 \text{ K cal mol}^{-1}$). λ is fairly larger than l . Hence, we cannot apply the classical mechanics at least for the hydrogen exchange reactions.

Of course, this conclusion will be modified for other reactions. First, λ becomes smaller because of two reasons: for the heavy molecules the relative masses become larger in general and for the complex molecules the energies of the most effective elementary systems become higher on account of the rapid increase of the level density of the activated state. Second, the value of l will become a little larger as a rule, which is determined by the degree of the valence bindings. Thus, we can expect the next relation

$$\lambda \sim l$$

for the present. This shows that the nuclear motions do not necessarily proceed classically. But it will not fit in with the experiences to emphasize the quantum effects too much. After all, we conclude that it is desirable to use the quantum language in the sense that the quantum treatment contains the classical treatment.

Next we shall go to the investigation of the third assumption — the equilibrium between the initial state and the activated one. Let us consider the reaction of the second order as an example. It is true that the reaction proceeds through the activated state. And the time which the system spends in the activated state is considerably long. To see this, it will be sufficient to trace the zigzag orbit of the classical motion of the representative point.⁽³⁾ But the life τ of the activated state is shorter than the mean period τ_c between the successive collisions at the ordinary conditions ($\tau_c \sim 10^{-10}$ sec.). Hence, the activated molecule does not experience any collision with other atoms or molecules among its existing period. Since the ther-

mal equilibrium can exist only for the ensemble of systems which can exchange their energies each other, the activated systems are not directly brought about into the canonical equilibrium (Boltzmann's distribution). And we should suppose that Eyring's theory need not claim this type of the thermal equilibrium of the activated state and what it requests is the microcanonical equilibrium between the activated and initial systems with the same energies. After all, this results in the canonical distribution of the activated systems.

However, as for the reactions of the second order, it is more reasonable to consider the following process as a single elementary (mechanical) process:



The discussions about the first order reactions given by Rice, Ramsperger and Kassel⁽⁴⁾ will be useful for the understanding of above circumstance. They supposed that the activated molecules might be considered as in the thermal equilibrium with the normal molecules in the complete first order reactions. The reason is as follows. The mean life τ of the activated molecules is very long compared with τ_c and so the mutual transitions between the activated molecules and the initial molecules are very frequent, while the decay processes are rare. Hence, the rate-determining step, is the decay processes of the activated molecules which take place practically without destroying the thermal distribution of the activated systems. Since, in the second order reactions the change from the initial state to the final state is a single one and makes the rate-determining step, it is natural to assume only the thermal equilibrium of the initial state which remains unchanged through the reaction processes. This assumption which we shall take here is not only reasonable but also necessary. Because, most experiments of the slow chemical reactions are performed under the condition of the constant temperature and this temperature should be taken as the measure of the thermal distribution of the reactants. Of course, there will be many cases in which the third assumption in question is approximately satisfied. In these cases, the whole rate is computed by means of the statistical mechanics and the evaluation of the so-called transmission coefficient κ is not important to give the approximate values of the rate constant. However, if we want to have the knowledges about the elementary processes and to take the data from the macroscopic experiments, we have to adopt the standpoint stated above.

§ 2. Qualitative discussions.

In § 1 we have been led to the following standpoints which are suitable for the theoretical treatment of the chemical reactions:

1. the reactions take place through the quantum-mechanical collision,
2. the initial state of the whole system is in the thermal equilibrium.

Hence the problem separates itself into two parts. The first part is the step in which we ask the crosssections of the elementary reactions by means of the quantum mechanics. And the second is the step in which we compute the rate constant k by the application of the statistical mechanics. Here we shall put another assumption explicitly:

3. the thermal reactions in gas phase may be treated as the statistically independent superposition of each elementary reaction.

This last assumption is applicable only in gas phase for the present. In general, the computation of k from the elementary crosssections can easily be carried out if the latter are known completely.

Thus our main subject is the derivation of the elementary crosssections. Here we find the close relations between the theory of the chemical reactions and the transitional theory of the quantum mechanics. Judging from the essential features of the present problem, next correspondence seems to be very natural:

The theory of the chemical structures

\longleftrightarrow the stationary theory of the quantum mechanics,

The theory of the chemical reactions

\longleftrightarrow the transitional theory of the quantum mechanics.

The former relation has been established since the explanation of the homopolar binding of H_2 by Heitler and London.⁽⁵⁾

Although the solutions of the individual problems can be obtained by the method of the numerical integrations, the results are necessarily special. In order to get the more general conclusions, we shall introduce the following assumption which is of importance very much:

4. the chemical reactions are the compound reactions.

The contents of this assumption have been already suggested by Eyring, especially in the word "activated complex". The compound reaction is the contrary to the surface reaction. In the compound reaction, the colliding particle gets into the interior of the collided particle and forms the quasi-stable molecule and then separates itself into the final products. We

shall begin the discussions by the explanation of the general reasons why the chemical reactions are compound. Those reasons are as follows:

- (a) the valence forces are of saturation character and of short range,
- (b) the collisions are caused by the thermal motion,
- (c) there exist several inner degrees of freedom which have to do with the reaction mechanism.

Owing to (a), the colliding particle practically does not interact with the collided particle at a distance larger than the molecular diameter, and within the range of force too, the atoms of both molecules cannot exert their influences additively. So the relative kinetic energy of the collision is transferred successively to the inner energies of the atomic motions in a large molecule composed of the colliding molecule and the collided one. Since the energy of the collision is usually much smaller than the bond energies of the stable molecule because of (b) and the masses of the atoms are all of the same order, the collision process proceeds in such a way that the excess energy of the collision be distributed among the various inner degrees of freedom of the large molecule ((c)). As the chance will be comparatively rare that the energies of each degree of freedom come together into a specified degree of freedom and the latter binding is destroyed, the collision of two molecules once makes up the state of quasi-stable molecule and the life of this state will be considerably long.

The more visible investigation is possible in the case where the adiabatic approximation is justified. When the reacting molecules are comparatively complex, we have the reaction tube like as Fig. 2 in the potential space of many dimensions. The important behavior of the middle part can be expected from the knowledges about the structure theory. First, in this place we have the semi-closed region of a considerable extent, for it has usually the opening ends only towards the two directions (to the initial and to the final side). This circumstance will be accelerated by the presence of the so-called potential basin.⁽⁶⁾ Second, this part is bended in general and especially is so when we adopt the oblique axes so as to obtain the normal form of the kinetic energy of the representative point.⁽⁷⁾ These two reasons make the life of the activated state be long and the reaction be compound.

For the treatment of the compound reactions it is convenient to borrow the theory of the nuclear reactions.⁽⁸⁾ (The latter are the typical examples of the compound reactions.) That is, we use the spacing d (or the density

ρ) and the width Γ of the compound state. To know d is equivalent to knowing the energy spectrum of the levels of the activated state and to know Γ is to know each decay probability of the activated levels. Particularly Γ/d is an important quantity as the characterizing measure of the reaction in question. $\Gamma \gg d$ is necessary for the classical treatment to be justified, while $\Gamma \ll d$ appears in the purely quantum mechanical phenomena. If the values of Γ/d of the chemical reactions are known, we shall obtain a feature of the chemical reactions. From the considerations in §1, we shall expect the relation $\Gamma \sim d$ except the reactions of complex molecules (which are probably classical), but this remains as a suggestion for the time being. The quantitative determination of Γ/d will be tried in the subsequent sections.

§ 3. First order reactions.

We shall consider the decomposition reaction of the first order



A molecule is excited to the unstable molecule A^* by means of the thermal collisions with other molecules. Since A^* molecule should spontaneously decay into the products B and C unless deactivated by other molecules, A^* state has its decay width Γ as well as its energy W . Hence, the following mathematical formulation is convenient which was used by Gamow and Breit⁽⁹⁾ in the nuclear reactions.

First we write down the Schrödinger equation which contains all valence electrons and atomic cores (atomic nuclei + core-electrons) of the molecules concerned in the elementary reaction (A in the present case). We designate the energy of this elementary system by E . Then we determine the boundary condition in such a way that the asymptotic forms of the wave function should be composed of only the diverging waves even if we made any separation of the system into two parts. This condition causes the eigenvalue problem and there exist the solutions of the Schrödinger equation only for the certain complex energies:

$$E^{n,J,M} = W^{n,J,M} - \frac{i}{2} \Gamma^{n,J,M} (= W^{n,J} - \frac{i}{2} \Gamma^{n,J}) \quad (3.1)$$

where J is the quantum number representing the total angular momentum,

M is the one specifying an independent state belonging to the same J and n is the other one than above two quantum numbers. At the fixed (J, M) , n runs from smaller $W^{J,M}$ to larger $W^{J,M}$. From the conservation law of the density, it is easily proved that

$$\Gamma^{n,J}/\hbar = \frac{1}{\tau^{n,J}} > 0 \quad (3.2)$$

is the decay frequency, i.e. the reciprocal of the life, of the (n, J, M) state.

According to Lindemann, Hinshelwood and others,⁽¹⁰⁾ $\tau^{n,J} \gg \tau_c$ is the necessary condition for the reaction to proceed as the first order one. And this condition assures the thermal equilibrium of A^* molecules with A molecules. Thus

$$\frac{N_{n,J,M}^*}{N_i} = \frac{Z_{n,J,M}^*}{Z_i} \quad (3.3)$$

where N is the number and Z is the thermal weight.

$$Z_{n,J,M}^* = \exp(-W^{n,J}/KT).$$

Combining (3.2) and (3.3), we obtain the general formula of the rate constant k of the first order reaction (the thermal decomposition of A molecules):

$$k = \sum_{n,J,M} \frac{\Gamma^{n,J,M}}{\hbar} \frac{Z_{n,J,M}^*}{Z_i} \quad (3.4)$$

It will be obvious that this formula does not correspond to the theory of Hinshelwood, but to the theory of Rice, Ramsperger and Kassel.⁽¹¹⁾

(3.4) can be transformed to the more convenient form. The states whose energies are smaller than a certain value W_0 will have the extremely small decay widths. For the decay processes are supposed to take place with the tunnel effects through the thick potential walls. So we neglect the contributions from these states and take the state having the energy $W^{n,0,0} = W_0$ as the origin of n and measure the energies of the activated levels from this W_0 as well as the energies of the initial states are measured from the minimum level. Since the partial width of $B+C$ decomposition is nearly equal to the total width $\Gamma^{n,J}$ in the states of energies $W^{n,J,M} > W_0$, we can speak of the activated molecules whose energies are near

to W_0 and whose decomposition widths to $B+C$ are approximately equal to I . These molecules are just the activated complexes which are mathematically well defined. In this approximation, (3.4) is rewritten as follows:

$$k = \sum_{n,J,M} \frac{\Gamma_{n,J,M}}{h} \frac{Z_{n,J,M}^*}{Z_i} \exp(-W_0/KT)$$

or

$$k = \nu \frac{Z^*}{Z_i} \exp(-W_0/KT) \quad (I)$$

where

$$\nu = \left(\frac{\Gamma_{n,J,M}}{h} \right)_{av.} \quad (3.5)$$

$$Z^* = \sum_{n,J,M} \exp(-W_{n,J,M}/KT). \quad (3.6)$$

Here $av.$ means the statistical average. All the quantities appearing in the above expression (I) have the clear meanings. ν is the average decay frequency and Z^* , Z_i are the partition functions of the activated state and the initial state respectively (each not containing the translational part). Each energy origin consists in its lowest energy level. W_0 may be called the (theoretical) activation energy.

As is well known, the experimental results or the rate constants are summed up as follows:

$$k = F \exp(-W_{ex}/KT). \quad (3.7)$$

where W_{ex} is the experimental activation energy and F is the frequency factor which may be taken as constant in the small range of temperatures.

In order to compare (I) with (3.7), we shall transform (I) as follows. First,

$$\begin{aligned} Z^* &= \sum_n \left\{ \sum_{J,M} \exp\left(-\frac{W_{n,J,M} - W_{n,0,0}}{KT}\right) \right\} \exp\left(-\frac{W_{n,0,0}}{KT}\right) \\ &\equiv \sum_n Z_r^{*(n)} \exp\left(-\frac{W_{n,0,0}}{KT}\right) \equiv Z_r^* \sum_n \exp\left(-\frac{W_{n,0,0}}{KT}\right) \\ &\equiv Z_r^* \cdot Z_v^*. \end{aligned} \quad (3.8)$$

where $Z_r^* \equiv (Z_r^{*(n)})_{av.}$ is the partition function due to the rotation of the

activated molecule as a whole and Z_v^* is the one due to the other freedoms which we may call the vibrational freedoms. As for Z_r^* , it is expected to have the value of the same order with $Z_{i,r}$, because A^* molecule is similar to A molecule with the high energy. Z_r^* is composed of two parts in general, one of which is concerned in the inner degrees of freedom having the close relations to the decay process, and another is due to the other degrees of freedom which have no direct relation to the decomposition process. However, when we make the ratio $Z_r^*/Z_{i,v}$, the latter is cancelled by the corresponding part of $Z_{i,r}$. Therefore, we shall speak only of the contributions from the effective inner degrees of freedom hereafter. Then we obtain

$$Z_v^* = \frac{KT}{d} \quad (3.9)$$

where d is the mean spacing of the activated levels. For, if $d \ll KT$,

$$d = \left(\frac{1}{\rho} \right)_{av.} = \int_0^{\infty} \frac{1}{\rho} e^{-W/KT} dW / \int_0^{\infty} e^{-W/KT} \rho dW = KT / Z_v^*.$$

Putting

$$\nu = \frac{\Gamma}{h} \quad (3.10)$$

we get after all

$$k \approx \left(\frac{KT}{hZ_{i,v}} \right) \left(\frac{\Gamma}{d} \right) \exp(-W_0/KT) \quad (3.11)$$

Although $\nu \frac{Z_v^*}{Z_i}$ is also the function increasing with the temperature T , its temperature-gradient is small. So we can utilize the next relation for the order estimation of Γ/d .

$$\left(\frac{KT}{hZ_{i,v}} \right) \left(\frac{\Gamma}{d} \right) < F. \quad (3.12)$$

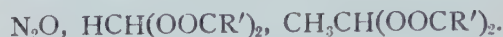
Using this equation, we obtain the next results from the experimental data in which the elementary reactions seem to act upon Lindemann's mechanism.⁽¹²⁾

$$\Gamma/d < DZ_{i,v}$$

(i) $D \doteq 0.1 \sim 1$



(ii) $D \doteq 10^{-5} \sim 10^{-4}$



At the present stage of the molecular theory, the computation of $Z_{i,v}$ is difficult for the polyatomic molecules except the simple ones. Here we take the two cases (CH_3I , N_2O) as an example.

The decomposition reaction: $\text{CH}_3\text{I} \rightarrow \text{CH}_3 + \text{I}$, $T = 300^\circ\text{C}$. If we consider that all inner degrees of freedom are concerned in the decomposition, we obtain $Z_{i,v} = 1.86$. And so $I'/d < 0.1$.

The decomposition reaction: $\text{N}_2\text{O} \rightarrow \text{N}_2 + \text{O}$, $T = 630^\circ\text{C}$. As like above, $Z_{i,v} = 3.18$. And we get $I'/d < 1.1 \cdot 10^{-4}$.

In other cases we don't know the precise values of $Z_{i,v}$ at present. But their values are nearly of 1 order for the simple molecules and become large for the complex molecules. Hence we shall be able to say as follows. When the molecule in question is simple and composed of light atoms with the strong couplings, the reaction is associated by the quantum effects as a rule (I'/d is small). On the other hand, the reaction may be treated classically, when the molecule becomes complex and contains the heavy atoms with the weak couplings (I'/d is large). The elementary reaction belonging to the class (i) contains both cases and lie halfway in general.

However the reactions in the class (ii) do not belong above region. Though we cannot conclude uniquely about the complex molecules $\text{R} \cdot \text{CH}(\text{OOCR}')_2$ for the present, N_2O reaction is, at least, in the purely quantum mechanical region. That reason probably consists in the electronic transition due to the small perturbation such as the spin-orbital coupling, namely this type of reactions is non-adiabatic. The differences between the class (i) and the class (ii) will mainly consist in the adiabatic character of their elementary reactions.

As the practical method of the computations of the rate constants, Eyring's approximation are generally used for the reactions of the class (i). It is based on the next assumptions:

(a) the classical approximation: i.e. the assumption that the reactions

proceed adiabatically and $d \ll KT$, $\Gamma \gg d$ are satisfied.

(b) the statistical approximation: i.e. the assumption that there exists the microcanonical equilibrium between the initial systems and the activated systems for each value of energies.

(a) is the assumption which enables us to use the phase space instead of the Hilbert space. Further, according to (b)

$$\frac{\Gamma^{J,M}(W)}{\hbar} \rho_{J,M}(W) \approx \int_0^{\sqrt{2\mu(W-W_J^+)}} v \frac{dp}{\hbar} \rho_{J,M}^+(W-W_J^+-W_K)$$

where $W_K = \frac{1}{2\mu} p^2$ is the kinetic energy of the system passing through the transition surface and W_J^+ is the energy at the pass on the potential energy surface (the potential is made up of the adiabatic potential and the rotational potential). Then v and p are the velocity and the momentum directed from the initial side to the final side parallel to the reaction path respectively. $\rho_{J,M}^+(W')$ is the level density in the transition surface with the energy W' . Above expression assumes that the rate for the representative points to go under the equilibrium condition from the initial side to the final side over the mountain pass is nearly equal to the real rate. Thus

$$\begin{aligned} k &= \sum_{J,M} \int_{W_J^+}^{\infty} \frac{Z_{J,M}^*(W)}{Z_i} \frac{\Gamma^{J,M}(W)}{\hbar} \rho_{J,M}(W) dW \approx \sum_{J,M} \int_{W_J^+}^{\infty} dW \frac{e^{-W/KT}}{Z_i} \times \\ &\quad \times \int_0^{W-W_J^+} \hbar^{-1} dW_K \rho_{J,M}^+(W-W_J^+-W_K) \\ &= \frac{1}{\hbar Z_i} \sum_{J,M} e^{-W_J^+/KT} \int_0^{\infty} dW' e^{-W'/KT} \int_0^{W'} dW'' \rho_{J,M}^+(W''). \end{aligned}$$

Integrating by parts,

$$k \approx \frac{KT}{\hbar Z_i} \sum_{J,M} e^{-W_J^+/KT} \int_0^{\infty} dW' \rho_{J,M}^+(W') e^{-W'/KT}.$$

Finally, k can be written as follows:

$$k \approx \frac{KT}{\hbar} \frac{Z^+}{Z_i} \exp(-W^+/KT) \quad (I')$$

where $W^+ \equiv W_0^+$ is the classical activation energy and Z^+ is the partition

function of the systems in the transition surface.

$$Z^+ = \sum_{J, M} e^{-(W_J^+ - W_0^+)/KT} \int_0^\infty dW' \rho_{J, M}^+(W') e^{-W'/KT}. \quad (3.13)$$

(I') is just the Eyring's formula.

Judging from the discussions in § 1, § 2 and the results we already obtained in this section, the assumptions (a) and (b) are not always satisfied. However, Eyring's formula is usually corrected in two points. First, the quantum mechanical partition function Z^+ is used instead of the classical one and therefore, the activation energy W^+ is corrected by the zero point energies. This has the effect mainly correcting the assumption (a). Second, the so-called transmission coefficient κ is introduced in the expression of k —which has the effect lowering the over-estimate due to the assumption (b). Thus (I') is probably useful as the order estimations of the rate constants for the reactions of the class (i), though κ is not uniquely determined from this standpoint. However, (I') is useless for the reactions belonging to the class (ii).
(to be continued.)

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Progress of Theoretical Physics Vol. IV, No: 1, Jan.~Mar., 1949.

Theory of the Interaction of Elementary Particles. IV.^{*}

— *The Problem of Vacuum Polarization* (1) —

Hiroomi UMEZAWA, Jirô YUKAWA and Eiji YAMADA.

Institute of Theoretical Physics, Nagoya University.

(Received Aug. 11, 1948)

§ 1. Introduction.

Anticipating that at least a part of the divergence difficulties in quantum field theory arises from the fact that the interaction between various fields and matter are discussed separately, losing sight of the internal correlations among them, Sakata⁽¹⁾ has proposed putting things in order by considering mixtures of several fields, and has shown that the difficulty of the self-energy of an electron can be dissolved by mixing a short-range neutral scalar meson field, the *C*-meson field, with the customary electromagnetic field.

Subsequently, the *C*-meson theory has been developed in various direction, such as the elastic scattering of electrons,⁽²⁾ the energy level shift of the hydrogen atom,⁽³⁾ etc. On the other hand, however, it is thought necessary to investigate how far the method of mixed fields is instrumental in clearing divergence difficulties of other types.

One of the major difficulties involving divergences of the type different from that of the self-energy of charged particles is the problem of vacuum polarization. This arises as a result of the requirement, by Dirac's hole theory, of the existence of an infinite number of electrons filling the negative energy levels in vacuum, and has hitherto been treated by Dirac, Heisenberg and others by the method of density matrix. But as a result of the development of Yukawa's meson theory, it was found that such a vacuum polarization occurs not only on account of a charged Fermi particle, but also by a virtual creation of a pair of Bose particles due to the absorption and emission of photons. Therefore if the existence of various charged particles is simultaneously considered there remains a possibility

* This article composes a part of the collaborated work by the *Elementary Particle Theory Group* of the Nagoya University, and was read at the annual meeting of the Physical Society of Japan held on 21—23, May, 1948.

that the terms diverging individually cancel each other as a whole, so that the method of mixed fields is expected to be effective in this case too.

As an example of a divergence of the vacuum-polarization-type, we first calculated the self-energy of a photon, taking the existence of various sorts of charged particles into account. In this case, since the form of mutual interaction is already uniquely known, there is little ambiguity, and consequently the number and type of particles required to exist in order to make the self-energy of the photon finite by this method comes to be determined. As a result of our calculation, the requisite is that Fermi particles and charged scalar (or pseudoscalar) particles be assumed to exist with a relative abundance of 1:2, while the existence of vector (or pseudo-vector) particles is precluded because the divergence involved by them is of a higher order. We next calculated the self-energy of the C -meson, but this could not be prevented from diverging by means of the above mixture. However, as there are still many unclarified points about the nature of C -mesons, especially as the relation between C -mesons and nuclear forces has not yet been sufficiently studied,* and the form of interaction is not as unique as in the case of photons, this problem requires further investigation.

Finally, we studied, as a simple case of a collision process involving vacuum polarization, the divergence appearing in the 4th order approximation of Compton scattering, taking the existence of the various charged particles mentioned above and the C -meson into account. The result was that among the 3 types of diverging terms appearing when only the interaction between electron and photon is considered, two vanish and only that proportional to e^3 remains.

As the present stage, this term can be made to converge by the method of charge renormalization, but a more fundamental solution is desirable. Summarizing our conclusions, it was found that (i) the method of mixed fields is effective to a certain extent in the problem of vacuum polarization too (ii) here again the C -meson plays the role of preventing the divergence of the electron-mass-term, (iii) no undesirable divergence is caused, apart from that of the self-energy of the C -meson itself, and (iv) the solution of this latter problem calls for a more scrutinizing investigation into the nature of the C -meson.

* Pais⁽⁴⁾ has indicated that the self-energies of nucleons due to various nuclear-force mesons can actually be made to converge by a neutral meson-field (the f -field).

§ 2. The Self-Energy of the Photon.

The interaction between a photon and a charged particle can in general be written as follows.

$$H' = eH_1 + e^2H_2,$$

where eH_1 , e^2H_2 are both quadratic in the wave function U of the charged particle, and respectively linear and quadratic in the vector potential \mathbf{A} of the electromagnetic field.

When a single photon of momentum \mathbf{l}_0 exists, the difference from vacuum is that an additional energy results from the occurrence of the following two of processes:

- (i) $\mathbf{l}_0 \rightarrow 0 \rightarrow \mathbf{l}_0$
 $\mathbf{p} + \mathbf{l}_0$
 $-\mathbf{p}$
- (ii) $\mathbf{l}_0 \rightarrow \mathbf{l}_0, \mathbf{l}_0 \rightarrow \mathbf{l}_0$
 $\mathbf{p} - \mathbf{l}_0$
 $-\mathbf{p}$

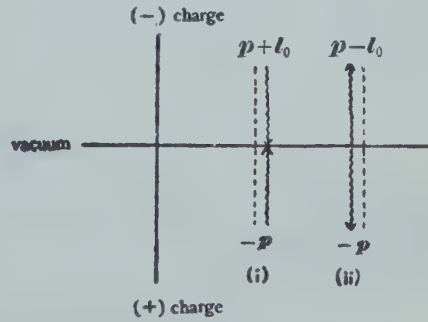


Fig. 1

The two processes giving rise to the self-energy of the photon.

where \mathbf{p} is the momentum of the charged particle. (Fig. 1)

Following Weisskopf,⁽³⁾ we identify this additional energy with the self-energy of the photon, and it becomes:

$$\begin{aligned} \varepsilon v = \sum_{\mathbf{p}} \left\{ \frac{e^2 (\mathbf{l}_0 | H_1 | \mathbf{p} + \mathbf{l}_0, -\mathbf{p}) (\mathbf{p} + \mathbf{l}_0, -\mathbf{p} | H_1 | \mathbf{l}_0)}{l_0 - E_{\mathbf{p} + \mathbf{l}_0} - E_{\mathbf{p}}} \right. \\ \left. + \frac{e^2 (\mathbf{l}_0 | H_1 | \mathbf{l}_0, \mathbf{p} - \mathbf{l}_0, -\mathbf{p}) (\mathbf{l}_0, \mathbf{p} - \mathbf{l}_0, -\mathbf{p} | H_1 | \mathbf{l}_0)}{-l_0 - E_{\mathbf{p} - \mathbf{l}_0} - E_{\mathbf{p}}} \right\} + e^2 (\mathbf{l}_0 | H_1 | \mathbf{l}_0) \end{aligned} \quad (1)$$

The 1st and 2nd terms are the contributions from processes (i) and (ii) respectively, E_p denotes the absolute value of the energy of the charged particle.

(a) The interaction between a Fermi particle and a photon is

$$e\bar{H}_1 = -e\{\psi^* \boldsymbol{\alpha} \mathbf{A} \psi, \quad e\bar{H}_2 = 0 \quad (2)$$

$$A = \sum_{l, \mu} \sqrt{\frac{2\pi}{l}} e(l, \mu) \{ e^{i(l \cdot r)} C_{l\mu} + e^{-i(l \cdot r)} C_{l\mu}^* \} \quad E_p = \sqrt{\mu^2 + p^2}$$

ψ , μ are the wave function and rest-mass of the Fermi particle respectively.

Substituting this into (1) and calculating the sum over spin directions of the Fermi particle by Casimir's method,

$$W_{r, r} = \frac{2\pi e^2}{l_0} \int \frac{(E_{p+l_0} + E_p) \{ 2(p \cdot e)^2 - p^2 - (p \cdot l_0) - \mu^2 - E_{p+l_0} E_p \}}{E_{p+l_0} E_p \{ p^2 + (p \cdot l_0) + \mu^2 + E_{p+l_0} E_p \}} \cdot \frac{p^2 dp \sin \theta d\theta d\varphi}{(2\pi)^2} \quad (3)$$

where e is the unit vector in the direction of polarization of the photon, θ the angle between p and l_0 , and φ that between e and the projection of p on the plane perpendicular to l_0 . Expanding this in terms of $1/p$ for large p 's and computing for the diverging terms, the logarithmic divergence vanishes on integration over the angles, and we obtain the expression

$$W_{r, r} = -\frac{2e^2}{3\pi l_0} p^2 + \text{finite term} \quad (4)$$

To investigate the nature of the finite term, we expand it for small l_0 and integrate, obtaining

$$W_{r, r} = -\frac{2e^2}{3\pi l_0} p^2 + O(l_0)$$

That is to say, the finite term tends to zero with l_0 .*

(b) The Lagrange function of a charged scalar (or pseudoscalar) meson U when an interaction exists between an electromagnetic field is

* The self-energy of the photon due to an electron has also been calculated by Serber,⁽⁶⁾ using a method which is an improvement, on the density matrix method of Dirac and Heisenberg. His result is

$$W = -\frac{a\hbar\nu}{2\pi} \left\{ \left(\frac{mc^2}{\hbar\nu} \right)^2 \left[8 \frac{Z^2}{R^4} + \left(\frac{4}{3} \right) \frac{x^2 Z^2}{R^4} - 2 \frac{Z^2}{R^2} \right] + \frac{1}{3} \frac{x^2}{R^2} + O(R) \right\}$$

where R is the off-diagonal distance $|\mathbf{X}|$, Z and x the components of \mathbf{X} in the directions of polarization and propagation respectively. As the introduction of \mathbf{X} corresponds to a shifting of the position \mathbf{X} through $\mathbf{X}/2$, if we introduce the cut-off momentum P instead of $1/r_0 = 2/R$ and integrate over all directions of \mathbf{X} , the diverging term coincides, in our units, with that of (4). However, since Serber regards only those terms independent of R as self-energy, the latter does tend to 0. (Serber, takes \hbar/mc , \hbar/mc^2 , m as the units of length, time and mass respectively.)

$$L = \frac{1}{4\pi} \left\{ \left(\frac{\partial U^*}{\partial t} + ieA_0 U^* \right) \left(\frac{\partial U}{\partial t} + ieA_0 U \right) - (\text{grad } U^* + ie\mathbf{A}U^*) (\text{grad } U - ie\mathbf{A}U) - \kappa^2 U^* U \right\}^\dagger$$

Deriving the Hamiltonian function from this, the interaction becomes

$$H' = -\frac{ie}{4\pi} \mathbf{A} (U^* \text{grad } U - \text{grad } U^* U) + \frac{e^2}{4\pi} \mathbf{A}^2 U^* U \quad (5)$$

$$= eH_1 + e^2 H_2,$$

with

$$U = i\kappa c \sum_{\mathbf{p}} \sqrt{\frac{2\pi}{E_{\mathbf{p}}}} (c_{\mathbf{p}} - d_{\mathbf{p}}^*) e^{i(\mathbf{p} \cdot \mathbf{r})} \quad (6)$$

Substituting (6) in (5),

$$H_1 = (2\pi)^{3/2} \sum_{\mathbf{p}, \mathbf{p}', l} \frac{2i(\mathbf{p}' \cdot \mathbf{e}(l))}{\sqrt{E_{\mathbf{p}} E_{\mathbf{p}'} l}} [(c_{\mathbf{p}}^* - d_{\mathbf{p}})(c_{\mathbf{p}'} - d_{\mathbf{p}'}^*) \cdot \{C_{l\mu}^* \delta(-\mathbf{p} + \mathbf{p}' - l) + C_{l\mu} \delta(-\mathbf{p} + \mathbf{p}' + l)\}]$$

$$H_2 = (2\pi)^2 \sum_{\mathbf{p}, \mathbf{p}', l, l'} \frac{(c_{\mathbf{p}}^* - d_{\mathbf{p}})(c_{\mathbf{p}'} - d_{\mathbf{p}'}^*)}{\sqrt{E_{\mathbf{p}} E_{\mathbf{p}'} l l'}} \{C_{l\mu}^* C_{l'\mu'}^* \delta(-\mathbf{p} + \mathbf{p}' - l - l') + C_{l\mu}^* C_{l'\mu'} \delta(-\mathbf{p} + \mathbf{p}' - l + l') + C_{l\mu} C_{l'\mu'}^* \delta(-\mathbf{p} + \mathbf{p}' + l - l') + C_{l\mu} C_{l'\mu'} \delta(-\mathbf{p} + \mathbf{p}' + l + l')\} e(l\mu) e(l'\mu') \quad (8)$$

Substituting these in (1) and expanding, as before, in terms of $1/p$ and integrating,

$$W_{P.S.} = -\frac{e^2}{2\pi l_0} \left(\frac{1}{3} p^2 - \mu^2 \log p \right) + \frac{e^2}{2\pi l_0} (p^2 - \mu^2 \log p) + O(l_0)$$

$$= \frac{e^2}{3\pi l_0} p^2 + O(l_0). \quad (9)$$

The expressions in the 1st and 2nd parentheses are respectively the contributions of eH_1 , and $e^2 H_2$. The finite term, as in (4), tends to zero with l_0 .

† As U is involved quadratically in the interaction, the difference of nature regarding reflection has no effect, and the interaction is identically the same for both scalar and pseudo-scalar cases.

(c) For the interaction of charged vector and pseudovector mesons U and U_0 with an electromagnetic field, we start, as before, from the Lagrangian function, derive the Hamiltonian, and eliminate U_0 , obtaining

$$\begin{aligned} H' = & 4\pi ic(U^+ \mathbf{A} \operatorname{div} \mathbf{U}^{*+} - \mathbf{U}^{*+} \mathbf{A} \operatorname{div} \mathbf{U}^+) \\ & + \frac{ie}{4\pi x^2} \{ [\mathbf{A} \mathbf{U}^*] \operatorname{curl} \mathbf{U} - [\mathbf{A} \mathbf{U}] \operatorname{curl} \mathbf{U}^* \} + 4\pi c^2 (\mathbf{A} \mathbf{U}^+) (\mathbf{A} \mathbf{U}^{*+}) \\ & + \frac{c^2}{4\pi x^2} [\mathbf{A} \mathbf{U}] [\mathbf{A} \mathbf{U}^*] = cH_1 + c^2H_2 \quad (10) \end{aligned}$$

$$\mathbf{U} = \sum_{\mathbf{p}} \left\{ \frac{i\sqrt{2\pi}}{\sqrt{E_p}} \sum_{\lambda=1,2} (a_{p,\lambda} - b_{p,\lambda}^*) \mathbf{e}(\mathbf{p}, \lambda) + \sqrt{2\pi} E_p (A_p + B_p^*) \mathbf{e}(\mathbf{p}) \right\} e^{i\mathbf{p} \cdot \mathbf{r}} \quad (11)$$

$$\mathbf{U}^+ = \sum_{\mathbf{p}} \left\{ \frac{\sqrt{E_p}}{\sqrt{8\pi x}} \sum_{\lambda=1,2} (a_{p,\lambda}^* + b_{p,\lambda}) \mathbf{e}(\mathbf{p}, \lambda) + \frac{i}{\sqrt{8\pi E_p}} (A_p - B_p) \mathbf{e}(\mathbf{p}) \right\} e^{-i\mathbf{p} \cdot \mathbf{r}} \quad (12)$$

Here \mathbf{U}^+ is the momentum conjugate to \mathbf{U} , $\mathbf{e}(\mathbf{p}, \lambda)$ and $\mathbf{e}(\mathbf{p})$ are the unit vectors in the directions of polarization and propagation respectively of the meson with momentum \mathbf{p} .

Substituting (10), (11) and (12) in (1) and performing the expansion and integration as before, the diverging terms of the self-energy become after laborious calculations,

$$\begin{aligned} W_{r,v} = & \left\{ \frac{6c^2}{8\pi l_0 x^2} \int_0^\infty p^3 dp + \frac{4c^2}{\pi l_0} \int_0^\infty p dp - \frac{7c^2 x^2}{4\pi l_0} \int_0^\infty \frac{dp}{p} \right\} - \left\{ \frac{8c^2}{3\pi l_0 x^2} \int_0^\infty p^3 dp \right. \\ & \left. + \left(\frac{7c^2 l_0}{3\pi x^2} + \frac{2c^2}{3\pi l_0} \right) \int_0^\infty p dp + \left(\frac{c^2 l_0^3}{42\pi x^2} + \frac{17l_0 c^2}{15\pi} - \frac{83c^2 x^2}{860\pi l_0} \right) \int_0^\infty \frac{dp}{p} \right\} \\ = & -\frac{2c^2}{3\pi l_0 x^2} \left(\frac{p^4}{4} \right) - \left(\frac{7c^2 l_0}{3\pi x^2} - \frac{10c^2}{3\pi l_0} \right) \left(\frac{p^2}{2} \right) - \left(\frac{c^2 l_0^3}{42\pi x^2} + \frac{17}{15} \frac{l_0 c^2}{\pi} \right. \\ & \left. + \frac{547}{360} \frac{c^2 x^2}{\pi l_0} \right) \times \int_0^\infty \frac{dp}{p} \quad (13) \end{aligned}$$

The expression in the 1st and 2nd parentheses are the contributions of cH_1 , and c^2H_2 respectively, and x the rest-mass of the meson. It is an effect of the longitudinal wave meson that the divergence has become of 4th order.

Comparing (4), (9) and (11), we see that when minding the fields of various charged particles in order to make the self-energy of the photon finite, the vector meson is precluded because the divergence it involves is

of higher order than that of any other field, and hence it becomes necessary that Fermi particles and scalar or pseudoscalar particles exist in nature with a relative abundance of 1:2. Therefore, there must exist 2 scalar (or pseudoscalar) particle against each electron of proton, and if any further charged Fermi particles exist, a similar counterbalance must hold for them.

§ 3. The Self-Energy of the *C*-meson.

If we regard \mathbf{l} as the momentum of the *C*-meson, here again the two types of intermediate states (i), (ii) which appeared in the calculation of the self-energy of the photon can be considered, and if the interaction is written

$$\bar{H}' = f\bar{H}_1 + g^2\bar{H}_2$$

the self-energy becomes, similar to (1),

$$W = \sum_{\mathbf{p}} \left\{ \frac{f^2(\mathbf{l}_0 | H_1 | \mathbf{p} + \mathbf{l}_0, -\mathbf{p})(\mathbf{p} + \mathbf{l}_0, -\mathbf{p} | H_1 | \mathbf{l}_0)}{\varepsilon_{l_0} - E_{\mathbf{p} + \mathbf{l}_0} - E_{\mathbf{p}}} + \frac{f^2(\mathbf{l}_0 | H_1 | 2\mathbf{l}_0, \mathbf{p} - \mathbf{l}, -\mathbf{p})(2\mathbf{l}_0, \mathbf{p} - \mathbf{l}_0, -\mathbf{p} | H_1 | \mathbf{l}_0)}{-\varepsilon_{l_0} - E_{\mathbf{p} - \mathbf{l}_0} - E_{\mathbf{p}}} + g^2(\mathbf{l}_0 | H_2 | \mathbf{l}_0) \right\} \quad (14)$$

Denoting the wave function and rest-mass of the *C*-meson by ϕ and M respectively,

$$\phi = \sum_i \sqrt{\frac{2\pi}{\varepsilon_i}} (q_i^* e^{-i\mathbf{l} \cdot \mathbf{r}} + q_i e^{i\mathbf{l} \cdot \mathbf{r}}) \quad \varepsilon_i = \sqrt{M^2 + \mathbf{l}^2}$$

Further, as the interaction between a charged particle and a *C*-meson, we use the expression that Hara⁽¹⁾ used in order to make the electromagnetic self-energy of the charged particle itself converge, but the form seems to be not so uniquely determined as in the case of the self-energy of the photon.

(a) The interaction between a Fermi particle and a *C*-meson is

$$\bar{H}' = f \int \psi^* \beta \psi \phi = f \bar{H}_1 \quad (15)$$

Substituting this in (13), taking $\mathbf{l}_0 = 0$ (the case of rest), and summing over the spin directions of the Fermi particle as in the case of the photon,

the integration can be exactly performed because $l_0=0$, and the self-energy of the C -meson becomes

$$W_{c.v} = -\frac{f^2}{\pi M} p^2 - \frac{f^2}{\pi M} \left(\frac{M^2}{2} - 3\mu^2 \right) \log \frac{2p}{\mu} \\ + \left\{ -\frac{f^2}{\pi M} \left[\frac{\mu^2}{2} - \frac{4}{M} \left(\frac{M^2}{4} - \mu^2 \right)^{\frac{3}{2}} \log \left(1 + \frac{M}{2\mu} \right) \left\{ 1 + \left(\frac{M^2}{4} - \mu^2 \right)^{\frac{1}{2}} \frac{M}{2} \right\} \right] \right. \\ \left. - \frac{f^2}{\pi M} \left[\frac{\mu^2}{2} - \frac{4}{M} \left(\mu^2 - \frac{M^2}{4} \right)^{\frac{3}{2}} \left\{ \sin^{-1} \frac{1}{\mu} \left(\mu^2 - \frac{M^2}{4} \right)^{\frac{1}{2}} \right. \right. \right. \right. \text{for } M > \mu \\ \left. \left. \left. - \sin^{-1} \mu \left(\mu^2 - \frac{M^2}{4} \right)^{-\frac{1}{2}} \right\} \right] \right\} \text{ for } \mu > M$$

There are two alternative forms for the finite term, depending on whether the mass of the charged particle is smaller or larger than that of the C -meson.

(b) The interaction between a scalar (or pseudoscalar) meson and a C -meson is

$$\bar{H}' = -\frac{f}{4\pi x} \int U^* U \phi - \frac{g^2}{4\pi x^2} \int U^* U \phi^2 = f\bar{H}_1 + g^2\bar{H}_2 \quad (17)$$

Substituting this in (13) with $l_0=0$ and calculating,

$$W_{c.s} = -\frac{g^2}{2\pi M} p^3 + \frac{x^2}{\pi M} \left(\frac{g^2}{2} - \frac{f^2}{4} \right) \log \frac{2p}{x} + \frac{f^2 x^2}{2\pi M^2} \sqrt{x^2 - \frac{M^2}{4}} \varphi(x, M) \\ \varphi(x, M) = \sin^{-1} \frac{1}{x} \left(x^2 - \frac{M^2}{4} \right)^{\frac{1}{2}} - \sin^{-1} x \left[x^2 - \frac{M^2}{4} \right]^{-\frac{1}{2}} \quad (18)$$

(c) The interaction between a vector (or pseudovector) meson and a C -meson is

$$\bar{H}' = -\frac{f}{4\pi x} \int U^* U \phi - \frac{g^2}{4\pi x^2} \int U^* U \phi^2 \\ - \frac{4\pi f^2}{x^2} \int \phi^2 \operatorname{div} U^+ \operatorname{div} U^+ = f\bar{H}_1 + g^2\bar{H}_{21} + f^2\bar{H}_{22} \quad (19)$$

Substituting this in (13) with $l_0=0$ and calculating,

$$W_{c.v} = -\frac{1}{\pi M x^2} \left(\frac{g^2}{4} + \frac{5f^2}{16} \right) p(p+x^2) - \frac{1}{\pi M} \left\{ \frac{7}{8} g^2 - \frac{f^2}{32} \left(13 - \frac{M^2}{x^2} \right) \right\} p(p^2+x^2)^{\frac{1}{2}} \\ + \frac{1}{\pi M} \left\{ -\frac{9}{8} x^2 g^2 - \frac{f^2 x^2}{32} \left(19 - \frac{M^2}{x^2} + \frac{M^4}{2x^2} \right) \right\} \log \frac{2p}{x} + \frac{x^2 f^2}{\pi M} \left\{ 1 + \frac{M^4}{8x^4} \sqrt{x^2 - \frac{M^2}{4}} \right\} \quad (20)$$

In this case too, as in that of the photon, the order of divergence becomes the 4th, due to the longitudinal wave of the vector meson.

Reviewing the above results we see that among the various charged particles, a vector meson gives a higher order divergence, and that even the existence of Fermi particle and Scalar mesons fails to make the self-energy of the C -meson finite, because the highest order terms in the two divergences are of the same sign. This state of affairs arises from the fact that the signs of the highest terms in the divergences of electromagnetic self-energy of charged particles which it is the function of C -mesons to cancel are all positive; but after all, we are not yet in a position to say anything definite about the self-energy of C -mesons, due to several reasons. That is to say, there might be, as mentioned before, some ambiguity as to the form of the interaction between a C -meson and a charged particle; and also, apart from this, there are many unknown aspects about the nature of the C -meson itself, the types of cohesive force fields, etc., and in particular it is altogether unknown what role the C -meson plays in the divergence of the self-energy of nucleons due to nuclear-force mesons. For instance, the C -meson itself might act as the field of cohesive force against this divergence, and even if this role were played by a cohesive force meson different from the C -meson now considered, it may be possible that an interaction exists between these two cohesive force mesons such that prevent each other's self-energy from diverging. Actually, the authors have considered two type of scalar cohesive force mesons for charged particles, and by assuming a direct interaction between the former themselves, have succeeded in making their self-energies finite too. (*to be continued.*)

Note added in proof. In this paper, on account of the imperfectness of usual perturbation calculation, we could not make detailed discussions of the finite terms. We found a new prescription to carry out perturbation calculations in a perfectly relativistic manner. By means of this prescription, we could calculate exactly every case of vacuum polarization including the finite terms. Detailed considerations and results based on this method will be given in the near future. Moreover, as a result of this calculation, we found that Eq. (13) was incorrect, and that the self-energy of the photon due to the vector meson field is three times that due to the scalar meson field, so that we need not preclude the vector meson field from the mixture of charged particles; and also that the finite term of the self-energy could not be written simply as $0(l_0)$.

Divergence Difficulty and Mixed Meson Theory. II.

Gentaro ARAKI.

*Department of Industrial Chemistry,
Kyoto University.*

(Received Sep. 1, 1948)

Introduction

In the first paper,⁽¹⁾ under the same title, the pseudoscalar-pseudovector mixture of meson fields was discussed in order to eliminate divergences from the magnetic moment of the nucleon and the two-nucleon potential. In this paper it was assumed that the interaction between pseudovector mesons and nucleons is the 6-vector coupling. On the other hand we can equally assume the interaction to be the pseudovector coupling.* We should also examine this case in order to work out all cases.

The purpose of the present paper is to examine all mixtures of two meson fields taking into account both the above mentioned couplings. We shall find the conclusion that there is only a case, a vector-pseudovector mixture, satisfying the above mentioned requirement in addition to the previously discussed case, pseudoscalar-pseudovector mixture.

§ 1. Source Density.

The source density functions of the meson field are given by Hermitian forms of the wave functions of nucleons which transforms according to irreducible representations of the Lorentz group.⁽²⁾ They are divided into five classes: a scalar (R), a 4-vector (\mathbf{M}, M_0), a 6-vector or antisymmetric tensor of the second rank (\mathbf{T}, \mathbf{S}), a pseudo-4-vector or antisymmetric tensor of the third rank (\mathbf{P}, P_0), and a pseudoscalar or antisymmetric tensor of the fourth rank (Q), where

$$R = \bar{\Psi} \dagger \tau \rho_3 \Psi \quad (1.1)$$

* S. Watanabe suggested me that it is necessary to examine a mixture of two pseudovector fields with such couplings for the sake of completeness.

$$\mathbf{M} = \Psi^\dagger \tau \rho_1 \sigma \Psi, \quad M_0 = \Psi^\dagger \tau \Psi \quad (1.2)$$

$$\mathbf{T} = -\Psi^\dagger \tau \rho_2 \sigma \Psi, \quad S = \Psi^\dagger \tau \rho_3 \sigma \Psi \quad (1.3)$$

$$\mathbf{P} = \Psi^\dagger \tau \sigma \Psi, \quad P_0 = \Psi^\dagger \tau \rho_1 \Psi \quad (1.4)$$

$$Q = \Psi^\dagger \tau \rho_2 \Psi \quad (1.5)$$

The notations in the right hand sides of these expressions are the same as those in the previous paper.

The source densities of the scalar meson field are given by R and (\mathbf{M}, M_0) , those of the vector field by (\mathbf{M}, M_0) and (\mathbf{T}, S) , those of the pseudovector field by (\mathbf{T}, S) and (\mathbf{P}, P_0) , and those of the pseudoscalar field by (\mathbf{P}, P_0) and Q . When the source density functions are given by (1.1), (1.2), (1.3), (1.4), and (1.5) we shall refer to the interactions between mesons and nucleons as the scalar, vector, 6-vector, pseudovector and pseudoscalar couplings respectively.

§ 2. Magnetic Moment of Nucleons.

It is generally assumed in the contemporary theory of elementary particles that the Hamiltonian of an isolated nucleon is given by Dirac's form. If the nucleon is placed in a magnetic field described by a vector potential function $\mathbf{A}(\mathbf{x})$ the magnetic energy part of its Hamiltonian becomes $-\tau^\dagger \tau \mu_n \sigma \mathbf{A}$. This is equivalent to $-\tau^\dagger \tau \mu_n \sigma \text{rot } \mathbf{A}$ for a nucleon at rest where $\tau^\dagger \tau$ denotes the projection (operator) to the proton state and μ_n denotes the nuclear magneton.

If we take into account the interaction of the nucleon with mesons the surplus magnetic energy $\tau^\dagger \tau \mu_n \sigma \text{rot } \mathbf{A}$ arises from the interaction for a nucleon at rest in the Schrödinger approximation. Therefore the magnetic moment of the proton amounts to $\mu_n + \mu_p$ and that of the neutron becomes a non-vanishing value $-\mu_p$.

Among the source densities given by (1.1)–(1.5) the scalar, the vector, and the pseudoscalar have no contribution to μ_p in the Schrödinger approximation. Accordingly μ_p vanishes for the scalar field. Only the 6-vector coupling contributes to μ_p for the vector field and only the pseudovector coupling does for the pseudoscalar field. In the case of the pseudovector field both the 6-vector and pseudovector couplings contribute to μ_p . The explicit forms of μ_p are given for three types of meson fields as follows:

(a) a vector meson field⁽¹⁾:

$$\mu_P = (c/\hbar c)(G/K_1)^2(B-K_1) \quad (2.1)$$

(b) a pseudovector meson field:

$$\mu_P = (c/\hbar c)(F/K)^2(K-3B/4) + (c/\hbar c)(g/K)^2(K-B)/2 \quad (2.2)$$

(c) a pseudoscalar meson field⁽¹⁾:

$$\mu_P = (c/\hbar c)(f/x)^2(B-x)/2 \quad (2.3)$$

where G is a constant of the vector coupling between vector mesons and nucleons, F and g are respectively constants of the pseudovector and 6-vector couplings between pseudovector mesons and nucleons, f is a constant of the pseudovector coupling between pseudoscalar mesons and nucleons, $\hbar K_1/c$, $\hbar K/c$, and $\hbar x/c$ are the masses of the vector, pseudovector and pseudoscalar mesons respectively, and

$$B = \frac{4}{3\pi} \int_0^\infty dk \quad (2.4)$$

is a divergent integral which we want to eliminate. The calculation of (2.2) will be accounted for in the last section.

§ 3. Two-Nucleon Potential.

The two-nucleon potential W according to the four types of the meson theory are well known.⁽³⁾ As we have just seen in the previous section the vector and pseudoscalar couplings have no contribution to the magnetic moment of a nucleon. Accordingly we omit these couplings altogether. The two-nucleon potential is then given for the three types of meson fields as follows:

(a) a vector meson field:

$$W = G^2(\vec{\tau}^{(1)}\vec{\tau}^{(2)}/2) \{ (\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}/3) 2\phi(K_1, r) - A\chi(K_1, r) + CA/(K_1^2 r^3) \} \quad (3.1)$$

(b) a pseudovector meson field:

$$W = -(2F^2 + g^2)(\vec{\tau}^{(1)}\vec{\tau}^{(2)}/2)(\boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)}/3)\phi(K, r) \\ + (F^2 - g^2)(\vec{\tau}^{(1)}\vec{\tau}^{(2)}/2) \{ A\chi(K, r) - CA/(K^2 r^3) \} \quad (3.2)$$

(c) a pseudoscalar meson field:

$$W = f^2 (\vec{r}^{(1)} \cdot \vec{r}^{(2)} / 2) \{ (\sigma^{(1)} \sigma^{(2)} / 3) \phi(x, r) + A \chi(x, r) - CA / (x^2 r^3) \} \quad (3.3)$$

where

$$A = 3(\sigma^{(1)} \mathbf{x})(\sigma^{(2)} \mathbf{x}) / r^2 - \sigma^{(1)} \sigma^{(2)} \quad (3.4)$$

$$\phi(x, r) = r^{-1} \exp(-xr) \quad (3.5)$$

$$\chi(x, r) = r^{-1} \{ 3^{-1} + (xr)^{-1} + (xr)^{-2} \} \exp(-xr) \quad (3.6)$$

$$C = \frac{2}{\pi} \int_0^\infty \left(\frac{1}{3} x \sin x + \cos x \right) dx \quad (3.7)$$

\mathbf{x} is a relative position vector between two nucleons, and r is its absolute value. The divergent terms of direct interaction type are here eliminated and the non-eliminable divergent parts are explicitly represented by using a divergent integral C . This parts are just those which we want to eliminate by mixing fields.

§ 4. Elimination of Divergence.

In the case of the scalar meson field there is no divergent term in the two-nucleon potential and μ_p vanishes. Accordingly there is no need to discuss this case, and we have only to examine remaining three cases of vector, pseudovector and pseudoscalar fields.

We see that the sign of the divergent integrals B and C involved in the magnetic moment and the two-nucleon potential are different for different types of meson fields. Therefore there is a possibility to eliminate both divergent integrals by mixing two fields.

We shall distinguish three kinds of the pseudovector mesons. The first is in interaction with nucleons by the β -vector coupling, the second by the pseudovector coupling, and the third by both couplings. They will be referred to as pvt , pvv and $p\bar{v}$ respectively. Further the vector and the pseudoscalar mesons will be referred to as v and ps respectively. If we consider a mixture of two fields in order to eliminate divergences in question we have eight possibilities of combinations as follows: (a) ps and v , (b) ps and pvt , (c) v and pvt , (d) $p\bar{v}$ and v , (e) $p\bar{v}$ and ps , (f) $p\bar{v}$ and pvt , (g) $p\bar{v}$ and ps , (h) $p\bar{v}$ and v . The case (a) is Schwinger's and the

cases (b) and (c) were discussed in the previous paper.

The coefficients of B and C for these various cases are given by the following table omitting common factors :

case	coeff. of B	coeff. of C
(a)	$2(G/K_1)^2 + (f/x)^2$	$(G/K_1)^2 - (f/x)^2$
(b)	$(f/x)^2 - (g/K)^2$	$(g/K)^2 - (f/x)^2$
(c)	$2(G/K_1)^2 - (g/K)^2$	$(G/K_1)^2 + (g/K)^2$
(d)	$4(G/K_1)^2 - 3(F/K')^2$	$(G/K_1)^2 - (F/K')^2$
(e)	$2(f/x)^2 - (F/K')^2$	$-(f/x)^2 - (F/K')^2$
(f)	$-2(g/K)^2 - 3(F/K')^2$	$(g/K)^2 - (F/K')^2$
(g)	$(f/x)^2 - (g/K)^2 - 3(F/K)^2/2$	$(g/K)^2 - (F/K)^2 - (f/x)^2$
(h)	$4(G/K_1)^2 - 2(g/K)^2 - 3(F/K)^2$	$(G/K_1)^2 + (g/K)^2 - (F/K)^2$

where the mass of $p\bar{\nu}\nu$ is denoted by $\hbar K'/c$ in order to distinguish it from the mass of $p\nu$, and the latter is denoted by $\hbar K/c$ as well as that of $p\nu$. It can easily be seen from this table which case satisfies the above mentioned requirement. In the cases (a) and (f) we can exclude C but do not B . In the cases (c) and (e) we can exclude B but do not C . In the case (d) we can do separately B or C but do not simultaneously both. In the case (g), F becomes zero if both the coefficients of B and C are equated to zero. Consequently this case reduces to (b).

In the cases (b) and (h) we can eliminate both B and C . The conditions for this requirement are

$$\text{Case (b)} \quad (g/K)^2 = (f/x)^2 \quad (4.1)$$

$$\text{Case (h)} \quad (G/K_1)^2 = (g/K)^2, \quad F^2 = 6g^2. \quad (4.2)$$

If these relations are satisfied the surplus magnetic moment of the proton is given by

$$\text{Case (b)} \quad \mu_p = (e/2x)(f^2/\hbar c)(K/x - 1) \quad (4.3)$$

$$\text{Case (h)} \quad \mu_p = (e/2K)(g^2/\hbar c)(13 - 10K_1/K) \quad (4.4)$$

Therefore, in order that μ_p is positive, K should be larger than x in the case (b) and K_1 should be smaller than $1.3K$ in the case (h).

On the other hand, by (4.1) and (4.2), the two-nucleon potential be-

comes as follows :

$$\text{Case (b)} \quad W = f^2 (\vec{\tau}^{(1)} \vec{\tau}^{(2)} / 2) [(\sigma^{(1)} \sigma^{(2)} / 3) \{ \phi(x, r) - (K/x)^2 \phi(K, r) \} \\ + A \{ \chi(x, r) - (K/x)^2 \chi(K, r) \}] \quad (4.5)$$

$$\text{Case (h)} \quad W = g^2 (\vec{\tau}^{(1)} \vec{\tau}^{(2)} / 2) [(\sigma^{(1)} \sigma^{(2)} / 3) \{ 10 (K_1/K)^2 \phi(K_1, r) \\ - 13 \phi(K, r) \} + 5A \{ \chi(K, r) - (K_1/K)^2 \chi(K_1, r) \}] \quad (4.6)$$

The coefficient of A should be negative in order to agree with the experiment of the quadrupole moment of the deuteron. In order to satisfy this requirement it is necessary that K is larger than x in the case (b) and K_1 is larger than K in the case (h). Then the spherically symmetric part is attractive in the case (b), and repulsive in the case (h) for larger r for 1S and $^3S+^3D$ states of the deuteron. If we require that it becomes attractive for small r in the case (h) we must restrict K_1 more severely so that

$$\text{Case (h)} \quad 1.14K < K_1 < 1.3K \quad (4.7)$$

$$(1.14 = \sqrt{1.3}).$$

The qualitative difference between potentials in the two cases (b) and (h) is in the behavior of their spherically symmetric parts.

We have thus examined all cases of possible mixtures of two meson fields and we see that only two cases (b) and (h) enable us to eliminate the divergences in question. If we want to know which assumption is in accordance with experiments the more detailed study is necessary.

§ 5. Magnetic Moment of Nucleon according to Pseudovector Theory.

The principle of calculating μ_p was already explained.⁽⁴⁾ We shall give a brief account for the calculation of it according to the pseudovector theory taking into account both the pseudovector and 6-vector couplings. According to the above mentioned principle the matrix element of the surplus magnetic energy operator in the case of the pseudovector field is given as follows :

$$U_{NA}^{(1)} = \tau_3 (e/K^2) (\pi/V) \sum_p (B_u + B_{\bar{u}} + B_n) / (EE')^2 \quad (5.1a)$$

$$U_{NA}^{(2)} = \tau_3 (e/K^2) (\pi/V) \sum_p (B_u' + B_{\bar{u}}' + B_n') / (EE')^2 \quad (5.1b)$$

where

$$B_u = (F\hbar cK/E' + ig)(F\hbar cK/E - ig)C_u \quad (5.2a)$$

$$\begin{aligned} B_u = & \{FE'/(\hbar cK) + ig\} \{F\hbar cK/E - ig\} (E/p'^2) \\ & \cdot \sum_e (\sigma \mathbf{p}') (\sigma \mathbf{e}) \{ (E p'^2 - E' \mathbf{p} \mathbf{p}') \mathbf{e} + E' (\mathbf{p}' \mathbf{e}) \mathbf{p} \} \mathbf{A}_{10} \\ & + (F\hbar cK/E' + ig) \{FE/(\hbar cK) - ig\} (E'/p^2) \\ & \cdot \sum_{e'} (\sigma \mathbf{e}') (\sigma \mathbf{p}) \{ (E' p^2 - E \mathbf{p} \mathbf{p}') \mathbf{e}' + E (\mathbf{p} \mathbf{e}') \mathbf{p}' \} \mathbf{A}_{10} \quad (5.2b) \end{aligned}$$

$$B_u = \{FE'/(\hbar cK) + ig\} \{FE/(\hbar cK) - ig\} C_u \quad (5.2c)$$

$$B_u' = \{F\hbar cK\}^2 / (EE') - g^2 + iFg\hbar cK(E - E') / (EE') \} C_u \quad (5.3a)$$

$$\begin{aligned} B_u' = & \{-F^2 E' / E + g^2 + iFg(\hbar cK/E + E' / \hbar cK)(E' - E) / (E' + E) \} \\ & \cdot (E/p'^2) \sum_e (\sigma \mathbf{p}') (\sigma \mathbf{e}) \{ (E p'^2 + E' \mathbf{p} \mathbf{p}') \mathbf{e} - E' (\mathbf{p}' \mathbf{e}) \mathbf{p} \} \mathbf{A}_{10} \\ & + \{-F^2 E / E' + g^2 + iFg(\hbar cK/E' + E / \hbar cK)(E' - E) / (E' + E) \} \\ & \cdot (E'/p^2) \sum_{e'} (\sigma \mathbf{e}') (\sigma \mathbf{p}) \{ (E' p^2 + E \mathbf{p} \mathbf{p}') \mathbf{e}' - E (\mathbf{p} \mathbf{e}') \mathbf{p}' \} \mathbf{A}_{10} \quad (5.3b) \end{aligned}$$

$$B_u' = \{-F^2 EE' / (\hbar cK)^2 + g^2 + iFg(E' - E) / (\hbar cK) \} C_u \quad (5.3c)$$

and the meaning of $U^{(1)}$, $U^{(2)}$, C_u , C_u , tt , tl , ll , and other notations are all the same as those in Part I.

The terms proportional to F^2 and Fg are new parts. The former part will be denoted by U^F . In this part the ll -contributions to $U^{(1)}$ and $U^{(2)}$ cancel each other, and the tt - and tl -contributions partly cancel. The matrix elements of U^F is thus reduced to

$$\begin{aligned} U_{BA}^F = & \tau_3 c (F/K)^2 (2\pi/V) \sum_{\mathbf{p}} [\{ 4(\hbar cK)^2 + (\mathbf{p}_1 - \mathbf{p}_0)^2 \} (\mathbf{p} \mathbf{A}_{10}) \\ & + \{ (\mathbf{p}_1 - \mathbf{p}_0)^2 / 2 - (E^2 + E'^2) / 2 + 2(\hbar cK)^2 \{ \hbar \sigma (\text{rot } \mathbf{A}_{10}) \} \} / (EE')^2] \quad (5.4) \end{aligned}$$

The first line of this expression exactly vanishes when the summation over the whole directions of \mathbf{p} is carried out because of the relation (2.9) in Part I. If we carry out further the summation over the whole values of $|\mathbf{p}|$ the terms proportional to $\text{rot } \mathbf{A}$ of U^F become

$$U^F = -\sigma \text{rot } \mathbf{A} \cdot \tau_3 \frac{c}{\hbar c} \left(\frac{F}{K} \right)^2 \frac{1}{\pi} \int_0^\infty \frac{k^4 - K^2 k^2}{(k^2 + K^2)^2} dk \quad (5.5)$$

from which the first part of (2.2) is easily obtained.

The terms proportional to Fg contains only higher derivatives of rot

A. Consequently they have no contribution to μ_p at all. The terms proportional to g^2 was calculated in Part I.

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Note added in proof: A brief account of the content of the present paper was published in *Phys. Rev.*, **74** (1948), 936.

Analytical Expression of Self-Consistent Functions and Doublet Intervals for F I, Ne II, Mg IV and Al V.

Watarô WATARI.

Department of Industrial Chemistry, Kyoto University.

(Received Sep. 9, 1948)

§ 1. Analytical Expression of Self-Consistent Functions.

For a quantitative study of atomic properties a good approximation to the atomic eigenfunctions is necessary. Such approximate functions for complex atoms are those of Hartree and Fock. These functions are, however, expressed in the tabular forms, and they are not suitable for the calculation of atomic or molecular properties. A method of obtaining simple analytical expressions for these tabular functions was given by Slater.⁽¹⁾

In this section we apply this method to the eigenfunctions of F I, Ne I, Mg III, and Al IV, for which self-consistent tabular functions have already been obtained.⁽²⁾ The results are shown in Table I.

One-electron functions are written in the form $R_{nl}(r)Y_{lm}(\theta, \varphi)$. In Table I the normalized function $rR_{nl}(r)=P_N(nl)$ are given, where R_{nl} is the radial function and r is measured in atomic units. Among these radial functions the $2s$ -functions are slightly modified so that they are orthogonal to the $1s$ -functions of the same configuration. The dependence of exponents and "intersections" (the ratio between two coefficients of $P_N(2p)$) on the atomic number is nearly linear for $P_N(1s)$ and $P_N(2s)$, but not for $P_N(2p)$.⁽¹⁾ The F I and Ne I functions belong to the less accurate class B of Hartree's classification.⁽³⁾

Table I.

(1) F I $1s^22s^22p^5$

$$P_N(1s)=53.10r \exp(-8.90r)$$

$$P_N(2s)=10.02r \exp(-7.74r)-12.45r^2 \exp(-2.55r)$$

$$P_N(2p)=r^2\{15.53 \exp(-3.8r)+2.043 \exp(-1.50r)\}$$

(2) Ne I $1s^22s^22p^5$

$$P_N(1s)=59.95r \exp(-9.65r)$$

$$P_N(2s) = 14.19r \exp(-8.26r) - 18.64r^2 \exp(-2.98r)$$

$$P_N(2s) = r^2 \{21.32 \exp(-4.1r) + 2.203 \exp(-1.61r)\}$$

$$(3) \text{ Mg III } 1s^2 2s^2 2p^1$$

$$P_N(1s) = 81.58r \exp(-11.8r)$$

$$P_N(2s) = 18.71r \exp(-10.2r) - 35.58r^2 \exp(-3.85r)$$

$$P_N(2p) = r^2 \{47.58 \exp(-8.5r) + 14.67 \exp(-2.90r)\}$$

$$(4) \text{ Al IV } 1s^2 2s^2 2p^1$$

$$P_N(1s) = 91.59r \exp(-12.8r)$$

$$P_N(2s) = 17.51r \exp(-11.0r) - 35.98r^2 \exp(-4.14r)$$

$$P_N(2p) = r^2 \{56.90 \exp(-11.5r) + 26.82 \exp(-3.60r)\}$$

§ 2. Doublet Intervals of Energy Levels.

The atoms F I, Ne II, Mg IV and Al V have one electron-hole in their atomic cores in their ground states. Consequently they have energy spectra of inverted doublets.⁽⁴⁾ According to Araki, the intervals of such inverted doublets are given by:

$${}^2L_{l-1/2} - {}^2L_{l+1/2} = -(2l+1)(Z-\sigma)\mu^2 \bar{r}^{-3} \quad (1)$$

where Z is the atomic number of the atom, μ is the Bohr magneton, and σ is a screening constant due to the spin-orbit interactions between the electron-hole and the core electrons. The parameters σ and \bar{r}^{-3} are given as follows:

$$\sigma = \sum_{n'l'} 2(2l'+1)(\zeta_{n'l'n'l} + 3\tilde{\zeta}_{n'l'n'l}) \quad (2)$$

$$\bar{r}^{-3} = \int_0^\infty \frac{1}{r^3} [R_{nl}(r)r]^2 dr \quad (3)$$

where the summation is to be taken over all orbitals in the atomic core, $n'l$ are the quantum numbers of the electron-hole, and

$$\zeta_{n'l',nl} = -\frac{1}{r^{-3}} \int_0^\infty \int_0^\infty \frac{1}{r_2} \frac{\partial a_0}{\partial r_2} [R_{n'l'}(1) R_{nl}(2) r_1 r_2]^2 dr_1 dr_2 \quad (4)$$

$$\tilde{\zeta}_{n'l',nl} = -\frac{1}{(2l+1)r^{-3}} \int_0^\infty \int_0^\infty \frac{a_l}{r_2} R_{n'l'}(1) \frac{dR_{n'l'}(2)}{dr_2} R_{nl}(1) R_{nl}(2) r_1^2 r_2^2 dr_1 dr_2 \quad (5)$$

$$\begin{aligned} \tilde{\zeta}_{n'l, n'l} = & -\frac{1}{(2l+1)r^{-3}} \int_0^\infty \int_0^\infty \left[\left(-\frac{1}{2l-1} \frac{\partial a_{l-1}}{\partial r_2} + \frac{1}{2l+3} \frac{\partial a_{l+1}}{\partial r_2} \right) \frac{R_{n'l}(1)R_{n'l}(2)}{r_2} \right. \\ & \left. + \left(\frac{l-1}{2l-1} a_{l-1} + \frac{l+2}{2l+3} a_{l+1} \right) \frac{R_{n'l}(1)}{r_2} \frac{dR_{n'l}(2)}{dr_2} \right] R_{n'l}(1)R_{n'l}(2)r_1^2 r_2^2 dr_1 dr_2 \quad (6) \end{aligned}$$

$$a_r = \begin{cases} r_1^l / r_2^{l+1} & \text{for } r_1 \leq r_2 \\ r_2^l / r_1^{l+1} & \text{for } r_2 < r_1 \end{cases} \quad (7)$$

The ground states 2P of F I, Ne II, Mg IV and Al V have a $2p$ -electron-hole in their atomic core $1s^2 2s^2 2p^6$. Using the analytical expression of the radial functions given in §1, the intervals of these 2P and the above mentioned parameters are calculated according to the general formula given by (1)-(7). The results are shown in Table II. The following value of the physical constant is adopted in the present calculation

$$a^2 R \text{ cm}^{-1} = 5,822 \text{ cm}^{-1}. \quad (8)$$

In case of hydrogen-like atoms r^{-3} is given by $Z^3/n^3(l+1)(l+1/2)l$ atomic units, so that a screening constant s of r^{-3} can be defined as follows:

$$\bar{r}^{-3} = \frac{(Z-s)^3}{n^3(l+1)(l+1/2)l} \quad (9)$$

The atomic one-electron functions which are used in the present calculations are not those of Ne II, Mg IV and Al V, but those of the closed shells of Ne I, Mg III and Al IV. Accordingly the screening constants should be corrected taking into account this difference. Such a correction is carried out according to one of the following two procedures.

The screening constant s is due to two $1s$, two $2s$ and five $2p$ -electrons. Therefore mean screening per an electron is $(1/9)s$ when the average are taken assuming that each electron equally contributes to s . In this case the correct value of \bar{r}^{-3} is given by (9) in which s is replaced by $s' = 8s/9$.

The correction may be carried out on another assumption. The screening of $1s$ electron is considered as larger than those of $2s$ and $2p$ electrons. Accordingly we can assume that the contribution from $1s$ electrons to s is 1 per an electron and that from $2s$ and $2p$ electrons is $(s-2)/7$ per an electron. In this case the correct value of \bar{r}^{-3} is given by (9) in which s is replaced by $s'' = s - (s-2)/7 = (6s+2)/7$. It may be expected

that either of these corrections gives the better results. For the parameters such as $\zeta_{n'l',nl}$ and $\tilde{\zeta}_{n'l',nl}$ the errors of this sort may be contained by a nearly equal percentage both in their numerator and denominator, and the final correction may be smaller. Thus we neglect such corrections altogether. The corrected results are also tabulated in Table II as well as uncorrected results.

We see that the calculated values of the intervals are generally smaller than the observed ones. The reason for these fact may be as follows: The self-consistent field method is a method of an approximation to the energy eigenvalue. The energy eigenvalue is approximately proportional to \bar{r}^{-1} . Therefore the approximation of the self-consistent function is the best for the range in which $r^{-1}|\phi|^2$ is large. In this range $r^{-3}|\phi|^2$ is small and the contribution to \bar{r}^{-3} mainly comes from the range in which the approximation of $r^{-3}|\phi|^2$ is not fairly good. The above obtained results may suggest that $|\phi|^2$ is too diffused.

Table II.

	F I	Ne II	Mg IV	Al V
ζ_{1021}	0.70863	0.70055	0.61844	0.61346
$\tilde{\zeta}_{1021}$	0.19983	0.20287	0.21064	0.19712
ζ_{2021}	0.069682	0.077966	0.065635	0.053829
$\tilde{\zeta}_{2021}$	-0.00083903	-0.0090080	0.012049	0.012128
ζ_{2121}	0.077465	0.080289	0.065681	0.070095
$\tilde{\zeta}_{2121}$	0.14724	0.11182	0.021155	0.010585
σ	5.8658	5.2148	3.4612	3.1812
s		3.7630	3.5687	3.7122
\bar{r}^{-3}	6.8983	10.1091	24.9725	33.3817
${}^2P_{1/2}-{}^2P_{3/2}$ (cm ⁻¹)	-188.8	-422.4	-1862	-2862
s'		3.3449	3.1722	3.2997
$(\bar{r}^{-3})'$		12.2816	28.6644	38.0308
$({}^2P_{1/2}-{}^2P_{3/2})'$ (cm ⁻¹)		-512.7	-2137	-3260
s''		3.5111	3.3446	3.4676
$(\bar{r}^{-3})''$		11.3839	27.0174	36.0435
$({}^2P_{1/2}-{}^2P_{3/2})''$ (cm ⁻¹)		-475.2	-2014	-309.0

observed ${}^2P_{1/2} - {}^2P_{3/2}$ (cm ⁻¹)	-467.0	-782.0	-2219	-3420
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In conclusion the author wishes to express his hearty thanks to Professor G. Araki for many valuable advices he has given in the course of the present calculation.

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Progress of Theoretical Physics Vol. IV, No. 1, Jan.~Mar., 1949.

A Self-Consistent Subtraction Method in the Quantum Field Theory. II.

Hiroshi FUKUDA, Yonezi MIYAMOTO and Sin-itiirō TOMONAGA.

*Physics Institutes, Tokyo University and
Tokyo Bunrika Daigaku.*

(Received Sep. 23, 1948)

§ 1. Introduction.

In the first paper of the same title^{*(1)} Tati and one of the present authors have proposed a subtraction procedure to be used in the treatment of quantum-theoretical problems involving infinite field reactions. This method consists in generalizing the method of canonical transformation used first by Bloch and Nordsieck⁽²⁾ and then by Pauli and Pierz⁽³⁾ in the treatment of similar problems in a non-relativistic approximation, and by means of this transformation the electron and the radiation fields are separated into two parts, one bound with each other and the other unbound and propagating freely in the absence of an external field.^{** (4)} Then infinite energies, one related to the infinite self-energy of the free electron and the other to that giving rise to the vacuum polarization of the radiation field, are separated as the interaction energies between bound parts of the fields. Then these infinite terms in the energy are dropped off considering that only the remaining finite terms have physical meaning.

Now when an external field is present, the unbound fields do no longer propagate freely, and, by virtue of the external field, transitions take place. The electron wave is able to change its state of propagation not only elastically affected by the external field but also emitting or absorbing unbound photons; thus an interaction appears between electron and radiation which were free from interaction with each other in the absence of the external field. This interaction causes now a radiation reaction upon the electron so that the motion of the electron will be modified. We shall

* This first paper will be cited as I.

** The usefulness of the canonical transformation in the field reaction problems was pointed out also by Schwinger, reference (4).

give in this paper an example how one can calculate this reaction and the result obtained is really finite as the consequence of our subtraction procedure introduced in I. Thus, for example, we are able to obtain a finite radiative level-shift of a bound electron in the external field and a finite e^2 -correction to the scattering cross section, the latter problem having been discussed recently by Koba and one of us.⁽⁶⁾

As will be shown below we can find finite effective interaction energies which describe the interaction between electron and radiation as mentioned above. One of these interaction energies is first order in e and is essentially of the form $e\psi^\dagger O A \mathcal{U}^0 \psi$, \mathcal{U}^0 being the potential (multiplied by e) of the external field and O some operator containing Dirac matrices and operators upon ψ^\dagger , A , \mathcal{U}^0 and ψ . This energy is responsible for the Bremsstrahlung of an electron in the field \mathcal{U}^0 or allied phenomena. Another interaction energy is of the second order in e and has the form $e^2\psi^\dagger O A \mathcal{U}^0 \psi$. This energy as well as the trivial zero order term $\psi^\dagger \mathcal{U}^0 \psi$ give rise to elastic scattering of the electron in the field \mathcal{U}^0 , the second order term giving rise to the e^2 -correction to the ordinary elastic scattering caused by the zero order term. The famous energy-level shift of a bound electron observed by Lamb and Retherford⁽⁴⁾ can be calculated as a combined effect of these two terms, namely as the first order modification of the energy-level due to the second order term combined with the second order effect due to the first order term.

Although we have thus successfully obtained finite answers for these field reaction problems and they are of the magnitude agreeing with experimental results, we must nevertheless confess that the calculation carried out in this paper is still unsatisfactory because we had to make a non-relativistic treatment in the evaluation of the effective energies, which have the form $\infty - \infty$. In the calculation with such improper expressions, from which we wish to draw a finite conclusion, the results are often affected by the way of calculation so that it will possibly occur that the approximate treatment would miss the essential point. But we think we have been able to confirm, at least, that the result converges by virtue of our subtraction procedure. A method more satisfactory from the relativistic point of view is now being investigated.

§ 2. General Formulation.

We use the natural unit system in which $\hbar=c=1$. The Greek suffices

λ, μ, ν, \dots which run from 1 to 4 are employed to specify the component of world vectors whose fourth components are measured in the imaginary unit. The space part of a world vector is denoted by a thick letter. The propagation vector of an electromagnetic wave we represent by the letter K and the corresponding vector for an electron wave by the letter P :

$$\left. \begin{aligned} K_\mu &= (\mathbf{k}, i|\mathbf{k}|) \\ P_\mu &= (\mathbf{p}, i\sqrt{\mathbf{p}^2 + \kappa^2}) \end{aligned} \right\} \quad (1)$$

κ being the mass of the electron. The fourth components of the propagation vectors are, when not noticed explicitly, assumed to be i times the positive value. To represent the coordinate of a world point we use the letter X :

$$X_\mu = (\mathbf{x}, ict). \quad (1')$$

We often write simply $A \cdot B$ for the scalar product $A_\lambda B_\lambda$.

Now the generalized Schrödinger equation in the presence of an external field is

$$\left\{ H(X) + H^0(X) + \frac{1}{i} \frac{\delta}{\delta C_X} \right\} \Psi[C] = 0 \quad (I)$$

where $H(X)$ is the density of the interaction energy between electron and radiation at the world point X . It is expressed as

$$H(X) = e\psi^\dagger(X)\gamma \cdot A(X)\psi(X). \quad (2)$$

$H^0(X)$, on the other hand, is the interaction energy density between the electron and the external field whose potential (multiplied by e) is denoted by \mathfrak{A}^0 . This energy is given by

$$H^0(X) = e\psi^\dagger(X)\gamma \cdot \mathfrak{A}^0(X)\psi(X). \quad (2')$$

In the following we often write, for simplicity, H, H', H'' etc. in place of $H(X), H(X'), H(X'')$ etc.

As mentioned in I we must separate each field into two parts, one oscillating with positive and the other with negative frequencies:

$$\psi = \psi^+ + \psi^{-\dagger}, \quad \psi^\dagger = \psi^{+\dagger} + \psi^-, \quad A = a^\dagger + a. \quad (3)$$

Then the commutation relations between fields are, for the electron,

$$\left. \begin{aligned} [\phi_r, \phi_s^\dagger]_+ &= \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right)_{rs} D_1(X-X') \\ [\phi_r^\dagger, \phi_s^{\dagger\dagger}]_+ &= - \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right)_{rs} D_1^\dagger(X-X') \\ [\phi_r^{\dagger\dagger}, \phi_s^{\dagger\dagger}]_+ &= \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right)_{rs} D_1^-(X-X') \end{aligned} \right\} \quad (4)$$

with

$$\left. \begin{aligned} D_1(X) &= \frac{1}{(2\pi)^3} \int \{ \exp(-iP \cdot X) - \exp(iP \cdot X) \} \frac{d\mathbf{p}}{2P_4} \\ D_1^\pm(X) &= \frac{1}{(2\pi)^3} \int \exp(\pm iP \cdot X) \frac{d\mathbf{p}}{2P_4}, \end{aligned} \right\} \quad (4')$$

and for the radiation they are

$$[A_\mu, A_\nu'] = -i\delta_{\mu\nu} D_0(X-X'), \quad [a_\mu, a_\nu^{\dagger'}] = i\delta_{\mu\nu} D_0^+(X-X') \quad (5)$$

with

$$\left. \begin{aligned} D_0(X) &= \frac{1}{(2\pi)^3} \int \{ \exp(-iK \cdot X) - \exp(iK \cdot X) \} \frac{d\mathbf{k}}{2K_4} \\ D_0^\pm(X) &= \frac{1}{(2\pi)^3} \int \exp(\pm iK \cdot X) \frac{d\mathbf{k}}{2K_4}. \end{aligned} \right\} \quad (5')$$

We now carry out the canonical transformation introduced in I:

$$\Psi[C] = \exp \left\{ - \int^c H(X') dX' \right\} \Psi_1[C]. \quad (6)$$

(dX' denote the imaginary volume element in the world, i. e. i times the volume element dV used in I). Then the Schrödinger equation for the transformed functional is, in the ϵ^2 approximation

$$\begin{aligned} & \left\{ H^0 + \left[\int^c H' dX', H^0 \right] + (1/2) \left[\int^c H'' dX'', \left[\int^c H' dX', H^0 \right] \right] \right. \\ & \quad \left. + (1/2) \left[\int^c H' dX', H \right] + \frac{1}{i} \frac{\partial}{\partial C_X} \right\} \Psi_1[C] = 0. \quad (II) \end{aligned}$$

The term H^0 in (II) is zero order in ϵ and describes the behavior of the electron under the action of the external field in the zeroth approximation. It describes e.g. the zero-order elastic scattering of the electron in the field of \mathfrak{A}^0 . The second term $\left[\int^c H' dX', H^0 \right]$, which is first order in ϵ , contains

besides \mathfrak{A}^0 the potential A describing the radiation field linearly so that this term is responsible for one-quantum phenomena which occur in the presence of the external field. If one rearranges factors in each term of $[\int^c H' dX', H^0]$ in the correct order as mentioned in I, one obtains a term of the form $e\psi^\dagger O\mathfrak{A}^0 A\psi$ which, as mentioned in §1, is responsible for the emission and absorption of Bremsstrahlung and allied phenomena. Explicitly written down this term is

$$\begin{aligned} [\int^c H' dX', H^0]_{\text{Brems}} = & - \int^c dX' e A_\mu' \mathfrak{A}_\nu^0 \\ & \cdot \left\{ \psi^\dagger \gamma_\mu \left(\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1(X-X') \gamma_\nu \psi + \psi^\dagger \gamma_\nu \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1(X-X') \gamma_\mu \psi' \right\}. \end{aligned} \quad (\text{III})$$

The third term in (II) contains various terms each consisting of one \mathfrak{A}^0 and an even number of ψ 's and A 's as factors. Explicitly they are given by

$$\begin{aligned} \frac{1}{2} \int^c dX' \int^c dX'' [H'', [H', H^0]] = & \frac{e^2}{2} \mathfrak{A}_\nu^0 \int^c dX' \int^c dX'' \\ & \cdot [\{a_\mu \psi^\dagger (a_\lambda \psi'' + a_\lambda'') + (a_\lambda \psi'' + a_\lambda'') a_\mu'\} [\psi^\dagger \gamma_\lambda \psi'', [\psi^\dagger \gamma_\mu \psi', \psi^\dagger \gamma_\nu \psi]] \\ & - i \{D_0(X'' - X') \psi^\dagger \gamma_\mu \psi'' [\psi^\dagger \gamma_\mu \psi', \psi^\dagger \gamma_\nu \psi] \\ & - D_0^-(X'' - X') [\psi^\dagger \gamma_\mu \psi'', [\psi^\dagger \gamma_\mu \psi', \psi^\dagger \gamma_\nu \psi]]\}]. \end{aligned} \quad (7)$$

If one rearranges the factors in each term of (7) in the correct order and collects the terms depending linearly on \mathfrak{A}^0 but containing no A and being bilinear in ψ , one obtains

$$\begin{aligned} \frac{e^2}{2i} \mathfrak{A}_\nu^0 \int^c dX' \int^c dX'' [\sum_\pm \pm D_0^\pm(X' - X'')] \\ \cdot \left\{ \psi^\dagger \gamma_\mu \left(\gamma \cdot \frac{\partial}{\partial X'} + x \right) D_1^\pm(X' - X'') \left(\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1(X - X') \gamma_\nu \psi \right. \\ + \psi^\dagger \gamma_\nu \left(\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1(X - X') \gamma_\mu \left(-\gamma \cdot \frac{\partial}{\partial X'} + x \right) D_1^\pm(X - X'') \gamma_\mu \psi'' \\ + \psi^\dagger \gamma_\mu \left(\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^\pm(X - X'') \gamma_\nu \left(-\gamma \cdot \frac{\partial}{\partial X'} + x \right) D_1(X - X') \gamma_\mu \psi' \\ \left. + \psi^\dagger \gamma_\nu \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1(X - X') \gamma_\mu \left(-\gamma \cdot \frac{\partial}{\partial X'} + x \right) D_1^\pm(X' - X'') \gamma_\mu \psi'' \right\} \end{aligned}$$

$$+ D_0(X'' - X') \phi^{\dagger''} \gamma_{\mu} \phi'' \sum_{\pm} \pm \text{Sp} \gamma_{\mu} \left(\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^{\pm}(X - X') \gamma_{\nu} \\ \cdot \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^{\pm}(X - X'), \quad (\text{IV}_1)$$

which together with the term of the same structure appearing in the later expression (IV₂), describes the e^2 -correction to the behavior of the electron in the field of \mathfrak{U}^0 .

The fourth term $(1/2) \left[\int^c H' dX', H \right]$ in (II) is the term discussed precisely in I. This is independent of the external field \mathfrak{U}^0 and contains terms each representing various e^2 -phenomena taking place independently of the external field such as scattering of a photon by a free electron or retarded interaction between two electrons. What is to be considered here is the term representing the self-energy of an electron. As mentioned in I this term is obtained by rearranging the factors in each term in $(1/2) \left[\int^c H' dX', H \right]$ and collecting together terms containing ψ 's bilinearly but no \mathcal{A} . We have seen in I that this term has the same form as the mass term in Dirac equation so that it represents the e^2 -correction to the electron mass. This term may be dropped off if one considers that this correction is already included in the electron mass and one uses the corrected mass everywhere in the calculations. According to I this e^2 -correction to the electron mass has the form

$$\delta x \phi^{\dagger}(X) \phi(X) \equiv \int^c H(X, X')_{\text{mass}} dX' \\ = \frac{e^2}{2i} \int^c dX' [D_0(X - X') \{ \phi^{\dagger'} \gamma_{\mu} \left(\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^{-}(X - X') \gamma_{\mu} \psi \\ + \phi^{\dagger'} \gamma_{\mu} \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^{-}(X - X') \gamma_{\mu} \psi' \} \\ + D_0^{+}(X - X') \{ \phi^{\dagger'} \gamma_{\mu} \left(\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^{+}(X - X') \gamma_{\mu} \psi \\ + \phi^{\dagger'} \gamma_{\mu} \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^{+}(X - X') \gamma_{\mu} \psi' \}]. \quad (8)$$

After the mass correction term has been dropped we must now carry out the second canonical transformation by means of

$$\Psi_1[C] = \exp \left[- \int^c dX' \int^c dX'' \{ (1/2) [H'', H'] - H(X', X'')_{\text{mass}} \} \right] \Psi_2[C] \quad (9)$$

The necessity of the second transformation corresponds to the similar situation mentioned in the last section of I. The Schrödinger equation for the functional $\Psi_2[C]$ is then found to be

$$\begin{aligned} & \left\{ H^0 + \int^c [H', H^0] dX' + (1/2) \int^c dX' \int^c dX'' [H'', [H', H^0]] \right. \\ & \left. + \int^c dX' \int^c dX'' [(1/2)[H'', H'] - H(X', X'')_{\text{mass}}, H^0] + \frac{1}{i} \frac{\partial}{\partial C_\lambda} \right\} \Psi_2[C] = 0. \end{aligned} \quad (\text{II}_2)$$

The fourth term appearing in (II₂) consists of various terms each containing one \mathfrak{U}^0 and an even number of ψ 's and A 's. Rearranging factors in each term in the correct order and collecting such terms which contain ψ 's bilinearly but no A , we obtain

$$\begin{aligned} & \frac{e^2}{2i} \mathfrak{U}^0 \int^c dX' \int^c dX'' D_0(X'' - X') \\ & \cdot \left\{ \sum_{\pm} \pm \psi^\dagger \gamma_\mu \left(\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^\mp(X - X') \gamma_\nu \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^\mp(X - X'') \gamma_\mu \psi'' \right. \\ & + \sum_{\pm} \pm \psi^\dagger \gamma_\mu \left(\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^\mp(X - X'') \gamma_\nu \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^\mp(X - X') \gamma_\mu \psi' \\ & - \psi^\dagger \gamma_\mu \psi'' \text{Sp} \sum_{\pm} \pm \gamma_\mu \left(\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^\mp(X - X') \gamma_\nu \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^\mp(X - X') \\ & \left. - \psi^\dagger \gamma_\mu \psi' \text{Sp} \sum_{\pm} \pm \gamma_\mu \left(\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^\mp(X - X'') \gamma_\nu \left(-\gamma \cdot \frac{\partial}{\partial X} + x \right) D_1^\mp(X - X') \right\} \end{aligned} \quad (\text{IV}_2)$$

which has the form $e^2 \psi^\dagger \mathcal{O} \mathfrak{U}^0 \psi$, and together with (IV₁), describes the e^2 -correction for the behavior of the electron in the field \mathfrak{U}^0 . Thus the e^2 -correction due to the radiation reaction for the elastic scattering of an electron is calculated directly from this term so far as the scattering first order in \mathfrak{U}^0 is concerned. If it is required to calculate the level shift of the electron in the field \mathfrak{U}^0 , one has to add to the first order shift due to this term the second order effect due to the term (III), which can contribute to the level shift by emission and reabsorption of a virtual photon. We have thus obtained the expressions which are to be regarded as the effective energy densities describing the interactions of the unbound parts of the electron and the radiation fields in the presence of the external field \mathfrak{U}^0 . The total effective energy is given by the sum of the expressions

(III), (IV₁) and (IV₂), and we shall use this energy to determine the e^2 -correction for the scattering cross section and the energy-level shift due to the reaction of the radiation field.

§ 3. Evaluation of the Effective Energies.

In evaluating the effective energies found in the preceding section it is desirable to carry out the calculation relativistically as far as possible. In applying our method to the calculation of the level shift or of the e^2 -correction for the scattering of an electron with a non-relativistic velocity, it is allowed to make a non-relativistic approximation in some sense. But it is important to introduce this approximation first in the end stage of the calculation after we have obtained a converging expression. This is because we are dealing with expressions of the form $\infty - \infty$, and the values obtained from such improper expressions depend so much on the way of evaluation that it is necessary to carry through the calculation on some definite prescription, which should be, of course, relativistically invariant. In such a situation there is much fear of that the calculation does not correspond to the true prescription if one introduces the non-relativistic approximation in a too early stage. We must confess, however, that we are yet unable to carry out the calculation relativistically throughout; we had to introduce a non-relativistic procedure from the beginning. Our work is, therefore, only of a provisional character. We believe, however, that we have been able to prove, at least, that our subtraction procedure is really effective in removing the divergence of the theory.

The first non-relativistic procedure we had to introduce is to consider that the variable surface C is a plane parallel to the xyz -plane. By this process our calculation becomes much parallel with the ordinary perturbational calculation.

Several calculations by the help of the relation

$$\int^C dX' \int^C dX'' F(X', X'') = \int^C dX' \int^{C'} dX'' \{F(X', X'') + F(X'', X')\} \quad (10)$$

shows that the expression (IV₁)+(IV₂) can be transformed into

$$\begin{aligned} (IV_1) + (IV_2) &= (a) + (b) + (c) + (d) + (e) + (f), \\ (a) &= \frac{e^2}{2i} \mathfrak{U}_v \circ \int^C dX' \int^C dX'' \sum_{\pm} 2D_{\pm}^{\pm}(X' - X'') \psi^\dagger \gamma_v \left(r \cdot \frac{\partial}{\partial X} + x \right) \end{aligned} \quad \Bigg|$$

$$\begin{aligned}
 & \cdot D_1^\mp(X-X')r_\nu\left(-r\cdot\frac{\partial}{\partial X}+z\right)D_1^\pm(X-X'')r_\mu\psi'' \\
 (b) &= \frac{e^2}{2i} \mathfrak{A}_\nu \circ \int dX' \int dX'' \sum_{\pm} -2D_0^\pm(X'-X'')\psi^\dagger r_\mu\left(r\cdot\frac{\partial}{\partial X}+z\right) \\
 & \quad \cdot D_1^\pm(X-X')r_\nu\left(-r\cdot\frac{\partial}{\partial X}+z\right)D_1^\pm(X-X'')r_\mu\psi'' \\
 (c) &= \frac{e^2}{2i} \mathfrak{A}_\nu \circ \int dX' \int dX'' \sum_{\pm} -2D_1^\pm(X'-X'')\psi^\dagger r_\mu\left(r\cdot\frac{\partial}{\partial X}+z\right) \\
 & \quad \cdot D_1^\pm(X-X'')r_\nu\left(-r\cdot\frac{\partial}{\partial X}+z\right)D_1^\pm(X-X')r_\mu\psi'' \\
 (d) &= \frac{e^2}{2i} \mathfrak{A}_\nu \circ \int dX' \int dX'' \sum_{\pm} \pm D_0^\pm(X'-X'')\psi^\dagger r_\mu\left(+r\cdot\frac{\partial}{\partial X}+z\right) \\
 & \quad \cdot D_1^\pm(X'-X'')r_\nu\left(r\cdot\frac{\partial}{\partial X}+z\right)D_1(X-X')r_\nu\psi \\
 (e) &= \frac{e^2}{2i} \mathfrak{A}_\nu \circ \int dX' \int dX'' \sum_{\pm} \pm D_0^\pm(X'-X'')\psi^\dagger r_\nu\left(-r\cdot\frac{\partial}{\partial X}+z\right) \\
 & \quad \cdot D_1(X-X')r_\mu\left(-r\cdot\frac{\partial}{\partial X}+z\right)D_1^\pm(X'-X'')r_\mu\psi'' \\
 (f) &= \frac{e^2}{2i} \mathfrak{A}_\nu \circ \int dX' \int dX'' \sum_{\pm} \pm 2D_0(X''-X')\psi^\dagger r_\nu\psi'' \\
 & \quad \cdot \text{Spr}_\nu\left(-r\cdot\frac{\partial}{\partial X}+z\right)D_1^\pm(X-X')r_\mu\left(r\cdot\frac{\partial}{\partial X}+z\right)D_1^\pm(X-X').
 \end{aligned} \tag{11}$$

The use of this form is more convenient than the use of (IV₁)+(IV₂) themselves because, if one will calculate on (IV₁)+(IV₂) directly, thereby the integration over dX' and dX'' being carried out according to (7) of I, one meet with vanishing denominators in the integrand of the integrals over the momentum space, which makes the calculation very complicated. We can avoid this difficulty by the use of (a)+(b)+(c)+...+(f) instead of (IV₁)+(IV₂) directly (but with a sacrifice of the relativistic invariance).

In order to carry through the integration over dX' and dX'' according to (7) of I we must now introduce the Fourier expansion of the D -functions and of ψ 's. The Fourier expansion of the D -functions are given in (4') and (5'). Let the Fourier expansion of ψ 's be

$$\psi(X) = \int \psi_{P_1} e^{iP_1 X} dP_1, \quad \psi^\dagger(X) = \int \psi_{P_2}^\dagger e^{-iP_2 X} dP_2 \tag{12}$$

where the fourth components of P_1 and P_2 may take both positive and ne-

gative imaginary value and, for simplicity, the symbol $\int d\mathbf{P}$ is used to include also the summation of terms oscillating with positive frequencies and those with negative frequencies.

The straightforward calculation of each term (a), (b), ... (e) according to (7) of I gives

$$\begin{aligned}
 (a) &= \frac{ie^2}{2} \mathfrak{A}_v \circ \int \frac{d\mathbf{K} d\mathbf{P}' d\mathbf{P}''}{(2\pi)^9} d\mathbf{P}_1 d\mathbf{P}_2 \\
 &\cdot \sum_{\pm} \phi_{P_2} \left\{ \frac{\gamma_{\mu} (\mp i\gamma \cdot \mathbf{P}' + x) \gamma_{\nu} (\mp i\gamma \cdot \mathbf{P}'' + x) \gamma_{\mu} \delta(\mathbf{K} + \mathbf{P}' \mp \mathbf{P}_2) \delta(\mathbf{K} + \mathbf{P}'' \mp \mathbf{P}_1)}{-4K_4 P_4' P_4'' (K_4 + P_4' \mp P_{24}) (K_4 + P_4' \mp P_{14})} \right\} \phi_{P_1} \\
 (b) &= \frac{ie^2}{2} \mathfrak{A}_v \circ \int \frac{d\mathbf{K} d\mathbf{P}' d\mathbf{P}''}{(2\pi)^9} d\mathbf{P}_1 d\mathbf{P}_2 \\
 &\cdot \sum_{\pm} \phi_{P_2} \left\{ \frac{\gamma_{\mu} (\pm i\gamma \cdot \mathbf{P}' + x) \gamma_{\nu} (\mp i\gamma \cdot \mathbf{P}'' + x) \gamma_{\mu} \delta(\mathbf{K} - \mathbf{P}' \pm \mathbf{P}_2) \delta(\mathbf{K} + \mathbf{P}'' \mp \mathbf{P}_1)}{-4K_4 P_4' P_4'' \{P_4' + P_4'' \pm (P_{24} - P_{14})\} (P_4' + K_4 \mp P_{14})} \right\} \phi_{P_1} \\
 (c) &= \frac{ie^2}{2} \mathfrak{A}_v \circ \int \frac{d\mathbf{K} d\mathbf{P}' d\mathbf{P}''}{(2\pi)^9} d\mathbf{P}_1 d\mathbf{P}_2 \\
 &\cdot \sum_{\pm} \phi_{P_2} \left\{ \frac{\gamma_{\mu} (\mp i\gamma \cdot \mathbf{P}' + x) \gamma_{\nu} (\pm i\gamma \cdot \mathbf{P}'' + x) \gamma_{\mu} \delta(\mathbf{K} + \mathbf{P}' \mp \mathbf{P}_2) \delta(\mathbf{K} - \mathbf{P}'' \pm \mathbf{P}_1)}{-4K_4 P_4' P_4'' (P_4' + K_4 \mp P_{24}) \{P_4' + P_4'' \mp (P_{24} - P_{14})\}} \right\} \phi_{P_1} \\
 (d) &= \frac{ie^2}{2} \mathfrak{A}_v \circ \int \frac{d\mathbf{K} d\mathbf{P}' d\mathbf{P}''}{(2\pi)^9} d\mathbf{P}_1 d\mathbf{P}_2 \\
 &\cdot \sum_{\pm} \phi_{P_2} \left\{ \frac{\gamma_{\mu} (\pm i\gamma \cdot \mathbf{P}' + x) \gamma_{\nu} \{ \pm i\gamma \cdot (\mathbf{K} + \mathbf{P}') + x \} \gamma_{\nu} \delta(\mathbf{K} + \mathbf{P}' \mp \mathbf{P}_2)}{8K_4 P_4' (K_4 + P_4' \mp P_{24}) (K \cdot \mathbf{P}')} \right\} \phi_{P_1} \\
 (e) &= \frac{ie^2}{2} \int \frac{d\mathbf{K} d\mathbf{P}' d\mathbf{P}''}{(2\pi)^9} d\mathbf{P}_1 d\mathbf{P}_2 \\
 &\cdot \sum_{\pm} \phi_{P_2} \left\{ \frac{\gamma_{\nu} \{ \mp i\gamma \cdot (\mathbf{K} + \mathbf{P}'') + x \} \gamma_{\mu} (\mp i\gamma \cdot \mathbf{P}'' + x) \gamma_{\mu} \delta(\mathbf{K} + \mathbf{P}'' \mp \mathbf{P}_1)}{8K_4 P_4'' (K_4 + P_4'' \mp P_{14}) (K \cdot \mathbf{P}')} \right\} \phi_{P_1}
 \end{aligned} \tag{13}$$

We omit here the calculation of (f) because this term represents the effect of vacuum polarization caused by the external field which should be dropped off with a similar reasoning as used in dropping off the mass correction term (8). (But see § 6).*

Transforming each numerator by the help of the formulas

$$\left. \begin{aligned}
 \gamma_{\mu} (\gamma \cdot A) \gamma_{\nu} (\gamma \cdot B) \gamma_{\mu} &= -2(\gamma \cdot B) \gamma_{\nu} (\gamma \cdot A) \\
 \gamma_{\mu} (\gamma \cdot A) \gamma_{\nu} \gamma_{\mu} &= 4A_{\nu} \\
 \gamma_{\mu} \gamma_{\nu} (\gamma \cdot B) \gamma_{\mu} &= 4B_{\nu} \\
 \gamma_{\mu} \gamma_{\nu} \gamma_{\mu} &= -2\gamma_{\nu}
 \end{aligned} \right\} \tag{13'}$$

* §§ 5 and 6 will appear in the next issue of this journal.

and performing the integration with respect to \mathbf{P}' and \mathbf{P}'' , the expressions in (13) are represented as

$$\begin{aligned}
 (a) &= \frac{e^2}{2i} \mathfrak{A}_v \circ \int \frac{d\mathbf{K}}{(2\pi)^3} d\mathbf{P}_1 d\mathbf{P}_2 \\
 &\quad \cdot \sum_{\pm} \psi_{P_2} \dagger \left[\frac{-i(\gamma \cdot P'') \gamma_v i(\gamma \cdot P') \mp 2ix(P' + P'')_v - x^2 \gamma_v}{2K_4 P_4' P_4'' (K_4 + P_4' \mp P_{24}) (K_4 + P_4'' \mp P_{14})} \right] \psi_{P_1} \\
 &\quad \text{with } \mathbf{P}' = -\mathbf{K} \pm \mathbf{P}_2, \quad \mathbf{P}'' = -\mathbf{K} \pm \mathbf{P}_1 \\
 (b) &= \frac{e^2}{2i} \mathfrak{A}_v \circ \int \frac{d\mathbf{K}}{(2\pi)^3} d\mathbf{P}_1 d\mathbf{P}_2 \\
 &\quad \cdot \sum_{\pm} \psi_{P_2} \dagger \left[\frac{i(\gamma \cdot P'') \gamma_v (i\gamma \cdot P') \mp 2ix(-P' + P'')_v - x^2 \gamma_v}{2K_4 P_4' P_4'' (K_4 + P_4'' \mp P_{24}) \{P_4' + P_4'' \mp (P_{24} - P_{14})\}} \right] \psi_{P_1} \\
 &\quad \text{with } \mathbf{P}' = \mathbf{K} \mp \mathbf{P}_2, \quad \mathbf{P}'' = -\mathbf{K} \mp \mathbf{P}_1 \\
 (c) &= \frac{e^2}{2i} \mathfrak{A}_v \circ \int \frac{d\mathbf{K}}{(2\pi)^3} d\mathbf{P}_1 d\mathbf{P}_2 \\
 &\quad \cdot \sum_{\pm} \psi_{P_2} \dagger \left[\frac{i(\gamma \cdot P'') \gamma_v (i\gamma \cdot P') \mp 2ix(P' - P'')_v - x^2 \gamma_v}{2K_4 P_4' P_4'' (K_4 + P_4' \mp P_{24}) \{P_4' + P_4'' \mp (P_{24} - P_{14})\}} \right] \psi_{P_1} \\
 &\quad \text{with } \mathbf{P}' = -\mathbf{K} \pm \mathbf{P}_2, \quad \mathbf{P}'' = \mathbf{K} \mp \mathbf{P}_1 \\
 (d) &= -\frac{e^2}{2i} \mathfrak{A}_v \circ \int \frac{d\mathbf{K}}{(2\pi)^3} d\mathbf{P}_1 d\mathbf{P}_2 \\
 &\quad \cdot \sum_{\pm} \psi_{P_2} \dagger \left[\frac{-i(\gamma \cdot P') (i(\gamma \cdot K) \pm x) \mp 2ix(\gamma \cdot K) + x^2}{4K_4 P_4' (K_4 + P_4' \mp P_{24}) (K \cdot P')} \right] \gamma_v \psi_{P_1} \\
 &\quad \text{with } \mathbf{P}' = -\mathbf{K} \pm \mathbf{P}_2 \\
 (e) &= -\frac{e^2}{2i} \mathfrak{A}_v \circ \int \frac{d\mathbf{K}}{(2\pi)^3} d\mathbf{P}_1 d\mathbf{P}_2 \\
 &\quad \cdot \sum_{\pm} \psi_{P_2} \dagger \gamma_v \left[\frac{-(i(\gamma \cdot K) \pm x) i(\gamma \cdot P'') \mp 2ix(\gamma \cdot K) + x^2}{4K_4 P_4'' (K_4 + P_4'' \mp P_{24}) (K \cdot P'')} \right] \psi_{P_1} \\
 &\quad \text{with } \mathbf{P}'' = -\mathbf{K} \pm \mathbf{P}_1
 \end{aligned} \tag{14}$$

where P_4' and P_4'' are given by

$$P_4' = i\sqrt{(\mathbf{K} \mp \mathbf{P}_2)^2 + x^2}, \quad P_4'' = i\sqrt{(\mathbf{K} \mp \mathbf{P}_1)^2 + x^2}. \tag{14'}$$

Then the use of the relations

$$\psi_{P_2} \dagger \{i(\gamma \cdot P_2) + x\} = 0, \quad \{i(\gamma \cdot P_1) + x\} \psi_{P_1} = 0 \tag{15}$$

and

$$(\gamma \cdot K)^2 = 0, \quad (\gamma \cdot P')^2 = (\gamma \cdot P'')^2 = -x^2 \quad (15')$$

gives rise to

$$\left. \begin{aligned} (a) + (b) + (c) &= \frac{ie^2}{2} \mathfrak{A}_v \circ \int \frac{dK}{(2\pi)^3} dP_1 dP_2 \psi_{P_2} \dagger (A + B + C + D) \psi_{P_1} \\ (d) &= \frac{ie^2}{2} \mathfrak{A}_v \circ \int \frac{dK}{(2\pi)^3} dP_1 dP_2 \psi_{P_2} \dagger (E) \psi_{P_1} \\ (e) &= \frac{ie^2}{2} \mathfrak{A}_v \circ \int \frac{dK}{(2\pi)^3} dP_1 dP_2 \psi_{P_2} \dagger (F) \psi_{P_1} \end{aligned} \right\} \quad (V)$$

with

$$\left. \begin{aligned} (A) &= \sum_{\pm} \{ -K_v (\gamma \cdot K) \mp \gamma_v (K \cdot (P_1 + P_2)) \pm (P_1 + P_2)_v (\gamma \cdot K) \\ &\quad + \gamma_v (P_1 \cdot P_2) - ix \delta_{v4} (\pm 2K_4 - P_{14} - P_{24}) \mp ix K_v \} \\ &\quad \cdot \frac{2 \{ (P'_4 + P''_4) (F'_4 + P''_4 + K_4) \mp (P'_4 P_{14} + P''_4 P_{24}) \}}{K_4 P'_4 P''_4 (K_4 + P'_4 \mp P_{14}) (K_4 + P''_4 \mp P_{24}) \{ (P'_4 + P''_4)^2 - (P_{24} - P_{14})^2 \}} \\ (B) &= \sum_{\pm} \{ \delta_{v4} (\gamma \cdot K) - \gamma_v K_4 + \gamma_4 K_v \mp \gamma_4 (P_1 + P_2)_v \pm \gamma_v (P_{14} + P_{24}) \mp ix \delta_{v4} \} \\ &\quad \cdot \frac{2(P'_4 + P''_4)}{K_4 P'_4 P''_4 \{ (P'_4 + P''_4)^2 - (P_{24} - P_{14})^2 \}} \\ (C) &= \sum_{\pm} (\gamma_v - 2\delta_{v4} \gamma_4) \cdot \frac{K_4 (P'_4 + P''_4) \mp (P_{24} P''_4 + P_{14} P'_4)}{K_4 P'_4 P''_4 \{ (P'_4 + P''_4)^2 - (P_{24} - P_{14})^2 \}} \\ (D) &= \sum_{\pm} 2i \delta_{v4} x \\ &\quad \cdot \frac{2P'_4 P''_4 (P'_4 + P''_4) \mp (P_{24} - P_{14}) (P''_4{}^2 - P'_4{}^2) - (P_{24} - P_{14}) (P'_4 P_{24} - P''_4 P_{14})}{K_4 P'_4 P''_4 (P'_4 + K_4 \mp P_{24}) (K_4 + P'_4 \mp P_{14}) \{ (P'_4 + P''_4)^2 - (P_{24} - P_{14})^2 \}} \\ (E) &= \sum_{\pm} \frac{\tilde{\gamma}_4 (\gamma \cdot K) \tilde{\gamma}_v \mp \tilde{\gamma}_4 \tilde{\gamma}_v ix + 2x^2 \tilde{\gamma}_v / (P'_4 + K_4 \mp P_{24})}{4K_4 P'_4 (K \cdot P')} \\ (F) &= \sum_{\pm} \frac{\tilde{\gamma}_v (\gamma \cdot K) \tilde{\gamma}_4 \mp \tilde{\gamma}_v \tilde{\gamma}_4 ix + 2x^2 \tilde{\gamma}_v / (P''_4 + K_4 \mp P_{14})}{4K_4 P''_4 (K \cdot P'')} \end{aligned} \right\} \quad (VI)$$

Thus far our expressions are non-relativistic only in the sense that we have taken the surface C of a special form i.e. a plane parallel to the xyz -plane. We now introduce in the next section the essentially non-relativistic approximation assuming that the velocity of the electron under consideration is far smaller than the light velocity.

(to be Continued.)

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Radiation Reaction in Collision Process. III.*

— First Radiative Correction for an Arbitrary Process including
Electrons, Positrons, and Light Quanta** —

Zirô Koba and Gyô Takeda.

*Institute of Physics, Faculty of Science,
Tokyo University.*

(Received Oct. 4, 1948)

§ 1. Introduction and Summary.

In the parts I and II of this work⁽¹⁾⁽²⁾ we have estimated by the straightforward application of the perturbation method the radiative corrections for the elastic scattering of an electron and for the Compton scattering respectively, and also have investigated from the view-point of the self-consistent subtraction method what modifications one has to introduce into the interaction Hamilton function in order to obtain finite results.

Thus we have arrived at the hypothesis that the Hamiltonian describing the electronic and electromagnetic fields in interaction is to be converted into

$$\begin{aligned}
 H = & \int u^* \{ (\boldsymbol{\alpha} \mathbf{p}) + m\beta \} u dV + (1/8\pi) \int (\mathbf{E}^2 + \mathbf{H}^2) dV + e \int (u^* \boldsymbol{\alpha} u, \mathbf{A}) dV \\
 & + (e^2/2) \iint u^* u u'^* u' | \mathbf{x} - \mathbf{x}' |^{-1} dV dV' - \delta m \int u^* \beta u dV - \delta e \int (u^* \boldsymbol{\alpha} u, \mathbf{A}) dV \\
 & - c \delta e \iint u^* u u'^* u' | \mathbf{x} - \mathbf{x}' |^{-1} dV dV' + (e^2/3\pi) \int r dr \int \mathbf{A}^2 dV \quad (1.1)
 \end{aligned}$$

with

$$\delta m = (3me^2/2\pi) \int^\infty dk/k \quad (1.2)$$

$$\delta e = -(e^2/3\pi) \int^\infty dk/k \quad (1.3)$$

* Parts I and II of this work have appeared in former issues of this journal; see reference 1, 2.

** The main content of this paper was read at the annual meeting of the Physical Society of Japan, May 22, 1948; also a preliminary report has been published in this journal, Vol. 3 (1948), 203.

Here u^* , u are quantized amplitudes of electron fields, A the transverse part of the electromagnetic potential. The natural unit system $\hbar=c=1$ is employed throughout this note. As discussed in detail at the end of II., this prescription is rather of a provisional character; especially as to the last term of (1.1), which, as pointed out by Oppenheimer,⁽³⁾ has a neither gauge- nor Lorentz-covariant form, a more profound consideration is required.

Setting aside this fundamental question, there is another point to be studied; whether our modified Hamiltonian (1.1), derived from the considerations about Compton scattering and mutual scattering of two electrons, is effective in eliminating the diverging radiations for a more general collision process. The present note is concerned with the latter problem.

In order to dispose of all possible radiative corrections in a general case, it is essential to analyze systematically the complicated chains connecting the initial and the final states not only through the least necessary number of intermediate states, but also through certain detours including the emission and reabsorption of a virtual photon or the creation and annihilation of a virtual electron-positron pair, which just account for the radiation reaction. For this purpose we have introduced a "transition diagram method", which turns out an effective tool for the discussion of higher order processes. The first half of this paper is appropriated to the illustration of this method, as our later reasoning is based entirely upon it.

In this way we believe to have clarified that by the use of the Hamilton function (1.1) one can indeed obtain a reasonable result, provided that one restricts oneself to the first radiative correction, i.e. the correction $\delta\sigma \sim e^{2n+2}$ to the zeroth cross-section $\sigma_0 \sim e^{2n}$ (n being the order of perturbation required to get a non-vanishing matrix element for the process considered). The detailed proof is given in the last half of this note.

§ 2. Transition Diagram.

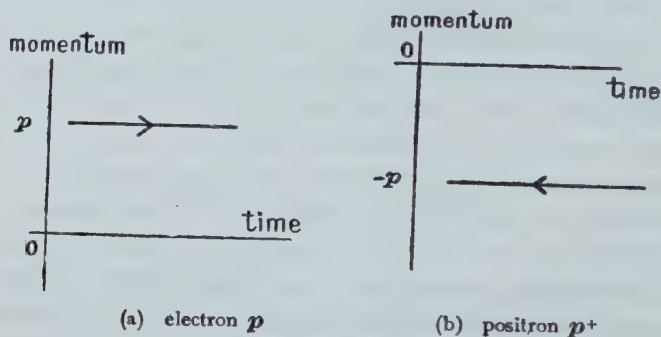
In order to command a view of a whole connection between the initial and the final states which appears in the perturbation calculus of a certain complicated process, we propose here an improved form upon its diagram expression heretofore in use.⁽⁴⁾⁽⁵⁾

i) Notations and rules.

An electron with momentum p is represented by a horizontal line-segment at the height p with an arrow-head that points to the right (Fig.

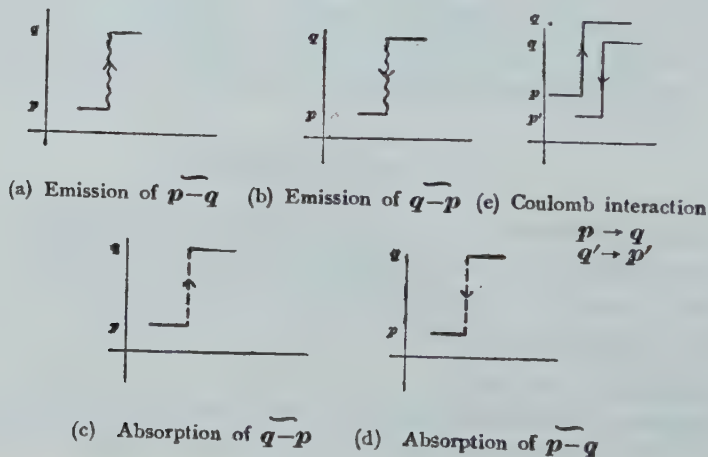
1 (a)), while a positron with momentum p — we denote it by p^+ — is represented again by a horizontal segment, but at the height $(-p)$ and with an arrow-head directed to the left (Fig. 1 (b)). By the "height" in these figures is meant, of course a three-dimensional position in the momentum space. The meaning of the arrow will be clarified later.

Fig. 1



The existence of photons, real or virtual, need not be shown explicitly; only their emission or absorption is indicated by the "leaps", i.e. vertical segments connecting the two horizontal lines which correspond to the two momenta of electron or positron concerned. The emission we denote by a ripple-line and the absorption by a dotted line. Here also an arrow-head is affixed to indicate the direction, so that we may distinguish

Fig. 2



a photon with momentum k — we denote it by \tilde{k} — from $(-\tilde{k})$. For instance a ripple-line from an electron-line at p to one at q will represent the emission of a photon $\widetilde{p-q}$ (Fig. 2 (a)), while the same ripple-segment from q to p that of a photon $\widetilde{q-p}$ (Fig. 2 (b)). A dotted line, on the other hand, from p to q (or from q to p) will stand for the absorption of a photon $\widetilde{q-p}$ (or $\widetilde{p-q}$) (Fig. 2 (c) or (d)).

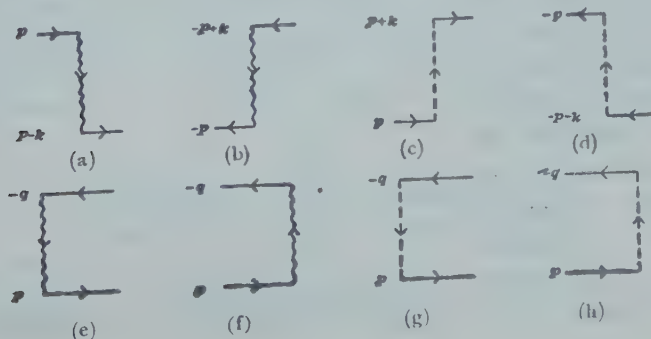
A momentum transfer through Coulomb interaction, which can be treated in parallel with the emission and the immediately following absorption of a photon, may be denoted by two solid vertical line-segments (Fig. 2 (e)).

ii) Fundamental processes.

By the fundamental processes are meant those which the interaction Hamiltonian itself (without iteration) brings about, or in other words, first order processes. They are namely :

- an electron p emits a photon \tilde{k} and goes over to $p-k$;
- a positron p^+ emits a photon \tilde{k} and goes over to $(p-k)^+$;
- an electron p absorbs a photon \tilde{k} and goes over to $p+k$;
- a positron p^+ absorbs a photon \tilde{k} and goes over to $(p+k)^+$;
- an electron-positron pair p, q^+ is created with the emission of a photon $\widetilde{(-p-q)}$;
- an electron-positron pair p, q^+ is annihilated with the emission of a photon $\widetilde{(p+q)}$;
- an electron-positron pair p, q^+ is created with the absorption of a photon $\widetilde{(p+q)}$;

Fig. 3



- h) an electron-positron pair p, q^+ is annihilated with the absorption of a photon ($-\widetilde{p-q}$).

The "diagrams" corresponding to the above eight processes are shown in the Figure 3 (a)—(h) respectively. (The coordinate axes will be omitted hereafter.)

There are also processes of momentum change or pair formation and destruction due to Coulomb interaction, whose expression can be obtained simply by replacing the vertical ripple- or dotted segment in the above figures by a solid one.

One can see from figure 3 at once that, with the conventions made in the preceding section, the condition of momentum conservation is automatically fulfilled if one connects the three segments, which represent a light quantum and two particles concerned, in a proper order, i. e. in such a way that the arrows always follow one another. It will be convenient for the later argument to call the arrow direction in the electron- or positron-line the direction of the "proper time" of that particle — in contrast to the "field time" t (abscissa), which flows always from left to right, and according to which the sequence of intermediate states should be traced. In this sense one might be allowed to regard a positron metaphorically as "an electron whose proper time flows in the opposite direction to the field time", and the pair creation or annihilation as a sudden inversion of the direction of the proper time of one (not two!) particle. Such a concept, once proposed by Zisman⁽⁶⁾ and Schönberg,⁽⁷⁾ turns out, at least, very useful in our further discussions.

iii) Simple examples.

As illustration we give here the diagrams for the four processes of Compton scattering (Fig. 4) and for the reaction of electromagnetic field on an electron (self-energy) (Fig. 5).

Fig. 4 Compton scattering: initial state p, \tilde{k} ,
final state q, \tilde{l} . ($p+k=q+l$, $E_p+k=E_q+l$)

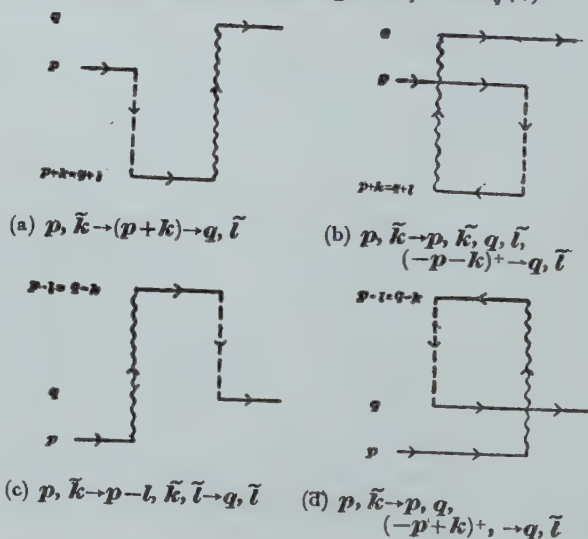
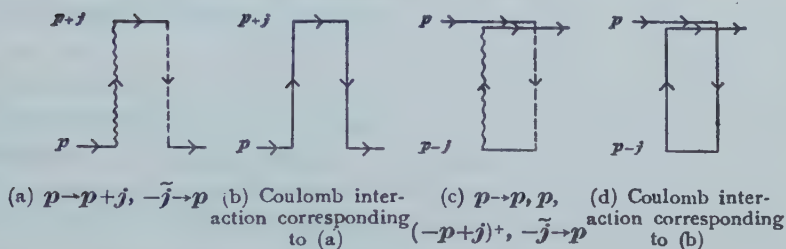


Fig. 5 Self-energy of an electron p


iv) Representation by sequences.

Further we can characterize such transition diagrams as figures 4 and 5 by means of two sequences. Take Fig. 4 (b), for instance; on one hand we may proceed along the electron-line guided by arrows and then have the sequence: "absorption of k ", "emission of l ". This we denote by the symbol (k, l^*) . We may, on the other hand, also trace the sequence of events from left to right according to their order in field time, and this time we have first "emission of l " and then "absorption of k " or abridged as before (l^*, k) . Thus it is natural to represent this connection by the symbol $(k, l^* | l^*, k)$, with the convention that the order in proper time is written at the left part, the order in field time at the right. Analogous considerations give for Fig. 4 (a), (c), and (d), $(k, l^* | k, l^*)$, $(l^*, k | l^*, k)$ and $(l^*, k | k, l^*)$ respectively.

It need hardly be pointed out that the order in proper time corresponds to that of projection operators λ^\pm — e.g. $\lambda_p^\pm = (1/2E_p) \{E_p \pm (\alpha p) + m\beta\}$ — which appear in the numerator of the transition matrix element, while the order in field time determines the successive intermediate states in the usual sense, and so the energy difference in the denominator of the transition matrix element is to be derived out of it. The sign, plus or minus, affixed to each matrix element accounting for the exchange of electrons or positrons, can be decided at once, if one only compares the above two sequences and determines if the one corresponds to an even or to an odd permutation of the other.

§ 3. Classification of Transition Processes, Number of Possible Connections.

With the help of our diagram expression, one can classify any irreducible process including electrons, positrons and photons by two numbers N

and L . By an irreducible process is meant one that cannot be reduced into two or more simpler processes occurring at the same time but independently one upon another.

N is defined as $1/2 \times$ (the number of electrons and positrons in the initial state + the number of those in the final state), thus representing how many sets of connected segments are found in the diagram. L denotes the total number of leaps (vertical segments) in the diagram considered. If the sum of the number of real photons at the initial and the final states be Q , so L , N and Q are related through the equation

$$L = Q + 2(N - 1) \quad (3.3)$$

so far as no radiative correction is taken into account.

Some examples of the N - L -classification are: Compton scattering ($N=1$, $L=Q=2$); double Compton scattering ($N=1$, $L=Q=3$); pair creation by two photons ($N=1$, $L=Q=2$); non-radiative collision of two electrons ($N=2$, $L=2$, $Q=0$); radiative collision of two electrons ($N=2$, $L=3$, $Q=1$).

For the given value of N , L or Q , one can now calculate the number of possible connections. First we treat the simple case $N=1$, $L(=Q)=$ arbitrary. Here the diagram consists of only one set of segments as in Fig. 4 and Fig. 5 and we have seen in §2 iv) that to such a possible connection a ways corresponds a pair of sequences each containing L elements. The converse is also true, as can be verified by drawing a scheme actually. The one-to-one correspondence being verified, the number of possible connections is nothing but that of possible ways of forming two sequences out of given L elements $(L!)^2$:—

Now we go over to the general case $N \geq 2$. In order that a process may be an irreducible one, it is necessary that a momentum transfer between two lines, either through emission and absorption of a virtual photon or through Coulomb interaction, takes place not less than $(N-1)$ times among the N lines. Since we confine ourselves to the perturbation of the lowest order, it cannot occur more than $(N-1)$ times; so we have $L = Q + 2(N-1)$ as stated before. When $N \geq 2$ one must also take note of the possible exchange of the particles themselves. There are namely $N!$ possible ways of combining into N sets the $2N$ ends of electron-positron-lines fixed in the initial and the final states. Out of these N cases we extract a certain one and consider the further varieties that can be produced by distri-

buting L vertical segments over these N lines.

As the further enumerations are elementary but tedious, we restrict ourselves here to a special case $N=2$. Of course the same procedure can be applied to cases $N \geq 3$ mutatis mutandis. First we divide the Q elements (absorption of the photons given in the initial state and emission of those given in the final) into two groups each consisting of, say, S and $Q-S$ elements respectively. There are ${}_QC_S$ ways of such a division. When the first set of S elements is allotted to the first electron-line, this line contains $(S+1)$ elements in all, because there is one more which represents the emission or absorption of the virtual photon. These $(S+1)$ elements can be ordered in $(S+1)!$ ways. Similarly we have on the second line $(Q-S+1)$ elements which can be arranged in $(Q-S+1)!$ ways. We obtain therefore as the possible number of sequences in the proper times

$$2! \sum_{S=0}^Q {}_QC_S (Q+1-S)! (S+1)! \quad (3.2)$$

the factor $2!$ accounting for the exchange of the electrons themselves.

As for the permutation in the field time, one can arrange $(Q+2)$ elements in all in an arbitrary way. Of the two elements concerning the virtual photon, the one which is earlier in the field time is to correspond to the emission and the other to its reabsorption. When these two events take place one immediately after another in the field time, they can also be replaced by Coulomb interaction. The number of possible sequences in the field time thus turns out:

$$\{2(Q+2)! + 2! (Q+1)!\} \quad (3.3)$$

The factor 2 of the first term represents two possible polarizations of virtual photon and the factor $2!$ in the second term comes from the fact that we have distinguished in a Coulomb interaction which of the two electrons has emitted and which of them absorbed the "longitudinal photon" in analogy to the case of a transverse one and thus counted the Coulomb interaction twice as many as in the usual sense.

The product of (3.2) and (3.3) gives the final result for $N=2$:

$$\begin{aligned} & 4\{(Q+2)! + (Q+1)!\} \sum_{S=0}^Q {}_QC_S (Q+1-S)! (S+1)! \\ &= \frac{2}{3} \{(Q+1)!\}^2 Q(Q+3)(Q+5) \end{aligned} \quad (3.4)$$

Similar considerations for $N=3$ yield as the number of connections:

$$\begin{aligned}
& 3! \cdot 3 \sum_{S=0}^Q \sum_{W=0}^S {}_2C_S \cdot {}_2C_W (Q-S+2)! (S-W+1)! (W+1)! \\
& \quad \cdot 4 \{ (Q+4)! + {}_2C_1 (Q+3)! + {}_2C_2 (Q+2)! \} \\
& = \frac{1}{5} (Q+2)! (Q+3)! Q(Q^2+9Q+19)(Q^2+15Q+44) \quad (3.5)
\end{aligned}$$

The formulas for $N \geq 4$ are very complicated and will not be given here. We have not yet succeeded to derive a general expression for arbitrary N .

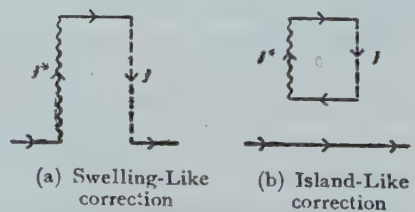
§ 4. Radiative Correction.

So far we have dealt only with the lowest order transitions and so disregarded the radiative correction to the given process at all. Now we shall proceed to the consideration of first radiative correction, introducing the emission and reabsorption of one more virtual photon \tilde{j} , say. In contrast to that virtual photon which is necessary in the case $N \geq 2$, already in the lowest order transition, to relate two electron-lines to each other, this virtual photon \tilde{j} implies some grades of freedom either in its own momentum or in the momenta of a virtual electron-positron pair, and to obtain the total contribution the integration with regard to them must be carried out, which, as is well known, turns out in many a case divergent.

Suppose that a certain process with its all possible connections is given. For simplicity we assume $N=1$, $L=Q$. In order to calculate the first radiative correction, we have to construct connections with $(L+2)$ elements, that is, the given Q and j^* and j out of the given connections with L elements. This can be performed in two ways: a) one can add the two new elements, j^* and j to the existing line, so that the latter receives a swelling-like correction. (Fig. 6 (a)); b) one can build apart from the existing one a new closed line including j^* and j , thus resulting in an island-like correction (Fig. 6 (b)). It will be seen in the following that the first kind correction is what we have called of the mass-type $[M]$, and the second what we have called of the polarization-type $[P]$ in II.

Out of the polarization type corrections, the most general one is, using the notations of II, the process

Fig. 6



$$\cdot \rightarrow r, (-r-j)^+, \tilde{j} \rightarrow \cdot$$

where both r and \tilde{j} are quite arbitrary. Such a process, however, cannot in general affect the cross section at all, because similar contributions, which come from various distribution in field time, when summed, vanish on account of interference with the original process, a result to be expected in general for any two independent events. The only polarization corrections which are effective are those including a virtual photon $\tilde{j}=\tilde{l}$ or $\tilde{j}=-\tilde{l}$, \tilde{l} being one of the real photons that appear in the considered process. This circumstance will be understood most easily by a simple example: In Fig. 7 (a) an electron p emits a photon \tilde{l} and goes over to $p-l$; in (b) a polarization type correction to the above process is illustrated: a real photon \tilde{l} is emitted through the pair creation of $r, (-r-l)^+$, then this pair is annihilated with the emission of a virtual photon $-\tilde{l}$. This $-\tilde{l}$ is absorbed by p , converting it into $p-l$.

In this example one of the elements l^* , which lay in the uncorrected case on the electron-line, has been removed on to the closed line which corresponds to the polarization effect, this is what we may call of $[P, I]$ type. It is possible that two, three, even all the Q elements

are transferred to the closed line, so that on the original "main" electron-line remain $Q-1, Q-2, \dots, 1$ leaps respectively. These cases may be denoted by $[P, II], [P, III], \dots, [P, Q]$.

The number of connections for the first radiative correction can be deduced in a similar way as stated in the last paragraph. So we give here the results immediately.

Mass type correction $[M]$.

$$(Q+2)! \{2(Q+2)! + 2! (Q+1)!\} \cdot (1/2)$$

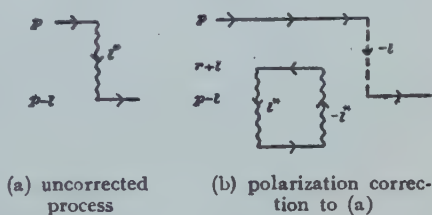
In the field time the element j^* must always precede j , whence the last factor $1/2$ in the above expression.

Polarization type correction $[P]$.

$$[P, I]: q C_1 Q! 1! \{2(Q+2)! + 2! (Q+1)!\}$$

.....

Fig. 7



$$\begin{aligned}
[P, S]: & {}_q C_s(Q+1-S)! S! \{2(Q+2)! + 2! (Q+1)!\} \\
& \dots\dots\dots \\
[P, Q]: & {}_q C_Q 1! Q! \{2(Q+2)! + 2! (Q+1)!\}
\end{aligned}$$

Summing up

$$\begin{aligned}
[P] &= \sum_{s=1}^Q {}_q C_s(Q+1-S)! S! \{2(Q+2)! + 2! (Q+1)!\} \\
&= Q! (Q+1)! \{2(Q+3)!\}
\end{aligned}$$

For the general case $N \geq 2$, one can get along the same course, the only difference lies in the existence of the correction of another type: an electron-line is related to another through emission and absorption of two photons, the algebraic sum of whose momenta is equal to that of the original one.

Thus we have obtained the total number of possible first radiative corrections. It amounts to an enormous number: 1080 for the Compton scattering, and 11,088 for the double Compton scattering etc. But of course not all of them do diverge. In the following two paragraphs we shall examine the diverging part of $[P]$ and $[M]$ in detail. (The third type "two-photon coupling" does not contribute to divergence.)

(to be continued.)

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Progress of Theoretical Physics Vol. IV, No. 1, Jan.~Mar., 1949.

Notes on the Self-Energy Problems.

Katurô SAWADA, Shûji TAKAGI and Minoru KOBAYASHI.
Institute of Physics, Kyoto University.

(Received Oct. 11, 1948)

§ 1. Introduction.

It is the well-known difficulty that the self-energy of an electron interacting with electromagnetic field when calculated following to the current quantum field theory becomes divergent. Dirac⁽¹⁾ has proposed new formalism of field quantization introducing negative energy photons, and has shown that the diverging difficulties of the self-energy could be removed by applying his method combined with the so-called λ -process which was developed by Wentzel and Dirac.

As Pauli⁽²⁾ has pointed out the λ -process serves to make convergent the "statical" self-energy which appears also in the classical theory, and the so-called "fluctuational" part of self-energy becomes to convergent expression by introducing negative energy photons. Since λ -process is the purely classical method and the fluctuational divergent terms are purely quantum mechanical nature, this result seems to be natural.

We shall show that the distinction between classical and quantum quantities is *a priori* possible by separating the c-numbers which appear in the commutation relations of the field variables into two parts. This separation is also possible for the electron field, and it is shown that the divergent terms in positron theoretical self-energy or vacuum polarization energy are related only to those quantum parts.

Further, it is shown that the quantum parts of c-numbers which appear on the right-hand side of the commutation rules do not affect the consistency of the theory such as, for instance, the integrability conditions of Schrödinger equation, and that only classical part concerns to this condition. The usual physical phenomena such as the scattering of photons by electrons or the potentials between electrons are also related only to the classical part.

Therefore, we can put the cut-off factor in this quantum part which

automatically necessitates to introduce negative energy photons obeying Dirac quantization, and Dirac's theory of negative energy photons is shown to be a special case of ours. It is also necessary, if we want to have finite self-energy in positron theory, to diminish quantum c-number of electron field.

§ 2. Formalism.

We adopt the super-many-time formalism developed by Tomonaga and others.⁽³⁾ The generalized Schrödinger equation is given as:

$$\left\{ H(P) - i \frac{\partial}{\partial C_P} \right\} \Psi[C] = 0 \quad (2.1)$$

where $H(P)$ is the interaction Hamiltonian density between electron field and radiation field, which is given by

$$H(P) = S_\mu(P) A_\mu(P) \quad (2.2)$$

where $A_\mu(P)$ are the 4-potentials of radiation field and $S_\mu(P)$ are the 4-currents of electron field:

$$S_\mu(P) = c i \psi^\dagger(P) \gamma^\mu \psi(P); \quad \psi^\dagger = \psi^* \gamma^4. \quad (2.3)$$

ψ^\dagger , ψ and A_μ satisfies respectively the equation of motion for free electron and the radiation field.

It is well known that to make the equation (2.1) soluble, surface C must be space-like, which is the consequence of the following integrability condition for (2.1):

$$[H(P), H(P')]_- = 0 \quad (2.4)$$

for any two world points P, P' on C .

Now, the field variables ψ 's and A 's satisfy the following commutation rules which follows from the canonical form of this dynamical system⁽⁴⁾:

$$[\psi_r(X), \psi_s^\dagger(X')]_+ = i \left(\gamma^\mu \frac{\partial}{\partial x_\mu} - \kappa \right)_{rs} D_\kappa(X - X') \quad (2.5)$$

$$[A_\mu(X), A_\nu(X')]_- = -i \delta_{\mu\nu} D_0(X - X') \quad (2.6)$$

where

$$D_{\pi}(X) = -\frac{1}{2(2\pi)^3} \int (e^{iL_{\lambda} X_{\lambda}} - e^{-iL_{\lambda} X_{\lambda}}) \frac{d\mathbf{l}}{L_4}, \quad L_4 = i\sqrt{\mathbf{l}^2 + \pi^2} \quad (2.7)$$

$$D_0(X) = -\frac{1}{2(2\pi)^3} \int (e^{iK_{\lambda} X_{\lambda}} - e^{-iK_{\lambda} X_{\lambda}}) \frac{d\mathbf{k}}{K_4}, \quad K_4 = i|\mathbf{k}| \quad (2.8)$$

π being the rest mass of the electron.

Here we make the resolution of field variables into two parts, one of which contains emission operators and the other absorption operators:

$$A_{\mu}(X) = U_{\mu}(X) + U_{\mu}^*(X) \quad (2.9)$$

where as usual

$$U_{\mu}(X) = i \int U_{\mu}(\mathbf{k}) \frac{e^{iK_{\lambda} X_{\lambda}}}{K_4} d\mathbf{k}, \quad U_{\mu}^*(X) = i \int U_{\mu}^*(\mathbf{k}) \frac{e^{-iK_{\lambda} X_{\lambda}}}{K_4} d\mathbf{k} \quad (2.10)$$

and this resolution gives particle aspect to the field.

$U_{\mu}(\mathbf{k})$, $U_{\mu}^*(\mathbf{k})$ satisfies the canonical commutation rules

$$[U_{\mu}(\mathbf{k}), U_{\nu}^*(\mathbf{k}')]_{-} = -i\delta_{\mu\nu} \frac{K_4}{2} \delta(\mathbf{k} - \mathbf{k}').$$

From this and (2.10) we can get the following commutation rules for $U_{\mu}(X)$, $U_{\mu}^*(X)$:

$$[U_{\mu}(X), U_{\nu}^*(X')]_{-} = -i\delta_{\mu\nu} \frac{1}{2} (\overset{(\circ)}{D}_0(X - X') - \frac{1}{i} \overset{(\circ)}{D}_{01}(X - X')) \quad (2.11)$$

where

$$D_{01}(X) = \frac{i}{2(2\pi)^3} \int (e^{iK_{\lambda} X_{\lambda}} - e^{-iK_{\lambda} X_{\lambda}}) \frac{d\mathbf{k}}{K_4} \quad (2.12)$$

is the usual D_1 -function.

The same decomposition can also be done for the electron field:

$$\psi(X) = \psi^{\dagger}(X) + \bar{\psi}^{\dagger}(X), \quad \psi^{\dagger}(X) = \psi^{\dagger}(X) + \bar{\psi}(X) \quad (2.13)$$

where

$$\begin{aligned} \psi^{\dagger}(X) &= i \int \psi^{\dagger}(\mathbf{l}) \frac{e^{iL_{\lambda} X_{\lambda}}}{L_4} d\mathbf{l}, & \bar{\psi}^{\dagger}(X) &= i \int \bar{\psi}^{\dagger}(\mathbf{l}) \frac{e^{-iL_{\lambda} X_{\lambda}}}{L_4} d\mathbf{l} \\ \psi^{\dagger}(X) &= i \int \psi^{\dagger}(\mathbf{l}) \frac{e^{-iL_{\lambda} X_{\lambda}}}{L_4} d\mathbf{l}, & \bar{\psi}(X) &= i \int \bar{\psi}(\mathbf{l}) \frac{e^{iL_{\lambda} X_{\lambda}}}{L_4} d\mathbf{l}. \end{aligned} \quad (2.14)$$

The commutation relations for ϕ^+ and $\bar{\phi}$ are as follows:

$$\begin{aligned} [\phi^+(X), \phi_s^\dagger(X')]_+ &= i \left(\gamma^\mu \frac{\partial}{\partial x_\mu} - x \right)_{rs} \frac{1}{2} \left(\overset{(c)}{D}_x(X-X') - \frac{1}{i} \overset{(q)}{D}_{x1}(X-X') \right) \\ [\bar{\phi}_s^\dagger(X), \bar{\phi}_t(X')]_+ &= i \left(\gamma^\mu \frac{\partial}{\partial x_\mu} - x \right)_{rs} \frac{1}{2} \left(\overset{(c)}{D}_x(X-X') + \frac{1}{i} \overset{(q)}{D}_{x1}(X-X') \right) \end{aligned} \quad (2.15)$$

We put (c) and (q) in (2.11) and (2.15) on the head of D - and D_1 -functions and call them as classical and quantum parts, because (q) part comes from the decomposition of (2.9) and (2.13) and (c) part from the canonical form of this dynamical system, and as will be shown in the following the D -part leads to the results which have classical analogues whereas D_1 -part leads to pure quantum theoretical difficulties. The latter can also be called as fluctuation part, since, if we set the time coordinate equal in (2.11), there remains only D_1 -part and this gives the fluctuation of the field.

The integrability condition (2.4) is also related to D -functions only, for we have

$$\begin{aligned} [H(P), H(P')]_- &= [S_\mu(P)A_\mu(P), S_\nu(P')A_\nu(P')]_- \\ &= \frac{1}{2} \{ [S_\mu(P), S_\nu(P')]_+ \cdot [A_\mu(P), A_\nu(P')]_- \\ &\quad + [S_\mu(P), S_\nu(P)]_- \cdot [A_\mu(P), A_\nu(P')]_+ \} \end{aligned} \quad (2.16)$$

which vanishes for space-like C by using the following relations:

$$\begin{aligned} [A_\mu(P), A_\nu(P')]_- &= -i \delta_{\mu\nu} \overset{(c)}{D}_0(P-P') \\ [S_\mu(P), S_\nu(P')]_- &= -i \{ \phi^\dagger(P) \gamma^\mu \left(\gamma^\nu \frac{\partial}{\partial P_\nu} - x \right) \overset{(c)}{D}_x(P-P') \gamma^\nu \phi(P') \\ &\quad - \phi^\dagger(P') \gamma^\nu \left(\gamma^\mu \frac{\partial}{\partial P_\mu} - x \right) \overset{(c)}{D}_x(P'-P) \gamma^\mu \phi(P) \}. \end{aligned} \quad (2.17)$$

§ 3. Self-energy.

According to Pauli and Fierz⁽⁵⁾ we separate free-field and bounded field by the following canonical transformation:

$$\Psi[C] = \exp \left\{ -i \int_C H(X) dV_4 \right\} \cdot \Phi[C], \quad (3.1)$$

then, Schrödinger equation (2.1) becomes

$$\left\{ \frac{i}{2!} \left[\int^c H(X) dV_4, H(P) \right] \right\} \Phi[C] = i \frac{\delta}{\delta C_P} \Phi[C] \quad (3.2)$$

in ϵ^2 -approximations.

The left-hand side of (3.2) can be written in analogous way as (2.16):

$$\begin{aligned} & \frac{i}{2!} \int^c \frac{1}{2} \{ [S_\mu(X), S_o(P)]_+ \cdot [A_\mu(X), A_o(P)]_- \quad (a) \\ & + [S_\mu(X), S_o(P)]_- \cdot [A_\mu(X), A_o(P)]_+ \} dV. \quad (b) \end{aligned} \quad (3.3)$$

Here, we define in the first term Wentzel Potential A_o^w as

$$A_o^w(P) = \frac{i}{2!} \int^c S_\mu(X) \cdot [A_\mu(X), A_o(P)]_- \cdot dV_4 \quad (3.4)$$

because they are written only in the electron variables, though they are the pure retarded potentials and satisfy the following equations:

$$\square A_o^w(P) = S_o(P), \quad \text{Div } A_o^w(P) = 0.$$

in contrast to the equations of the potentials introduced originally by Wentzel and Dirac.* Similarly in the second term of (3.3) we define Wentzel current as

$$S_o^w(P) = \frac{i}{2!} \int^c [S_\mu(X), S_o(P)]_- \cdot A_\mu(X) \cdot dV_4. \quad (3.5)$$

Then (3.3) can be written as:

$$\left\{ \frac{1}{2} [S_o(P), A_o^w(P)]_+ + \frac{1}{2} [S_o^w(P), A_o(P)]_+ \right\} \Phi[C] = i \frac{\delta}{\delta C_P} \Phi[C] \quad (3.6)$$

where the first term of the left-hand side represents the interaction energy between Wentzel field and the current and the second term that between Wentzel current and the radiation field.

It is easily shown that (3.3a) gives self-energy and Möller potentials. We note that it contains only the classical c-number of photon, for

$$A_o^w(P) = \frac{i}{2!} \int^c S_\mu(X) \cdot (-i) \delta_{\mu o} D_0^{(c)}(X-P) dV_4$$

* If we take $[A_\mu(X), A_\nu(X')]_- = -i \delta_{\mu\nu} (1/2) (D_0(X-X') + \lambda) + D_0(X-X' - \lambda)$ according to Dirac, it can easily be seen that we get the original Wentzel's potentials.

$$= \frac{1}{2!} \int S_0(X) \cdot \overset{(c)}{D}_0(X-P) dV_4. \quad (3.7)$$

Substituting (2.3), (3.3a) becomes to

$$\begin{aligned} -\frac{1}{2} [S_0(P), A_0^W(P)]_+ &= \frac{1}{2 \cdot 2!} \int [S_0(X), S_0(P)]_+ \cdot \overset{(c)}{D}_0(X-P) dV_4 \\ &= \frac{e^2}{2 \cdot 2!} \int \left\{ -i(\psi^\dagger(X) \gamma^0 \left(\gamma^\tau \frac{\partial}{\partial x_\tau} - x \right) \overset{(c)}{D}_\tau(X-P) \gamma^0 \psi(P) \right. \\ &\quad \left. + \psi^\dagger(P) \gamma^0 \left(\gamma^\tau \frac{\partial}{\partial P_\tau} - x \right) \overset{(c)}{D}_\tau(P-X) \gamma^0 \psi(X) \right\} \\ &\quad + 2\psi_r^\dagger(X) \cdot \gamma_{rs}^0 \psi_i^\dagger(P) \cdot \psi_s(X) \gamma_{im}^0 \psi_m(P) \} \overset{(c)}{D}_0(X-P) dV_4. \end{aligned} \quad (3.8)$$

In one-electron theory, the first two terms of (3.8) give the self-energy, whose principal term is

$$e^2 \sum_{\mathbf{k}} \frac{1}{2k^2} \quad (\text{due to } D_\tau \cdot D_0) \quad (3.9)$$

In positron-theoretical treatment, the last term of (3.8) gives also the self-energy :

$$\begin{aligned} &\frac{e^2}{2 \cdot 2!} \int 2\psi_r^\dagger(X) \gamma_{rs}^0 \psi_i^\dagger(P) \cdot \psi_s(X) \gamma_{im}^0 \psi_m(P) \cdot \overset{(c)}{D}_0(X-P) dV_4 \\ &\approx \frac{e^2}{2 \cdot 2!} \int 2i \left\{ \psi^\dagger(X) \gamma^0 \left(\gamma^\tau \frac{\partial}{\partial x_\tau} - x \right) \frac{1}{2} (\overset{(c)}{D}_\tau(X-P) + \frac{1}{i} \overset{(g)}{D}_{\tau 1}(X-P)) \right. \\ &\quad \cdot \gamma^0 \psi(P) + \psi^\dagger(P) \gamma^0 \left(\gamma^\tau \frac{\partial}{\partial P_\tau} - x \right) \frac{1}{2} (\overset{(c)}{D}_\tau(P-X) + \frac{1}{i} \overset{(g)}{D}_{\tau 1}(X-P)) \\ &\quad \left. \cdot \gamma^0 \psi(X) \right\} \overset{(c)}{D}(P-X) dV_4. \end{aligned} \quad (3.10)$$

which leads to the following divergent expressions :

$$-e^2 \sum_{\mathbf{k}} \frac{1}{2k^2} \quad (\text{due to } \overset{(c)}{D}_\tau \cdot \overset{(c)}{D}_0) \quad (3.11)$$

and

$$e^2 \sum_{\mathbf{k}} \frac{x}{E_k k^2} - e^2 \sum_{\mathbf{k}} \frac{E_k}{2x k^2} \quad (\text{due to } \overset{(g)}{D}_{\tau 1} \cdot \overset{(c)}{D}_0) \quad (3.12)$$

so that in positron theoretical treatment (3.9) and (3.11) cancel each other

and the remaining diverging terms of the self-energy due to Wentzel-field and -current is

$$e^2 \sum_{\mathbf{k}} \frac{\mathbf{x}}{E_{\mathbf{k}} k^2} - e^2 \sum_{\mathbf{k}} \frac{E_{\mathbf{k}}}{2\mathbf{x} k^2} \quad (3.13)$$

which is due to $\overset{(g)}{D}_{\mathbf{x}_1}$ and $\overset{(e)}{D}_0$ and does not vanish by λ -limiting process.

(3.3b) gives self-energy which does not converge by applying the λ -process in one electron treatment, Compton scattering and the vacuum polarization :

$$\begin{aligned} S_o^w(P) &= \frac{i}{2!} \int^c [S_{\mu}(X), S_o(P)]_- A_{\mu}(X) dV_4 \\ &= \frac{e^2}{2!} \int^c \{ \psi^{\dagger}(X) \cdot \gamma^{\mu} \left(\gamma^{\tau} \frac{\partial}{\partial x_{\tau}} - x \right) \overset{(e)}{D}_{\mathbf{x}}(X-P) \gamma^{\sigma} \psi(P) \\ &\quad - \psi^{\dagger}(P) \gamma^{\sigma} \left(\gamma^{\tau} \frac{\partial}{\partial P_{\tau}} - x \right) \overset{(e)}{D}_{\mathbf{x}}(P-X) \gamma^{\mu} \psi(X) \} A_{\mu}(X) dV_4. \end{aligned} \quad (3.14)$$

This part of the self-energy of the electron can be obtained by computing the expectation value

$$\begin{aligned} &\ll \frac{1}{2} [S_o^w(P), A_o(P)]_+ \gg_{AV} \\ &= \frac{e^2}{2 \cdot 2!} \int^c \{ \psi^{\dagger}(X) \gamma^{\mu} \left(\gamma^{\tau} \frac{\partial}{\partial x_{\tau}} - x \right) \overset{(e)}{D}_{\mathbf{x}}(X-P) \gamma^{\sigma} \psi(P) \\ &\quad - \psi^{\dagger}(P) \gamma^{\sigma} \left(\gamma^{\tau} \frac{\partial}{\partial P_{\tau}} - x \right) \overset{(e)}{D}_{\mathbf{x}}(P-X) \gamma^{\mu} \psi(X) \} \gg \cdot \\ &\quad \cdot \ll [A_{\mu}(X), A_o(P)]_+ \gg_{AV} dV_4 \end{aligned} \quad (3.15)$$

where

$$\ll [A_{\mu}(X), A_o(P)]_+ \gg_{AV} = \delta_{\mu o} \overset{(g)}{D}_{01}(X-P). \quad (3.16)$$

As the result the infinite term of the self-energy is given by

$$e^2 \sum_{\mathbf{k}} \frac{1}{2k\mathbf{x}} \quad (\text{due to } \overset{(e)}{D}_{\mathbf{x}} \cdot \overset{(g)}{D}_{01}) \quad (3.17)$$

which is identical both in one-electron and positron theoretical treatment.

Polarization energy of vacuum is also given by (3.15) if we take $\ll \gg_{AV}$ for the state where no electron is present, namely

$$\begin{aligned}
\ll S_\sigma^\mu(P) \gg_{vac} &\approx \frac{e^2}{2!} \int \frac{1}{2} S_{F\mu\nu} \left(\left(\gamma^\tau \frac{\partial}{\partial P_\tau} - x \right) D_{\kappa_1}^{(q)}(P-X) \gamma^\mu \cdot \right. \\
&\cdot \left(\gamma^\nu \frac{\partial}{\partial x_\nu} - x \right) D_\kappa^{(e)}(X-P) \gamma^\sigma - \left(\gamma^\tau \frac{\partial}{\partial P_\tau} - x \right) D_\kappa^{(e)}(P-X) \gamma^\mu \cdot \\
&\cdot \left. \left(\gamma^\nu \frac{\partial}{\partial x_\nu} - x \right) D_{\kappa_1}^{(q)}(X-P) \gamma^\sigma \right) A_\mu(X) dV_4
\end{aligned} \quad (3.18)$$

which is due to $D_\kappa^{(e)} D_{\kappa_1}^{(q)}$.

§ 4. Cut-off method in positron theory.

From the above considerations, it becomes clear that the positron theoretical divergences always appear in the place where D - and D_1 -functions are combined.

As the physical phenomena, such as the scattering or the potentials between electrons, arise only through D -functions of both fields, we think that D_1 -part is to be modified to obtain a satisfactory theory.

Since D_1 -functions do not concern with the integrability conditions, we shall be able to put cut-off factor in this place without destroying consistency of the theory. Therefore we take instead of (2.11)

$$\begin{aligned}
[U_\mu(X), U_\nu(X')]_- &= -i\delta_{\mu\nu} \frac{1}{2} (\bar{D}(X-X')) \\
&- \frac{1}{i} \frac{i}{2(2\pi)^3} \int f(k) (e^{iK_\lambda(X_\lambda-X'_\lambda)} + e^{-iK_\lambda(X_\lambda-X'_\lambda)}) \frac{d\mathbf{k}}{K_4}. \quad (4.1)
\end{aligned}$$

For $f(K)$ we should choose as follows:

$$f(K) = 1 \text{ for } K < K_0; \quad f(K) = 0 \text{ for } K > K_0 \quad (K_0: \text{cut-off momentum}) \quad (4.2)$$

since the purpose of the introduction of this factor is to diminish the contribution of D_1 -function in high energy region. The commutation relations between A_μ 's do not change by this modification.

The decomposition of the field into \mathbf{k} -space is no longer (2.10), since there appears different time factor in (4.1). It is necessary to decompose U once again into two parts:

$$\begin{aligned}
U_\mu(X) &= \bar{U}_\mu^+(X) + \bar{U}_\mu^-(X), \\
\bar{U}_\mu^+(X) &= i \int \bar{U}_\mu^+(\mathbf{k}) \frac{e^{iK_\lambda X_\lambda}}{K_4} d\mathbf{k}, \quad \bar{U}_\mu^-(X) = i \int \bar{U}_\mu^-(\mathbf{k}) \frac{e^{-iK_\lambda X_\lambda}}{K_4} d\mathbf{k}. \quad (4.3)
\end{aligned}$$

Then we obtain by comparing the time factor of both side of (4.1) the following relations :

$$[\dot{U}_\mu^+(X), \dot{U}_\nu^*(X')]_- = -i\delta_{\mu\nu} \frac{1}{2} \left(-\frac{1}{2(2\pi)^3} \right) \int (1+f(K)) \frac{e^{iK_\lambda(X_\lambda - X_{\lambda'})}}{K_4} d\mathbf{K} \quad (4.4)$$

$$[\bar{U}_\mu^+(X), \bar{U}_\nu^*(X')]_- = i\delta_{\mu\nu} \frac{1}{2} \left(-\frac{1}{2(2\pi)^3} \right) \int (1-f(K)) \frac{e^{-iK_\lambda(X_\lambda - X_{\lambda'})}}{K_4} d\mathbf{K} \quad (4.5)$$

while other commute each other.

If we write in \mathbf{k} -space, we obtain

$$[\dot{U}_\mu^+(\mathbf{k}), \dot{U}_\nu^*(\mathbf{k}')]_- = \delta_{\mu\nu} \frac{k_0}{2} \frac{(1+f(K))}{2} \delta(\mathbf{k}-\mathbf{k}') \quad (4.6)$$

$$[\bar{U}_\mu^+(\mathbf{k}), \bar{U}_\nu^*(\mathbf{k}')]_- = -\delta_{\mu\nu} \frac{k_0}{2} \frac{(1-f(K))}{2} \delta(\mathbf{k}-\mathbf{k}') \quad (4.7)$$

so that \bar{U} -field follows Dirac's new quantization. When we write in ordinary formalism as

$$\dot{U}_\mu^+(\mathbf{k}) = \dot{W}_\mu^+(\mathbf{k}) \cdot \sqrt{\frac{k_0}{2}}, \quad \bar{U}_\mu^+(\mathbf{k}) = \bar{W}_\mu^+(\mathbf{k}) \cdot \sqrt{\frac{k_0}{2}}$$

Hamiltonian should have the following form

$$\begin{aligned} H_0 &= \frac{1}{2} \sum_{\mathbf{k}} (\dot{W}_\mu^+(\mathbf{k}) \dot{W}_\mu^+(\mathbf{k}) + \dot{W}_\mu^+(\mathbf{k}) \bar{W}_\mu^*(\mathbf{k})) k_0 \frac{2}{1+f(\mathbf{k})} \\ &\quad + \frac{1}{2} \sum_{\mathbf{k}} (\bar{W}_\mu^*(\mathbf{k}) \bar{W}_\mu^+(\mathbf{k}) + \bar{W}_\mu^*(\mathbf{k}) \dot{W}_\mu^+(\mathbf{k})) k_0 \frac{2}{1-f(\mathbf{k})}, \\ H_{int} &= ic\psi^\dagger \gamma^\mu \psi \cdot \int \frac{d\mathbf{k}}{\sqrt{2k_0}} [(\dot{W}_\mu^+(\mathbf{k}) + \bar{W}_\mu^*(\mathbf{k})) e^{iK_\lambda X_\lambda} \\ &\quad + (\dot{W}_\mu^*(\mathbf{k}) + \bar{W}_\mu^+(\mathbf{k})) e^{-iK_\lambda X_\lambda}] \end{aligned} \quad (4.8)$$

and the commutation relations are

$$\begin{aligned} [\dot{W}_\mu^+(\mathbf{k}), \dot{W}_\nu^*(\mathbf{k}')]_- &= \delta_{\mu\nu} \frac{1+f(K)}{2} \delta(\mathbf{k}-\mathbf{k}') \\ [\bar{W}_\mu^+(\mathbf{k}), \bar{W}_\nu^*(\mathbf{k}')]_- &= -\delta_{\mu\nu} \frac{1-f(K)}{2} \delta(\mathbf{k}-\mathbf{k}') \end{aligned} \quad (4.9)$$

If we take as the special case $f=0$, then (4.8) (4.9) is identical with Dirac's negative energy photon.

However, $f(k)$ is not necessarily zero as far as it satisfies (4.2) and is relativistic invariant.

(4.8) and (4.9) can also be written by the emission and absorption operators :

$$\begin{aligned} [\bar{V}_\mu(k), \bar{V}_\nu^*(k')]_- &= \delta_{\mu\nu} \delta(k-k') \\ [\bar{V}_\mu(k), \bar{V}_\nu^*(k')]_- &= -\delta_{\mu\nu} \delta(k-k') \end{aligned} \quad (4.10)$$

$$\bar{V}_\mu(k) = \bar{V}_\mu(k) \cdot \sqrt{\frac{1+f(k)}{2}}; \quad \bar{W}_\mu(k) = \bar{V}_\mu(k) \cdot \sqrt{\frac{1-f(k)}{2}}. \quad (4.11)$$

Then we have

$$\begin{aligned} H_0 &= \frac{1}{2} \sum_{\mathbf{k}} (\bar{V}_\mu^*(\mathbf{k}) \bar{V}_\mu(\mathbf{k}) + \bar{V}_\mu(\mathbf{k}) \bar{V}_\mu^*(\mathbf{k})) k_0 \\ &\quad + \frac{1}{2} \sum_{\mathbf{k}} (\bar{V}_\mu^*(\mathbf{k}) \bar{V}_\mu(\mathbf{k}) + \bar{V}_\mu(\mathbf{k}) \bar{V}_\mu^*(\mathbf{k})) k_0 \\ H_{int} &= c \phi^\dagger \gamma^\mu \phi \int \frac{d\mathbf{k}}{\sqrt{2k_0}} [(\bar{V}_\mu(\mathbf{k}) e^{iK_\lambda X_\lambda} + \bar{V}_\mu^*(\mathbf{k}) e^{-iK_\lambda X_\lambda}) \sqrt{\frac{1+f(k)}{2}} \\ &\quad + (\bar{V}_\mu^*(\mathbf{k}) e^{iK_\lambda X_\lambda} + \bar{V}_\mu(\mathbf{k}) e^{-iK_\lambda X_\lambda}) \sqrt{\frac{1-f(k)}{2}}]. \end{aligned} \quad (4.12)$$

This reduces the part of self-energy which comes from $\bar{D}_{01}^{(q)} \cdot \bar{D}_\pi^{(e)}$ (3.17) to :

$$c^2 \sum_{\mathbf{k}} \frac{f(k)}{2kx}. \quad (4.13)$$

Therefore the cut-off of self-energy is equivalent to the introduction of negative energy Dirac photons.

The remaining self-energy in the positron-theoretical treatment contains $\bar{D}_{\pi 1}^{(q)}$ and this part should be modified also, but since photon cannot have self-energy, theory which does not contain $\bar{D}_{\pi 1}^{(q)}$ for electron field should be formulated, and this leads to vacant negative energy state for electron and positron.

§ 5. Conclusion.

The divergent difficulties in positron theory are discussed, and it is shown that they are all connected to D_1 -functions which appear by the

quantization of the field variables. The D_1 -functions do nothing to the actual physical phenomena and the integrability conditions, so we can modify this part and get finite self-energy without affecting other physical features. By doing so, we should necessarily introduce negative energy photons.

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The Level Shift and the Anomalous Magnetic Moment of the Electron.

Yôichiro NAMBU.

Department of Physics, Tokyo University.

(Received Oct. 25, 1948)

§ 1. Introduction.

The recent achievements both in experiment and theory have confirmed that the reaction of the radiation field is a really observable phenomenon and that one can calculate this effect in our present formalism of quantum theory if one employs some subtraction prescription. Thus Bethe⁽¹⁾ computed successfully the level shift of the hydrogen atom revealed by the experiment of Lamb and Retherford.⁽²⁾ Following Bethe's idea, Tomonaga⁽³⁾ has developed, independently of American authors, a so-called "self-consistent" subtraction method which aims at disposing of the infinities in a self-consistent manner and obtaining finite results for various processes. In fact he and collaborators could show that the radiative corrections for the elastic scattering of the electron and the Compton scattering were made finite by this method.

Although Tomonaga's theory has recourse to a relativistic canonical transformation which may be regarded as a generalization of the transformation used by Bloch and Nordsieck and by Pauli and Fierz, yet his method can equally be applied to the conventional perturbation formalism. We have here calculated the radiative correction of an electron moving in an external (electromagnetic) field along the line of the ordinary perturbation theory. To get definite numerical results we had to content ourselves with non-relativistic approximation for the initial and final states.

According to Tomonaga there are three kinds of infinities, that is, the electron's self energy, or the mass type infinity, the photon's self energy, or the vacuum polarization type infinity,⁽⁴⁾ and an infinity depending on the external field. We shall see that the subtraction of the first and the third type infinity is necessary in our case.

A similar calculation making use of the canonical transformation has

been carried out by Fukuda, Miyamoto and Tomonaga.⁽⁵⁾ Comparison of the obtained results is not without interest in view of the provisional character of the present theory.

§ 2. The perturbation scheme.

The outline of the calculation developed here is more or less identical with that which Dancoff⁽⁶⁾ followed in his estimation of the radiative correction for the elastic scattering of the electron and which has also been discussed by Koba and Tomonaga,⁽⁷⁾ and Endo, Kinoshita and Koba in the light of the new subtraction theory. The difference lies, however, in that they investigated the scattering cross section and therefore considered only transitions between positive energy states while we are now going to ask the transition matrix itself, thus including the transitions for the positrons as well as the creation and annihilation of pairs. Moreover we want to extend the calculation to the case of a magnetic external potential, or in general, an external static electromagnetic field

$$V - (eA) \equiv H.$$

The state of an electron (or more precisely, of an electron field) in interaction with its own electromagnetic field may conveniently be expressed by the following transition scheme⁽⁸⁾:

$$\begin{array}{c}
 \nearrow \mathbf{p}-\mathbf{k}, \tilde{\mathbf{k}} \longrightarrow \left\{ \begin{array}{l} \mathbf{p} \\ \mathbf{p}-\mathbf{k}, \mathbf{l}, (-\mathbf{l}+\mathbf{k})^+ \end{array} \right. \\
 \mathbf{p} \rightarrow \mathbf{p}, \mathbf{l}, (-\mathbf{l}+\mathbf{k})^+, -\tilde{\mathbf{k}} \rightarrow \mathbf{p}, \mathbf{l}, (-\mathbf{l})^+ \\
 \boxed{\phantom{\mathbf{p}, \mathbf{l}, (-\mathbf{l}+\mathbf{k})^+}} \searrow \mathbf{p}, \mathbf{l}-\mathbf{k}, (-\mathbf{l}+\mathbf{k})^+
 \end{array} \quad (1)$$

where \mathbf{p} is the momentum of the electron in the unperturbed state, $\tilde{\mathbf{k}}$ is that of the photon emitted, and \mathbf{l} and $(-\mathbf{l}+\mathbf{k})^+$ are momenta of the virtual pair. A corresponding scheme may be drawn for the positron. What is to be considered next is the matrix element of the external field H with respect to the transition of the electron from momentum \mathbf{p} to momentum \mathbf{q} when the interaction with the electromagnetic field is taken into account according to the above scheme.

In the c^2 -approximation here concerned, there are two kinds of matrix elements, namely

$$(i) \text{ first order in } e: (\mathbf{p}, s; 0 | H | \mathbf{q}, t; \tilde{\mathbf{k}}, \lambda_k), \quad (2)$$

(ii) second order in e : $(p, s; 0 | H | q, t; 0)$, (3)

where s and t denote the spin variable of the electron, and λ_k the polarization state of the emitted photon. The first kind corresponds to the Bremsstrahlung of the electron in the field H . The required correction to the zero order matrix H_{pq} comes out of the combined effect of the first order correction due to (3) and the second order correction due to (2).

Let the interaction Hamiltonian be given by

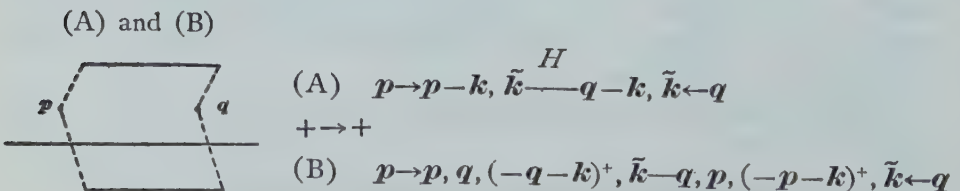
$$H = -e \int \psi^* \alpha \psi A dv + \frac{e^2}{2} \int \psi^* \psi \cdot \psi'^* \psi' / |\mathbf{r} - \mathbf{r}'| \cdot d\mathbf{v} d\mathbf{v}' + \int \psi^* (V - \alpha A) \psi dv, \quad (4)$$

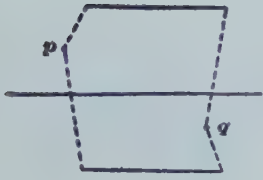
where the first two terms are the interaction of the electron field with its own electromagnetic field, while the third is the interaction with the external potential, the coefficient of which we need not assume to be small.

By decomposing the field variables into Fourier components, the matrix element (2) is immediately obtained:

$$(p; 0 | H | q; \tilde{k}, \lambda_k) = -\frac{e}{2\pi} \sum_{\pm} \pm \left\{ \frac{a_{\lambda} \cdot \lambda^{\pm} (p - q) H_{p-k, q}}{E_p \mp E_{p-k} - k} + \frac{H_{p, q-k} \lambda^{\pm} (q - k) a_{\lambda}}{E_q \mp E_{q-k} - k} \right\}. \quad (5)$$

For the calculation of the matrix elements (3) it is convenient to classify various transition types into several groups as was done by Dancoff. Here, however, the classification is not identical with his, a fuller account of which will be given in the next section. We shall further introduce diagrams illustrating various possible paths connecting the initial and final state. In these diagrams the abscissae mean the momentum of the electron, while the ordinates are used to discriminate positive and negative energy states. Dotted lines imply transitions due to emission or absorption of a photon (Coulomb interaction included), and full lines those due to the external field.





$$\begin{aligned}
 &+ \rightarrow - \quad p, (-q)^+ \rightarrow p-k, (-q)^+, \tilde{k} \longrightarrow q-k, \\
 &\quad (-q)^+, \tilde{k} \leftarrow. \\
 &+ \rightarrow - \quad p, (-q)^+ \rightarrow p, (-q-k)^+, \tilde{k} \longrightarrow p, \\
 &\quad (-p-k)^+, \tilde{k} \leftarrow.
 \end{aligned}$$

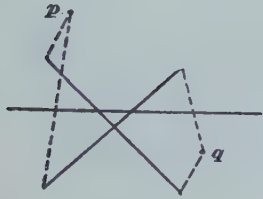


$$\begin{aligned}
 &- \rightarrow - \quad (-q)^+ \rightarrow (-q)^+, (-p)^+, p-k, \\
 &\quad \tilde{k} \longrightarrow (-q)^+, (-p)^+, q-k, \tilde{k} \leftarrow (-p)^+ \\
 &- \rightarrow - \quad (-q)^+ \rightarrow (-q-k)^+, \tilde{k} \longrightarrow (-p-k)^+, \\
 &\quad \tilde{k} \leftarrow (-p)^+
 \end{aligned}$$

(C)



$$+ \rightarrow + \quad p \begin{cases} \nearrow p-k, \tilde{k} \\ \searrow p, q, (-q+k)^+, -\tilde{k} \end{cases} \xrightarrow{H} p-k, q, (-q+k)^+ \leftarrow q$$



$$+ \rightarrow - \quad p, (-q)^+ \begin{cases} \nearrow p-k, (-q)^+, \tilde{k} \\ \searrow p, (-q+k)^+, -\tilde{k} \end{cases} \xrightarrow{H} p-k, (-q+k)^+ \leftarrow.$$

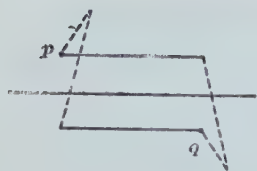


$$- \rightarrow - \quad (-q)^+ \begin{cases} \nearrow (-p)^+, (-q)^+, p-k, \tilde{k} \\ \searrow (-q+k)^+, -\tilde{k} \end{cases} \xrightarrow{H} \begin{cases} (-p)^+, \\ (-q+k)^+, \\ p-k, \\ -(-p)^+ \end{cases}$$

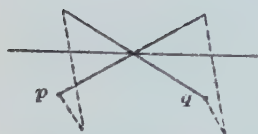
(D)



$$+ \rightarrow + \quad p \rightarrow p, q-k, (-q)^+, \tilde{k} \xrightarrow{H} q-k, \tilde{k} \rightarrow q$$

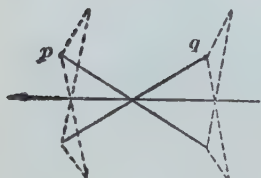


$$+\rightarrow- \quad p, (-q)^+ \rightarrow p, (-q+k)^+, -\tilde{k} \longrightarrow q, \\ (-q+k)^+, -\tilde{k} \leftarrow.$$

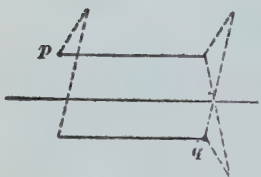


$$--\rightarrow- \quad (-q)^+ \rightarrow (-q+k)^+, -\tilde{k} \longrightarrow q, \\ (-q+k)^+, (-p)^+, -\tilde{k} \leftarrow (-p)^+$$

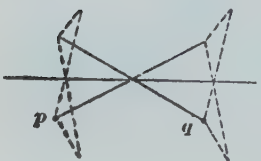
(E)



$$+\rightarrow+ \quad p \begin{cases} \rightarrow p, q-k, (-q)^+, \tilde{k} \\ \rightarrow p, q, (-q+k)^+, -\tilde{k} \end{cases} \xrightarrow{H} p, q, (-q)^+ \longrightarrow q$$

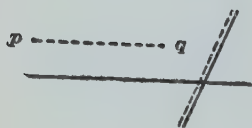


$$+\rightarrow- \quad p, (-q)^+ \xrightarrow{H} q, \\ (-q)^+ \leftarrow \begin{cases} q-k, (-q)^+, \tilde{k} \\ (-q+k)^+, q, -\tilde{k} \end{cases}$$

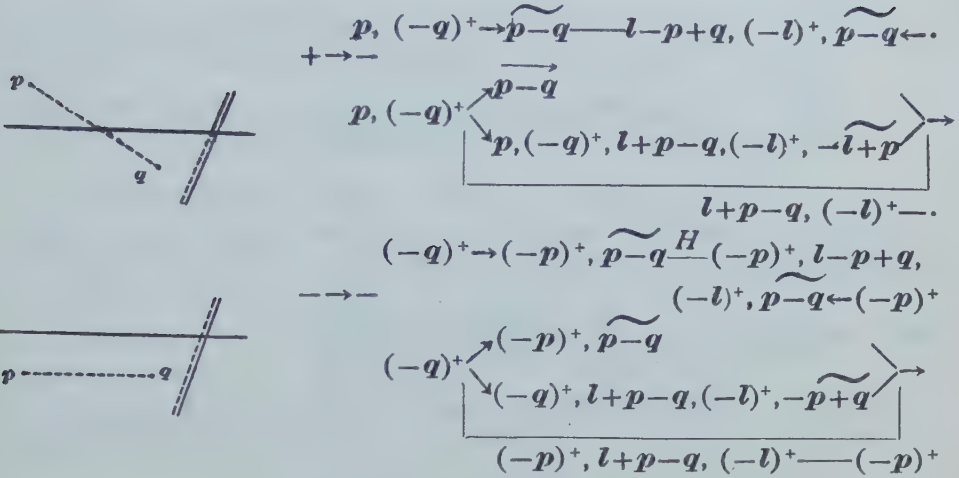


$$--\rightarrow- \quad (-q)^+ \longrightarrow q, (-p), \\ (-q)^+ \leftarrow \begin{cases} q-k, (-q)^+, (-p)^+, \tilde{k} \\ q, (-q+k)^+, (-p)^+, -\tilde{k} \end{cases} \longrightarrow (-p)^+$$

(F)



$$+\leftarrow+ \quad p \rightarrow q, \widetilde{p-q} \xrightarrow{H} q, l-p+q, (-l)^+, \widetilde{p-q} \leftarrow q \\ p \begin{cases} \rightarrow q, \widetilde{p-q} \\ \rightarrow p, l+p-q, (-l)^+, -\widetilde{p+q} \end{cases} \xrightarrow{H} q, l+p-q, \\ (-l)^+ \longrightarrow q$$



$$(A) \quad \frac{1}{4\pi^2} \frac{e^2}{\hbar c} \sum_{\pm, \pm'} \int \frac{d\mathbf{k}}{k^3} \frac{\lambda^{\pm}(\mathbf{p})(\mathbf{a} \times \mathbf{k})\lambda^{+}(\mathbf{p}-\mathbf{k})H_{pq}\lambda^{+}(\mathbf{q}-\mathbf{k})(\mathbf{a} \times \mathbf{k})\lambda^{\pm'}(\mathbf{q})}{(\pm E_p - E_{p-k} - k)(\pm' E_q - E_{q-k} - k)} \\ - \frac{1}{2} \frac{1}{4\pi^2} \frac{e^2}{\hbar c} \sum_{\pm} \left\{ \int \frac{d\mathbf{k}}{k^3} \frac{\lambda^{\pm}(\mathbf{p})(\mathbf{a} \times \mathbf{k})\lambda^{+}(\mathbf{p}-\mathbf{k})(\mathbf{a} \times \mathbf{k})\lambda^{\pm}(\mathbf{p})H_{pq} + \text{conj.}}{(\pm E_p - E_{p-k} - k)(\pm E_p - E_{p-k} - k)} \right\} \quad (6)$$

$$(B) \quad \frac{1}{4\pi^2} \frac{e^2}{\hbar c} \sum_{\pm, \pm'} \int \frac{d\mathbf{k}}{k^3} \frac{\lambda^{\pm}(\mathbf{p})(\mathbf{a} \times \mathbf{k})\lambda^{-}(\mathbf{p}-\mathbf{k})H_{pq}\lambda^{-}(\mathbf{q}-\mathbf{k})(\mathbf{a} \times \mathbf{k})\lambda^{\pm'}(\mathbf{q})}{(\mp E_p - E_{p-k} - k)(\mp' E_q - E_{q-k} - k)} \\ - \frac{1}{2} \frac{1}{4\pi^2} \frac{e^2}{\hbar c} \sum_{\pm} \left\{ \int \frac{d\mathbf{k}}{k^3} \frac{\lambda^{\pm}(\mathbf{p})(\mathbf{a} \times \mathbf{k})\lambda^{-}(\mathbf{p}-\mathbf{k})(\mathbf{a} \times \mathbf{k})\lambda^{\pm}(\mathbf{p})H_{pq} + \text{conj.}}{(\mp E_p - E_{p-k} - k)(\mp E_p - E_{p-k} - k)} \right\} \quad (7)$$

$$(C_1) \quad - \frac{1}{4\pi^2} \frac{e^2}{\hbar c} \sum_{\pm, \pm'} \left\{ \int \frac{d\mathbf{k}}{k^3} \frac{\lambda^{\pm}(\mathbf{p})(\mathbf{a} \times \mathbf{k})\lambda^{-}(\mathbf{p}-\mathbf{k})H_{pq}\lambda^{+}(\mathbf{q}-\mathbf{k})(\mathbf{a} \times \mathbf{k})\lambda^{\pm'}(\mathbf{q})}{(\mp' E_q - E_{q-k} - k)(\pm' E_q - E_{q-k} \mp E_p - E_{p-k})} \right. \\ \left. + \text{conj.} \right\} \\ - \frac{1}{4\pi^2} \frac{e^2}{\hbar c} \sum_{\pm, \pm'} \left\{ \int \frac{d\mathbf{k}}{k^3} \frac{\lambda^{\pm}(\mathbf{p})(\mathbf{a} \times \mathbf{k})\lambda^{+}(\mathbf{p}-\mathbf{k})H_{pq}\lambda^{-}(\mathbf{q}-\mathbf{k})(\mathbf{a} \times \mathbf{k})\lambda^{\pm'}(\mathbf{q})}{(\mp' E_q - E_{q-k} - k)(\pm E_p - E_{p-k} \mp' E_q - E_{q-k})} \right. \\ \left. + \text{conj.} \right\} \quad (8)$$

$$(C_2) \quad - \frac{1}{2} \frac{1}{4\pi^2} \frac{e^2}{\hbar c} \sum_{\pm, \pm'} \left\{ \int \frac{d\mathbf{k}}{k^2} \frac{\lambda^{\pm}(\mathbf{p})\lambda^{-}(\mathbf{p}-\mathbf{k})H_{pq}\lambda^{+}(\mathbf{q}-\mathbf{k})\lambda^{\pm'}(\mathbf{q})}{\pm' E_q - E_{q-k} \mp E_p - E_{p-k}} + \text{conj.} \right\} \\ - \frac{1}{2} \frac{1}{4\pi^2} \frac{e^2}{\hbar c} \sum_{\pm, \pm'} \left\{ \int \frac{d\mathbf{k}}{k^2} \frac{\lambda^{\pm}(\mathbf{p})\lambda^{+}(\mathbf{p}-\mathbf{k})H_{pq}\lambda^{-}(\mathbf{q}-\mathbf{k})\lambda^{\pm'}(\mathbf{q})}{\pm E_p - E_{p-k} \mp' E_q - E_{q-k}} + \text{conj.} \right\}$$

$$(D) \quad -\frac{1}{4\pi^2} \cdot \frac{e^2}{\hbar c} \sum_{\pm} \left\{ \int \frac{d\mathbf{k}}{k^3} \frac{\lambda^{\pm}(\mathbf{p})(\boldsymbol{\alpha} \times \mathbf{k}) \lambda^{\pm}(\mathbf{p}-\mathbf{k})(\boldsymbol{\alpha} \times \mathbf{k}) \lambda^{\mp}(\mathbf{p})}{(E_p - E_{p-k} - k)(-E_p - E_{p-k} - k)} H_{pq} + \text{conj.} \right\} \quad (9)$$

$$(E) \quad -\frac{1}{4\pi^2} \cdot \frac{e^2}{\hbar c} \sum_{\pm} \left\{ \int \frac{d\mathbf{k}}{k^3} \frac{\lambda^{\pm}(\mathbf{p})(\boldsymbol{\alpha} \times \mathbf{k})(\lambda^{\pm}(\mathbf{p}-\mathbf{k}) - \lambda^{\mp}(\mathbf{p}-\mathbf{k}))(\boldsymbol{\alpha} \times \mathbf{k}) \lambda^{\mp}(\mathbf{p})}{(-2E_p)(-E_p - E_{p-k} - k)} H_{pq} + \text{conj.} \right\} \\ -\frac{1}{4\pi^2} \cdot \frac{e^2}{\hbar c} \sum_{\pm} \left\{ \int \frac{d\mathbf{k}}{k^3} \frac{\lambda^{\pm}(\mathbf{p})(\lambda^{\pm}(\mathbf{p}-\mathbf{k}) - \lambda^{\mp}(\mathbf{p}-\mathbf{k})) \lambda^{\mp}(\mathbf{p})}{-2E_p} H_{pq} + \text{conj.} \right\} \quad (10)$$

$$(F) \quad +\frac{1}{4\pi^2} \cdot \frac{e^2}{\hbar c} \sum_{\pm, \pm'} \left\{ \int \frac{d\mathbf{l}}{|\mathbf{p}-\mathbf{q}|^3} \frac{\lambda^{\pm}(\mathbf{p})(\boldsymbol{\alpha} \times \overline{\mathbf{p}-\mathbf{q}}) \lambda^{\pm'}(\mathbf{q})}{\pm E_p \mp' E_q - |\mathbf{p}-\mathbf{q}|} \right. \\ \cdot \frac{\text{Tr}[\lambda^{+}(\mathbf{l}-\mathbf{p}+\mathbf{q})(\boldsymbol{\alpha} \times \overline{\mathbf{p}-\mathbf{q}}) \lambda^{-}(\mathbf{l}) H_{pq}]}{-E_l - E_{l-p+q} - |\mathbf{p}-\mathbf{q}|} + \text{conj.} \Big\} \\ +\frac{1}{4\pi^2} \cdot \frac{e^2}{\hbar c} \sum_{\pm, \pm'} \left\{ \int \frac{d\mathbf{l}}{|\mathbf{p}-\mathbf{q}|^3} \frac{\lambda^{\pm}(\mathbf{p})(\boldsymbol{\alpha} \times \overline{\mathbf{p}-\mathbf{q}}) \lambda^{\pm'}(\mathbf{q})}{\pm E_p \mp' E_q - |\mathbf{p}-\mathbf{q}|} \right. \\ \cdot \frac{\text{Tr}[\lambda^{-}(\mathbf{l})(\boldsymbol{\alpha} \times \overline{\mathbf{p}-\mathbf{q}}) \lambda^{+}(\mathbf{l}+\mathbf{p}-\mathbf{q}) H_{pq}]}{\pm E_p - E_l \mp' E_q - E_{l+p-q}} + \text{conj.} \Big\} \\ +\frac{1}{4\pi^2} \cdot \frac{e^2}{\hbar c} \sum_{\pm, \pm'} \int \frac{d\mathbf{l}}{|\mathbf{p}-\mathbf{q}|^3} \frac{\lambda^{\pm}(\mathbf{p})(\boldsymbol{\alpha} \times \overline{\mathbf{p}-\mathbf{q}}) \lambda^{\pm'}(\mathbf{q})}{-E_l - E_{l+p-q} - |\mathbf{p}-\mathbf{q}|} \\ \cdot \frac{\text{Tr}[\lambda^{-}(\mathbf{l})(\boldsymbol{\alpha} \times \overline{\mathbf{p}-\mathbf{q}}) \lambda^{+}(\mathbf{l}+\mathbf{p}-\mathbf{q}) H_{pq}]}{\mp E_p - E_l \mp' E_q - E_{l+p-q}} + \text{conj.} \Big\} \\ +\frac{1}{4\pi^2} \cdot \frac{e^2}{\hbar c} \sum_{\pm, \pm'} \int \frac{d\mathbf{l}}{|\mathbf{p}-\mathbf{q}|^3} \frac{\lambda^{\pm}(\mathbf{p}) \lambda^{\pm'}(\mathbf{q}) \text{Tr}[\lambda^{-}(\mathbf{l}) \lambda^{+}(\mathbf{l}+\mathbf{p}-\mathbf{q}) H_{pq}]}{\pm E_p - E_l \mp' E_q \mp' E_{l+p-q}}. \quad (11)$$

§ 3. Subtraction of infinite terms.

The terms (A) and (B) in the last section correspond to (A) and (B) in Dancoff's paper; that is, (A) includes only even transitions by H , where intermediate states are all positive, and (B) is the same but with all negative intermediate states. (C) corresponds to odd transitions, or Dancoff's (b), (b'), (c), (c') of (C), where H plays a rôle either at the first or at the last stage of the transition chain, while (D) has, being also odd, H in the middle (Dancoff's (a) and (a')).

(E) is the infinite mass terms. In fact, we can show that these terms are of the same form as caused by a mass type perturbation $\delta m\beta$. To

subtract this part we may add a counter term $-\delta m\beta$, where δm is the diverging self energy of the electron:

$$\delta m = \lim_{K \rightarrow \infty} \frac{e^2}{\pi \hbar c} \left(\frac{3}{2} \ln 2K - \frac{1}{4} \right). \quad (12)$$

But we shall here neglect entirely this part (E), assuming that it is exactly equivalent to a perturbation due to the self energy of the electron.

The last part (F) is responsible for the infinite polarization of vacuum caused by the external field. This should also be subtracted as a whole by a similar reasoning that it simply amounts to a renormalization of the coupling constant or reinterpretation of the external field.⁽¹⁰⁾

Thus we are left with the terms (A), (B), (C) and (D), which we now proceed to evaluate.

§ 4. Calculation for a non-relativistic electron.

The integrations involved in the terms (A) to (D) are in general so complicated that it seems very difficult to obtain a rigorous expression of the correction which is valid throughout the whole range of the electron's energy. We have therefore restricted ourselves to non-relativistic cases, that is, small initial and final momenta compared with the rest mass of the electron, although intermediate states may have any high energy and hence must be treated relativistically.

We expand all quantities appearing in these integrals in powers of \mathbf{p} and \mathbf{q} and retain only terms up to quadratic. Thus for instance,

$$E_{p-k} = \sqrt{m^2 + \mathbf{p}^2 + \mathbf{k}^2 - 2(\mathbf{p}\mathbf{k})} = E_k \left\{ 1 - (\mathbf{p}\mathbf{k})/E_k^2 + \mathbf{p}^2/2E_k^2 - (\mathbf{p}\mathbf{k})^2/2E_k^4 \right\},$$

$$\frac{1}{E_p - E_{p-k} - k} \equiv \frac{1}{F_p} = \frac{1}{F_0} \left\{ 1 - \frac{(\mathbf{p}\mathbf{k})}{EF_0} + \frac{(\mathbf{p}\mathbf{k})^2}{E^2 F_0^2} + \frac{P^2 E^2 - (\mathbf{p}\mathbf{k})^2}{2E^3 F_0} - \frac{\mathbf{p}^2}{2mF_0} \right\} \quad (13)$$

Then we can carry out the integration in the photon momentum \mathbf{k} in terms of elementary functions. Each individual term may be eventually divergent as $\int^\infty d\mathbf{k}/k$, but after combining all such contributions we see that these divergencies just cancel. In the case of the Coulomb type external potential V this occurs already for the individual parts (A) and (B), no other terms being divergent, while for the magnetic type potential $\mathbf{a}\mathbf{A}$ convergence is obtained only after the combination of (A), (B) and (C).

We add all the matrix elements obtained for transitions $+\rightarrow+$, $+\rightarrow$

$-$, $- \rightarrow +$ and $- \rightarrow -$ together, and collect similar terms from (A) to (D). We give here the results for the Coulomb type and the magnetic type separately since they have different forms.

(i) Coulomb type, V . The correction consists of three terms :

$$\frac{e^2}{\pi \hbar c} V_{p,q} \times \left\{ \begin{aligned} & \left(x - \frac{1}{2} \right) \beta(a, p-q), & (14) \\ & \left(\frac{1}{3} \ln 2\epsilon - \frac{7}{24} \right) (p-q)^2, & (14') \\ & \left(-\frac{1}{12} - \frac{1}{3} \ln 2 \right) i\sigma \cdot (p \times q), & (14'') \end{aligned} \right.$$

where ϵ is the cut-off momentum of the photon on the low energy side. This term arises from (A) and (B), and corresponds to the well-known infra-red catastrophe. This will be compensated if we add a correction due to (2). The term denoted by x is also divergent on the low energy side, but has another origin. It is that term which arises from the transition $+ \rightarrow -$ or $- \rightarrow +$ in (C_2) , and is related with the fact that the Born approximation becomes invalid for low energies in the case of Coulomb interaction.

(ii) Magnetic type, aA . There arise more terms than in the former case, namely a_n , $\beta(p+q, n)$, $i\beta\sigma \cdot (\overline{p-q} \times n)$, $(pq)a_n$, $(p^2+q^2)a_n$, $(pn)(qa) + (qn)(pa)$, $(pn)(pa) + (qn)(qa)$ and $i\rho_1(p \times q) \cdot n$, where n is the unit vector in the direction of A . We give here the coefficients only for terms up to the first order in p or q .

$$\frac{e^2}{\pi \hbar c} A_{p,q} \times \left\{ \begin{aligned} & \left(\frac{1}{6} - \frac{2}{3} \ln 2K + x \right) a_n, & (15) \\ & \left(\frac{4}{9} + \frac{1}{3} \ln 2K - \frac{x}{2} \right) \beta(p+q, n), & (15') \\ & \left(\frac{25}{36} + \frac{1}{3} \ln 2K - \frac{x}{2} \right) i\beta\sigma \cdot (\overline{p-q} \times n). & (15'') \end{aligned} \right.$$

x is of the same nature as before, while the logarithmic terms are divergent on the ultra-violet side. This divergence, however, need not be taken up seriously. For these various kinds of terms are not rigorously relativistic in spite of their apparent Dirac matrix expression, and if we reduce them to Pauli approximation, the divergencies then cancel each other.

§ 5. Derivation of the correction formulae.

Now we transform the above results to a representation in the ordinary space :

$$\begin{aligned}\beta(\alpha, \mathbf{p}-\mathbf{q}) V_{\mathbf{p}, \mathbf{q}} &\rightarrow -\rho_2(\boldsymbol{\sigma} \mathbf{E}), \\ i\boldsymbol{\sigma} \cdot (\mathbf{p} \times \mathbf{q}) V_{\mathbf{p}, \mathbf{q}} &\rightarrow -\mathbf{E} \cdot (\boldsymbol{\sigma} \times \mathbf{p}), \\ (\mathbf{p}-\mathbf{q})^2 V_{\mathbf{p}, \mathbf{q}} &\rightarrow -\Delta V, \\ i\beta\boldsymbol{\sigma} \cdot (\overline{\mathbf{p}-\mathbf{q}} \times \mathbf{n}) A &\rightarrow \beta(\boldsymbol{\sigma} \mathbf{H}).\end{aligned}\tag{16}$$

Looking at this, we notice that the electron possesses an additional interaction with the magnetic field which may be interpreted as a correction to the magnetic moment of one Bohr magneton. The above results, however, have not a relativistically invariant form. Indeed there are some terms in the correction to αA which have no partners for V . The coefficients of $\rho_2(\boldsymbol{\sigma} \mathbf{E})$ and $\rho_3(\boldsymbol{\sigma} \mathbf{H})$ are also different. This may be attributed to our non-relativistic treatment in the process of calculation as well as in the entire formalism.

The above mentioned unpaired terms (15) and (15') have non-vanishing values for $\mathbf{p}=\mathbf{q}$, and the first of them has the same form as the original interaction αA . Therefore we can omit this term by renormalizing the external field or its coupling constant. The same will be done for the other term if we divide it into αA and $\beta(\boldsymbol{\sigma} \mathbf{H})$ in the order of the Pauli approximation and retain only the second term :

$$\beta(\mathbf{p}+\mathbf{q}, \mathbf{n})=2a_n-i\beta\boldsymbol{\sigma} \cdot (\overline{\mathbf{p}-\mathbf{q}} \times \mathbf{n}).\tag{17}$$

Then the additional magnetic moment becomes

$$\frac{1}{4} \frac{e^2}{\pi \hbar c} \frac{\hbar c}{mc} = \frac{1}{2\pi} \frac{c^2}{\hbar c} \mu.\tag{18}$$

This is in agreement with Schwinger's result.⁽⁹⁾

Before we apply the above results to the level shift of the hydrogen atom we have to take into account the contribution from (2), or the emission of real photons. The calculation is similar but more complicated because it is a fourth order perturbation. But as we know that this correction is large only in the neighbourhood of the infra-red limit, we use Bethe's

non-relativistic formula⁽¹⁾ for the contribution of low energy photons. Then the lower limit of integration in (14') and the upper limit in Bethe's formula cancel each other. Further, if we make Pauli approximation,

$$\lambda^+(\mathbf{p})\beta(\boldsymbol{\alpha}, \mathbf{p}-\mathbf{q})\lambda^+(\mathbf{q}) = -\frac{1}{2}(\mathbf{p}-\mathbf{q})^2 + i\boldsymbol{\sigma}(\mathbf{p} \times \mathbf{q}). \quad (19)$$

The first term is added to the main term (14') which does not depend on the spin variable. The second term is combined with (14''), and represents a spin-orbit interaction. In the case of the hydrogen-like atom,

$$\mathbf{E} = -e^2 Z \mathbf{r} / r^3, \quad (20)$$

$$\begin{aligned} \mathbf{E} \cdot (\boldsymbol{\alpha} \times \mathbf{p}) &= \boldsymbol{\sigma} \cdot (\mathbf{p} \times \mathbf{E}) = -e^2 Z \boldsymbol{\sigma} \cdot (\mathbf{p} \times \mathbf{r}) / r^3 = -e^2 Z (\boldsymbol{\sigma} \mathbf{l}) / r^3 \\ &= -e^2 Z \{j(j+1) - l(l+1) - 3/4\} / 2r^3, \end{aligned} \quad (21)$$

where l is the orbital, and j the total angular momentum. Thus we get for the level shift the following formula:

$$\begin{aligned} \delta E_{nj} &= \frac{8}{3\pi} \left(\frac{e^2}{\hbar c} \right)^3 Z^4 R_Y \left\{ \left(\ln \frac{mc^2}{\langle E_0 - E \rangle_{AV}} - \ln 2 + \frac{1}{8} \right) \frac{1}{n^3} \delta_{lc} \right. \\ &\quad \left. + \left(\frac{1}{2} \ln 2 + \frac{7}{8} \right) \frac{j(j+1) - l(l+1) - 3/4}{n^3 l(l+1)(2l+1)} (1 - \delta_{lc}) \right\}. \end{aligned} \quad (22)$$

Especially the level difference of $2S_{1/2}$ and $2P_{1/2}$ becomes for $Z=1$

$$\begin{aligned} 3E_{2S_{1/2}} - \delta E_{2P_{1/2}} &= \frac{1}{3\pi} \left(\frac{e^2}{\hbar c} \right)^3 Z^4 R_Y \left(\ln \frac{mc^2}{\langle E_0 - E \rangle_{AV}} \right. \\ &\quad \left. - \frac{5}{6} \ln 2 + \frac{5}{12} \right) = 1019 \text{ mc.} \end{aligned} \quad (23)$$

§ 6. Discussion.

Our results do not completely agree with the calculation of Fukuda, Miyamoto and Tomonaga, who used the canonical transformation method. One might doubt that the origin of this discrepancy lies in the ambiguity of the subtraction prescription and the non-relativistic character of the calculation. In fact, if we actually calculate the part (E), we see that it is not completely compensated by the self energy term (E'):

$$(E) = -\frac{3}{4} \ln 2K + \frac{7}{8}, \quad (E') = \frac{3}{4} \ln 2K - \frac{1}{8} \quad (24)$$

Moreover, tracing back the calculation, we find a difference already existing in the starting expressions (A) to (D). The corresponding expression derived by the canonical transformation has the part (D) smaller by a factor 1/2, and contains unfamiliar terms which have no equivalent in our case :

$$-\frac{1}{2} \int \frac{1}{2k^3} \frac{\lambda^-(p)(\alpha \times k)\lambda^+(p-k)(\alpha \times k)\lambda^+(p)}{(E_p - E_{p-k} - k)(-E_p - E_{p-k} - k)} dk \cdot H_{p,q} + \text{conj.}$$

$$-\frac{1}{2} \int \frac{1}{2k^3} \frac{\lambda^-(p)(\alpha \times k)\lambda^-(p-k)(\alpha \times k)\lambda^-(p)}{(E_p - E_{p-q} - k)(-E_p - E_{p-k} - k)} dk \cdot H_{p,q} + \text{conj.} \quad (25)$$

This situation may be explained as follows. Hitherto we have been doing with the static perturbation. This is sufficient for the calculation of the diagonal part of an observable, e.g. the self energy in the ordinary sense. But in such cases as ours in which the transition due to the self energy and its elimination is concerned, the time dependent perturbation is required for a more careful definition of the self energy. The perturbed wave function is then given by

$$a_n^{(2)}(t) = a_n^{(0)} + \frac{H_{nn_0}}{E_{n_0} - E_n} a_{n_0}^{(0)} (\exp i(E_{n_0} - E_n)t/\hbar - 1)$$

$$+ \sum_{n'} \frac{H_{nn'} H_{n'n_0}}{E_{n_0} - E_{n'}} a_{n_0}^{(0)} \left[\frac{\exp i(E_{n_0} - E_n)t/\hbar - 1}{E_{n_0} - E_n} - \frac{\exp i(E_{n'} - E_n)t/\hbar - 1}{E_{n'} - E_n} \right]. \quad (26)$$

We divide it in two parts :

$$a_n^{(2)}(t) = \left[a_n^{(0)} + \frac{H_{nn_0} a_{n_0}^{(0)}}{E_{n_0} - E_n} (\exp i(E_{n_0} - E_n)t/\hbar - 1) \right. \\ \left. + \frac{1}{2} \sum_{n'} \frac{H_{nn'} H_{n'n_0} a_{n_0}^{(0)}}{(E_{n_0} - E_{n'}) (E_{n'} - E_n)} (\exp i(E_{n'} - E_n)t/\hbar - 1) \right. \\ \left. \cdot (\exp i(E_{n_0} - E_{n'})/\hbar - 1) \right] \\ + \frac{1}{2} \sum_{n'} \frac{H_{nn'} H_{n'n_0} a_{n_0}^{(0)}}{(E_{n_0} - E_{n'}) (E_{n'} - E_n)} \left[\frac{2E_{n'} - E_{n_0} - E_n}{E_{n_0} - E_n} (\exp i(E_{n_0} - E_n)t/\hbar - 1) \right. \\ \left. - (\exp i(E_{n'} - E_n)t/\hbar - \exp i(E_{n_0} - E_{n'})t/\hbar) \right]. \quad (27)$$

The first is the renormalized wave function while the second is *defined* to be the perturbation due to the self energy term. This procedure, however, is different from that of the static perturbation in which the renormalized wave function would be

$$a_n^{(2)} = \left[a_n^{(0)} + \frac{H_{nn_0}}{E_{n_0} - E_n} a_n^{(0)} + \sum_{n' \neq n_0} \frac{H_{nn'} H_{n'n_0}}{(E_{n_0} - E_{n'}) (E_{n_0} - E_n)} \right] \\ \times \left[\sum_{n'} \left(a_{n'}^{(0)} + \frac{H_{n'n_0}}{E_{n_0} - E_{n'}} a_{n_0}^{(0)}, a_{n'}^{(0)} + \frac{H_{n'n_0}}{E_{n_0} - E_{n'}} a_{n_0}^{(0)} \right) \right]^{-1/2} \quad (28)$$

This means a non-linear hence non-unitary, correspondence between the perturbed and unperturbed wave function.

If we make necessary modifications according to the above circumstance, the level shift formulae become

$$\delta E_{n,j} = \frac{8}{3\pi} \left(\frac{c^2}{\hbar c} \right)^2 Z^4 R_Y \left\{ \left(\ln \frac{mc^2}{\langle E_0 - E \rangle_{AV}} - \ln 2 + \frac{7}{8} \right) \frac{1}{n^3} \delta_{l_0} \right. \\ \left. + \left(\frac{1}{2} \ln 2 + \frac{1}{8} \right) \frac{j(j+1) - l(l+1) - 3/4}{n^3 l(l+1)(2l+1)} (1 - \delta_{l_0}) \right\}, \quad (29)$$

$$\delta E_{2S1/2} - \delta E_{2P1/2} = 1087 \text{ mc.}^* \quad (30)$$

The author expresses cordial thanks to Professor Tomonaga who suggested this work and to Messrs Fukuda, Miyamoto and Tani for kind helps and valuable discussions.

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- (4) On this point, see a note by J. R. Oppenheimer supplemented to Tomonaga's letter, and Tati and Tomonaga's remark at the end of their paper, (3).
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* Added in Proof

Complete neglect of the vacuum polarization due to the external field is open to question. A deviation from mere charge-renormalizing correction would modify the level shift, Eq. (30), to 1060mc.

Letters to the Editor

Reaction of Meson Field on Nucleon.

K. Baba and D. Itô.

*Physics Institute, Tokyo
Bunrika Daigaku.*

Jan. 4, 1949

In this letter we describe briefly the results obtained by applying Tomonaga's so called "Self-Consistent" subtraction⁽¹⁾ method for the case when scalar meson field and nucleon field are interacting.

The mass-correction effect for the nucleon and the vacuum polarization of the scalar meson field due to their interaction with each other are calculated using the "super-many-time" formalism⁽²⁾ and the method of canonical transformation introduced by Tomonaga⁽¹⁾ and Schwinger. The order of approximation is the square of the interaction constant, and nucleon is of course treated in the manner of "hole theory".

The terms containing the scalar mesonic self-energy of the nucleon was unified in the type of neutron and proton mass as had been expected. The mass-correction term of the nucleon in Hamilton density which must be subtracted according to the idea of "Self-Consistent" subtraction is as follows:

$$\delta M_n \psi_n^\dagger(X) \psi_n(X) + \delta M_p \psi_p^\dagger(X) \psi_p(X),$$

where

$\psi_n(X)$, $\psi_p(X)$: wave function describing the neutron state and the proton state of nucleon.

$$\begin{aligned} \delta M_n = & \frac{-1}{8\pi^2} \int_0^\infty \frac{dp}{\sqrt{P} \sqrt{K}} \left[f^2 \left\{ \frac{\sqrt{P} + x_p}{\sqrt{P} + \sqrt{K} - x_n} \right. \right. \\ & \left. \left. - \frac{\sqrt{P} - x_p}{\sqrt{P} + \sqrt{K} + x_n} \right\} \right. \\ & + \left(\frac{g}{x} \right)^2 \left\{ \left(\frac{\sqrt{P} + x_p}{\sqrt{P} + \sqrt{K} - x_n} \right. \right. \\ & \left. \left. - \frac{\sqrt{P} - x_p}{\sqrt{P} + \sqrt{K} + x_n} \right) (p^2 + x^2) \right. \\ & + \left(\frac{1}{\sqrt{P} + \sqrt{K} - x_n} - \frac{1}{\sqrt{P} + \sqrt{K} + x_n} \right) \\ & \left. p^2 \sqrt{K} + \left(\frac{1}{\sqrt{P} + \sqrt{K} - x_n} \right. \right. \\ & \left. \left. + \frac{1}{\sqrt{P} + \sqrt{K} + x_n} \right) (x_n - x_p) p^2 \right\}] \\ \sqrt{P} \equiv & \sqrt{p^2 + x_p^2}, \quad \sqrt{K} \equiv \sqrt{p^2 + x^2}, \\ x_p = & \frac{M_p c}{\hbar}, \quad x_n = \frac{M_n c}{\hbar}, \quad x = \frac{M_1 c}{\hbar} \end{aligned}$$

M_p : proton mass, M_n : neutron mass,

M_1 : meson mass,

f, g : scalar and vector interaction constant.

The expression for the diverging part of the integral is as follows:

$$\begin{aligned} \delta M_n \approx & \frac{-1}{4\pi} \left[f^2 (2x_p + x_n) \int \frac{dp}{p} \right. \\ & \left. + \left(\frac{g}{x} \right)^2 \cdot 4x_n \int p dp \right] \end{aligned}$$

The expression for δM_p is obtained simply by substituting $n \rightarrow p$ and $p \rightarrow n$ in the expression for the δM_n .

The terms corresponding to the vacuum polarization of scalar meson field owing to the vacuum nucleons was found:

$$\begin{aligned} \delta H_{pot} = & \frac{1}{8\pi^2 \hbar c} \left\{ f^2 U \phi^* \phi \right. \\ & \left. - \left(\frac{g}{x} \right)^2 V (\text{Grad } \phi^* \text{ Grad } \phi) \right\} \end{aligned}$$

where

ϕ^*, ϕ : wave function of meson,

Grad: four dimensional gradient.

$$U \equiv \int_0^\infty \frac{x_p x_n - (l^2 + \sqrt{l^2 + x_p^2} \sqrt{l^2 + x_n^2})}{\sqrt{l^2 + x_p^2} \sqrt{l^2 + x_n^2}} A(l) dl$$

$$V \equiv \int_0^\infty \frac{x_p x_n + (l^2 - \sqrt{l^2 + x_p^2} \sqrt{l^2 + x_n^2})}{\sqrt{l^2 + x_p^2} \sqrt{l^2 + x_n^2}} A(l) dl$$

$$A(l) \equiv \frac{1}{\sqrt{l^2 + \kappa_p^2} + \sqrt{l^2 + \kappa_n^2} - x} + \frac{1}{\sqrt{l^2 + \kappa_p^2} + \sqrt{l^2 + \kappa_n^2} + x}$$

The expression for the diverging part of the integral is:

$$\delta H_{Pol} \cong \frac{1}{8\pi^3 \hbar c} \left\{ \int^2 U' \phi^* \phi - \left(\frac{q}{x} \right)^2 V'' (\text{Grad } \phi^* \text{ Grad } \phi) \right\}$$

where

$$U' \equiv -8\pi \int_0^\infty l dl - 2\pi (x_n - x_p)^2 \int_0^\infty \frac{dl}{l}$$

$$V' \equiv -2\pi (x_n - x_p)^2 \int_0^\infty \frac{dl}{l}$$

It is remarkable that in the case of the meson field the vacuum polarization term can be calculated without any difficulty, as has appeared in the case of quantum electro-dynamics. The reason is that the meson has a non-vanishing mass and so there exists always a Lorentz system in which meson is at rest and we may carry out the integration over the nucleonic moment assuming the spherical symmetry in this system.

Detail of the calculation together with the discussions about the nuclear forces will be published in this journal.

In conclusion we wish to express our cordial thanks to Professor S. Tomonaga for his many useful guidance throughout

this work.

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Second Configuration Space and Third Quantization.

Y. Nambu.

Department of Physics,
Tokyo University.

Feb. 7, 1949

Various divergence difficulties in quantized field theory are closely connected with the zero point fluctuation of the field. Formally, it is due to the fact that a quantity composed of q numbers has sometimes a non-vanishing expectation value even in the lowest state considered. To dispose of this zero point fluctuations, we may decompose field quantities into creation and annihilation operators, and rearrange products of these operators in such a way that creation operators (with asterisks) stand always to the left of annihilation operators (without asterisks), thereby separating pure fluctuation terms from the rest. This idea, or essentially the same as this, was applied with brilliant success to quantum electrodynamics by Schwinger⁽¹⁾ and Tomonaga.⁽²⁾ If we make the convention that every quantity composed of quantized fields should be well ordered in the manner above mentioned, multiplication of two such quantities will require in general a rearrangement.

Now let us regard a product of quantized fermion fields $\phi_1^* \phi_2^* \dots \phi_n^* \phi_1' \phi_2' \dots \phi_n'$, where the suffixes denote space-time

as well as spin coordinates, as a *state* or a *wave function* specified by these coordinates, and write for short

$$\phi_1^* \phi_2^* \dots \phi_n^* \phi_1' \phi_2' \dots \phi_n' \equiv (1, 2, \dots n; 1', 2', \dots n'). \quad (1)$$

Multiplication by ϕ_r^* or ϕ_r either from the left or from the right will yield, after rearrangement, the following result:

$$\begin{aligned} \vec{\phi}_r^* (1, 2, \dots n; 1', 2', \dots n') \\ = (r, 1, 2, \dots n; 1', 2', \dots n'), \\ \vec{\phi}_r (1, 2, \dots n; 1', 2', \dots n') \\ = \sum_{i=1}^n (-1)^{i-1} \{r, i\} (1, 2, \dots i-1, i+1, \dots n; 1', 2', \dots n') \\ + (-1)^n (1, 2, \dots n; r, 1', 2', \dots n'), \{r, i\} \equiv [\phi_r, \phi_i^*]_+, \\ \overleftarrow{\phi}_r (1, 2, \dots n; 1', 2', \dots n') \\ = (1, 2, \dots n; 1', 2', \dots n', r), \quad (2) \\ \overleftarrow{\phi}_r^* (1, 2, \dots n; 1', 2', \dots n') \\ = \sum_{i'=n'}^1 (-1)^{n'-i'} \{i, r\} (1, 2, \dots n; 1', 2', \dots i'-1, i'+1, \dots n') \\ + (-1)^{n'} (1, 2, \dots n, r; 1', 2', \dots n'). \end{aligned}$$

If we introduce such operators that create (+) or annihilate (-) the fields ϕ^* and ϕ , and denote them by $\vec{\phi}^*$, $\vec{\phi}$, $\overleftarrow{\phi}^*$ and $\overleftarrow{\phi}$, Eqs. (2) may be written as

$$\begin{aligned} \vec{\phi}_r^* = \vec{\phi}_r^*, \quad \vec{\phi}_r = \vec{\phi}_r + \sum_i \{r, i\} \vec{\phi}_i^* \\ \overleftarrow{\phi}_r = \overleftarrow{\phi}_r, \quad \overleftarrow{\phi}_r^* = \overleftarrow{\phi}_r^* + \sum_i \overleftarrow{\phi}_i \{i, r\} \quad (3) \end{aligned}$$

We have included in the above defined operators the sign functions, which necessitates the distinction between ϕ and $\tilde{\phi}$. They are connected by simple relations:

$$\begin{aligned} \pm \pm' \phi_r \phi_s &= -(\pm)(\pm') \tilde{\phi}_r \tilde{\phi}_s, \\ \pm \pm' \phi_r^* \phi_s^* &= -(\pm)(\pm') \tilde{\phi}_r^* \tilde{\phi}_s^*, \\ \pm \pm' \phi_r^* \phi_s &= -(\pm)(\pm') \tilde{\phi}_r^* \tilde{\phi}_s, \quad (4) \end{aligned}$$

Further defining new operators by

$$\begin{aligned} \vec{\Psi}_r^* &\equiv \vec{\phi}_r^*, \quad \vec{\Psi}_r \equiv \sum_i \{r, i\} \vec{\phi}_i^*, \\ \overleftarrow{\Psi}_r^* &\equiv \overleftarrow{\phi}_r^*, \quad \overleftarrow{\Psi}_r \equiv \sum_i \overleftarrow{\phi}_i \{i, r\}, \quad (5) \end{aligned}$$

which obey the same commutation relations as the original wave field:

$$\{\vec{\Psi}_r^*, \vec{\Psi}_s^*\} = \{\vec{\Psi}_r, \vec{\Psi}_s\} = \{r, s\}, \quad (6)$$

Eqs. (2) become

$$\begin{aligned} \vec{\phi}_r &= \vec{\Psi}_r^*, \quad \vec{\phi}_r = \vec{\Psi}_r + \vec{\Psi}_r^*, \\ \overleftarrow{\phi}_r &= \overleftarrow{\Psi}_r, \quad \overleftarrow{\phi}_r^* = \overleftarrow{\Psi}_r^* + \overleftarrow{\Psi}_r. \quad (7) \end{aligned}$$

Similar operators can be defined for a boson field, φ^* and φ , provided that we regard $\frac{1}{\sqrt{n_1! n_2! \dots}} \varphi_1^* \varphi_2^* \dots \varphi_n^* \varphi_1' \varphi_2' \dots \varphi_n'$ as a state, where n_1, n_2 , etc. are the number of times it contains φ_1^*, φ_2^* , etc. respectively. Corresponding to Eqs. (2) to (7), we get

$$\begin{aligned} \vec{\varphi}_r^* &= \vec{\varphi}_r^* = \vec{\Phi}_r^*, \\ \vec{\varphi}_r &= \vec{\varphi}_r + \sum_i \{r, i\} \vec{\varphi}_i^* = \vec{\Phi}_r + \vec{\Phi}_r^*, \\ \overleftarrow{\varphi}_r &= \overleftarrow{\varphi}_r = \overleftarrow{\Phi}_r, \\ \overleftarrow{\varphi}_r^* &= \overleftarrow{\varphi}_r^* + \sum_i \overleftarrow{\varphi}_i \{i, r\} = \overleftarrow{\Phi}_r^* + \overleftarrow{\Phi}_r, \quad (8) \\ [\vec{\Phi}_r^*, \vec{\Phi}_s^*] &= [\vec{\Phi}_r, \vec{\Phi}_s] = [r, s] = [\varphi_r, \varphi_s^*]_-. \end{aligned}$$

Now an equation of motion for the transformation matrix $U[C, C_0]$ defined by $\Psi[C] = U[C, C_0] \Psi[C_0]$ is of the form

$$i \frac{\delta}{\delta C} U = H U, \quad -i \frac{\delta}{\delta C} U^* = U^* H \quad (9)$$

in Tomonaga's super-many time formalism. This may be regarded as an equation of motion for the *wave function* U if the operator H is expressed in terms of the above mentioned operators:

$$i \frac{\delta}{\delta C} U = \vec{H} U, \quad -i \frac{\delta}{\delta C} U^* = \vec{H} U^*. \quad (10)$$

In the same manner, an equation of Hei-

senberg type

$$-i\frac{\delta}{\delta C}A=[H, A] \quad (11)$$

goes over into an equation of Schrödinger type

$$-i\frac{\delta}{\delta C}A=(\vec{H}-\hat{H})A. \quad (12)$$

Moreover, it is possible to make \vec{H} or \hat{H} Hermitian by a simple transformation, and the form of the resulting Hamiltonian very much resembles the original one *II*. On these grounds we should like to call the present formalism the method of the *third quantization*. A quantity composed of the operators of the second quantization (for short, q^2 -numbers) may be interpreted as a wave function in the *second configuration space* on which, when transformed to the third quantization representation, the q^3 -numbers can operate. Application of this method to the theory of Schwinger and Tomonaga enables one to carry out the canonical transformations involved in a quick way and gives a clear perspective of the whole theory. It also enables one to solve the generalized wave equation proposed by Snyder⁽³⁾ and Yukawa.⁽⁴⁾

A paper dealing with mathematical refinement and various applications is now being worked out.

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Semi-Classical Derivation of the Anomalous Magnetic Moment of the Electron.

Z. Koba.

Institute of Physics, Faculty of Science, Tokyo University.

Feb. 28, 1949.

Resently Welton⁽¹⁾ has given an intuitive explanation to the electromagnetic shift of energy levels by calculating the mean square amplitude of an electron coupled to the zero point fluctuations of the electromagnetic field. His method, however, failed in the treatment of the anomalous magnetic moment of the electron, his result of the relative correction being $-e^2/2\pi$ (we use the unit $\hbar=c=1$ throughout), i.e. of the right order of magnitude but with the opposite sign. So we propose here a new classical model, which being a natural extension of Welton's idea takes into account the effect of virtual pair creation and annihilation and gives a correct result also for the radiative correction of the magnetic moment.

According to the old work of Schrödinger⁽²⁾ the behavior of Dirac electron is best characterised by its Zitterbewegung; in other words the velocity and coordinate operators can be divided into two parts, one which commutes with the Hamiltonian H (we denote this part by $-$), and the other which anticommutes with H (we denote this part by \sim)

$$\alpha_j = \bar{\alpha}_j + \tilde{\alpha}_j, \quad x_j = \bar{x}_j + \tilde{x}_j;$$

where

$$\bar{\alpha}_j = H^{-1}p_j, \quad \bar{x}_j = \bar{x}_{0,j} + H_0^{-1}p_j t;$$

and

$$\begin{aligned}\tilde{a}_j &= \tilde{a}_{0,j} \exp(-2iHt) = \exp(2iHt) \tilde{a}_{0,j}, \\ \tilde{x}_j &= -\frac{1}{2i} \tilde{a}_{0,j} H^{-1} \exp(-2iHt) \\ &= -\frac{1}{2i} \tilde{a}_j H^{-1} = \frac{1}{2i} H^{-1} \tilde{a}_j.\end{aligned}$$

The magnitude of these anticommutable parts are given by

$$\begin{aligned}\tilde{a}_j^2 &= 1 - H^{-2} p_j^2, \\ \tilde{x}_j^2 &= \frac{1}{4} H^{-2} (1 - H^{-2} p_j^2)\end{aligned}$$

It may be noticed that for a rest electron $|\tilde{x}_j| \approx 1/2m$, i.e. the magnetic moment of the electron. (This fact can be proved easily.)

Now Welton assumes that x is coupled to vacuum fluctuation of the electric field $\sum E_k \exp(ikt)$:

$$\frac{d}{dt} \tilde{x} = \frac{e}{m} \sum \frac{E_k \exp(ikt)}{ik} \quad (1)$$

Here we have used a first order differential equation because it corresponds to the quantum theory better. To this we add a further assumption that a similar relation holds for \tilde{x} too.

$$\frac{d}{dt} \tilde{x} + \tilde{x}(2im) = \frac{e}{m} \sum \frac{E_k \exp(ikt)}{ik} \quad (2)$$

The second term on the left-hand side assures that when the external force is absent \tilde{x} performs an oscillation with frequency $2m$.

The solution of (1) and (2) with the boundary condition $\tilde{x}=0$, $\dot{\tilde{x}}=1/2m \exp(2imt)$ for $e \rightarrow 0$ can easily be obtained and by means of a reasoning quite analogous to that of Welton, the corrected value of $|\tilde{x}|$ turns out

$$\begin{aligned}|\tilde{x}| &= \frac{1}{2m} \left\{ 1 + 4e^2 \sum_{k^2} \frac{E_k^2}{(2m+k)} \right\} \\ &= \frac{1}{2m} \left\{ 1 + \frac{8e^2}{3\pi} \int_0^m \frac{dk}{2m+k} \right\}\end{aligned}$$

$$= \frac{1}{2m} \left(1 + \frac{e^2}{\pi} \right)$$

Thus this effect, when added to the value obtained by Welton, yields the result: $e^2/2\pi$, that agrees with the experiment.

Details will appear in a later issue. Further applications are now being developed.

- (1) T. A. Welton, Phys. Rev. **74** (1948), 1157.
(2) E. Schrödinger, Sitzber. d. preuss. Akad. Berlin (1936) 418.

Effect of the C-Meson Field on the Anomalous Magnetic Moment of the Electron.

Z. Koba, Y. Nambu and S. Tati.

Institute of Physics, Faculty of Science, Tokyo University.

Mar. 15, 1949.

The cohesive force field introduced by Nagoya group and Pais⁽¹⁾ has proved useful in eliminating the so-called mass-type divergence in the quantum electromagnetics, but up to the present one has found no definite case where this hypothesis leads to appreciable deviation from ordinary quantum theory of interaction between electron and radiation and thus enables one really to detect such a heavy quantum.

This time we have investigated the modifications of the electromagnetic properties of the electron due to the reaction of this C-meson field and have again verified that these corrections are quite negligible compared with the radiative corrections. It may be noted that we have subtracted the finite self-energy which can be obtained by this hypothesis, because it brings about a mass correction and this renormalized mass is to be iden-

tified with the experimental one, so that no observable effect can be expected so far as we have no neutral electron for comparison.

The anomalous magnetic moment of the electron caused by the C -meson field has been calculated in two different ways: Schwinger's⁽²⁾ formalism and Luttinger's⁽³⁾ method. The result is, in the unit $\hbar = c = 1$,

$$\frac{\Delta\mu}{\mu} = \frac{f^2}{\pi} \delta^2 \left\{ \log \frac{1}{\delta} - \frac{7}{12} \right\} + O(\delta^3),$$

where δ is the ratio of the electron mass to that of the C -meson. Since we are to put $f^2 = 2e^2$, we have for $\delta = 1/100$

$$\Delta\mu/\mu \approx (e^2/2\pi) \times 1.6 \times 10^{-3},$$

and for $\delta = 1/200$

$$\Delta\mu/\mu \approx (e^2/2\pi) \times 4.7 \times 10^{-4}.$$

In the course of calculation it has become clear that the positive sign of the correction is due to the scalar type coupling and the small absolute magnitude is due to the large mass of the C -meson.

On the basis of this and similar consideration recently carried out by our group⁽⁴⁾ we should like to conclude in general: C -meson has hardly any detectable effect in the region where quantum electrodynamics has proved valid, because its range is far smaller than the Compton wave length of the electron; to seek for its experimental evidence one has to turn to problems of the heavy particles.

Detailed accounts on related problems will appear later.

(1) O. Hara, *Prog. Theor. Phys.* **3** (1948), 188.
A. Pais, *Phys. Rev.* **68** (1945), 227.

(2) J. Schwinger, unpublished. We are very much obliged to Prof. Schwinger for having sent his manuscript to Prof. Tomonaga.

(3) J. M. Luttinger, *Helv. Phys. Acta* **21** (1948) 483. We wish to express our sincere thanks

to Dr. Luttinger for having sent us a copy of his paper.

(4) Cf. e.g. S. Endô, T. Kinoshita and Z. Koba, letter to the editor of this journal to be published in this issue.

Errata: "Reactive Corrections for the Elastic Scattering of an Electron."

S. Endô, T. Kinoshita and Z. Koba.

Institute of Physics, Faculty of Science, Tokyo University.

Mar. 22, 1949.

In the previous letter,⁽¹⁾ we gave a faulty conclusion with regard to the correction for elastic scattering cross-section in the case when interaction both with electromagnetic field and C -meson field were considered. The result (2) given there was namely obtained by merely adding the diverging radiative correction to the C -mesonic correction which contains a divergence of the same magnitude and opposite sign. It is a well-established fact, however, that interaction with a surrounding self-field modifies the mechanical mass of an electron and that the observed mass must be identified with this modified one. In the case above mentioned we overlooked the *finite* mass correction which has to be taken into account in order to compare the result with experiments.

The relative correction due to the change of mass is

$$\frac{\delta\sigma}{\sigma_0} = \frac{1}{\sigma_0} \frac{d\sigma_0}{dm} (\delta m(\text{rad}) + \delta m(C\text{-meson})) \quad (1)$$

where $\delta m(\text{rad})$ and $\delta m(C\text{-meson})$ are rest mass modifications of an electron by respective fields. Now, it is shown that

the C -mesonic part of correction to the elastic scattering cross-section is almost exactly cancelled by $1/\sigma_0 \cdot d\sigma_0/dm \cdot \delta m$ (C -meson). Thus, subtracting (1) from $(\delta\sigma/\sigma_0)$ C -meson of the previous letter, we get a result which is nearly the same the one obtained considering radiation field alone and subtracting the *infinite* mass correction. Therefore, we can conclude that the C -meson field has hardly any detectable effect on the correction for the scattering cross-section, contrary to the conclusion formerly stated. The detailed calculation will be published in a later issue.

(1) S. Endo, T. Kinoshita and Z. Koba, Prog. Theor. Phys. **3** (1948), 320.

Interpretation of Bursts in Thin-Walled Chambers.

Y. Fujimoto and Y. Yamaguchi.

Physical Institute, Tokyo University.

Mar. 28, 1949.

Recently Carmichael⁽¹⁾ has published his experimental data on cosmic-ray bursts, and clarified the nature of bursts in unshielded ionization chambers. According to his conclusion, bursts with smaller size are mainly due to stars, and bursts with larger size are due to air showers. Although this conclusion may be valid in most, it seems to be better to improve his treatments of smaller bursts in the following points.

1) He interpreted that most of small bursts are caused by single α -particles, which are produced in stars from wall. This interpretation is based on the assumption, that the shape of the energy

spectrum of such single α -particles is equal to that of single protons, which was theoretically derived by Bagge.⁽²⁾ As is well known, Bagge's proton spectrum is composed of two parts, i.e. low energy part of evaporated protons and high energy part of directly ejected protons. From experiments of photographic plates,⁽³⁾ it is no doubt that $1/3 \sim 1/4$ of star particles are α -particles. These α -particles are perhaps evaporated from excited nuclei, and not directly ejected. So, the energy spectrum of α -particles will be cut down at $20 \sim 30$ Mev. which is of the order of the temperature of excited nuclei, and will essentially differ from the proton spectrum at higher energies. From this point of view, we cannot accept his opinion that single α -particles are main agents of small bursts.

2) In his large ionization chamber, gas pressure is rather low, and a single proton loses at most $10 \sim 12$ Mev. in gas by ionization. From considerations of 1), we think a group of several protons is the main agent of a burst. Of course, evaporated α -particles also contribute, but they will play a little role because of their rather small energy and short range. To fit the experimental data, Carmichael estimated;

Single α -particle intensity from wall
 $\sim 1.77 \cdot 10^{-2}$ per cm^2 , day. (1)

Considering the mean proton energy as $15 \sim 20$ Mev. estimated intensity (1) may be consistent with the star frequency in emulsion.⁽⁴⁾

3) In his small ionization chamber, gas pressure is so high that most of bursts will be caused by stars in gas rather than in wall against his opinion. He estimated from his experiment;

Single proton intensity from wall
 ~ 0.67 per cm^2 , day. (2)

This should be interpreted as the frequency of small stars produced in gas and wall. Simple calculations show that this is also consistent with the photographic data.

4) Carmichael considered that a burst with size 50~200 Mev. in his small chamber was caused mainly by a single α -particle. As we mentioned in 1), we can hardly understand that such a high energy α -particle is ejected from a nucleus. It may be suspected to be contri-

buted by a double star.⁽⁵⁾ If a high energy nucleon enters into the chamber, it will sometimes produce two or more stars in gas and wall and make a very large burst. This interpretation gives the correct frequency of large fragmentation bursts in the order of magnitude.

We wish to thank Mr. Hayakawa for his kind interests taken in this work.

(1) H. Carmichael, Phys. Rev. **74** (1948), 1667.

(2) E. Bagge, Ann. d. Phys. **37** (1941), 512.

(3) D. H. Perkins, Nature **160** (1947), 229.

(4) B. Rossi, Rev. Mod. Phys. **20** (1948), 537.

(5) This was suggested by Mr. Hayakawa.

Errata: (Volume 3, 1948)

On the β -Spectra of the Cu^{64}S. Gotô, E. Kanai and M. Kobayasi

p. 449 for $\begin{cases} \text{--- Exp.} \\ \text{--- Theor.} \\ \text{.... Fermi,} \end{cases}$ read $\begin{cases} \text{--- Theor.} \\ \text{--- Fermi} \\ \text{.... Exp.} \end{cases}$ in Fig. I, II.

Possible Types of Nonlocalizable Fields.....H. Yukawa

p. 453 L. (left) 23, for $\exp(ik_\mu x^\mu/\lambda)$, read $\exp(ik_\mu X^\mu/\lambda)$,
 for $\exp(-ik_\mu x^\mu/\lambda)$, read $\exp(-ik_\mu X^\mu/\lambda)$
 in equation (11).

L. (lef.) 31, for (11), read (12),

L. (right) 9, for (10), read (11).

L. (") 13, for (14), read (15),

L. (") 41, for in press, read 1864,

L. (") bottom, for 435, read 450.

On the Energy Level Shifts of Atoms due to the Interaction between the Electron and the Electromagnetic Field.*

Osamu HARA and Takasi TOKANO.

*Institute of Theoretical Physics,
Nagoya University.*

(Received Aug. 11, 1948)

§ 1. Introduction.

It was ascertained, by the experiment of Lamb and Retherford,⁽¹⁾ that the $2S\frac{1}{2}$ and $2P\frac{1}{2}$ levels of hydrogen, which are necessarily expected, from Dirac's theory, to coincide, are actually separated by about 1000 M cycles; and it has ever since been attempted by Bethe⁽²⁾ and many other authors to interpret this as being due to the interaction between the electron and the electromagnetic field. Bethe's attack leaves a questionable point in that he calculated non-relativistically, limiting the energy of the photons concerned below mc^2 (m : the electron mass) under the anticipation that a relativistic calculation would reveal that processes involving photons above mc^2 did not count in the final result. Dyson's calculation⁽³⁾ uses the Pauli-Weisskopf electron and is consequently not in conformity with fact. We have, therefore, attempted a relativistic calculation with the Dirac electron.

The level shift is to be obtained by subtracting the term corresponding to the additional mass from the electromagnetic self-energy of a bound electron, but, contrary to Bethe's expectation, the remainder after separating the mass-term by means of a canonical transformation diverges logarithmically even when a relativistic calculation is made, and the subtraction done positron-theoretically.

But it can be shown that this diverging term is proportional to the external potential and is ignored if we renormalize the electronic charge in a suitable way. Therefore process involving a photon of energy $\epsilon > mc^2$ bears no effect on level shift, which explains why Bethe's calculation had succeeded. Thus the observable level shift becomes finite and on carrying

* This is one of the series of the theory of the interaction of elementary particles which are worked by the *Elementary Particle Theory Group* of the Nagoya University.

out the calculation neglecting recoil and retardation in the intermediate state a result identically coinciding with that of Bethe is obtained.

§ 2. The Fundamental Equation and the Canonical Transformation.

In order to perform calculations positron-theoretically, we describe the electron system* in terms of quantized field variables, introduce the electromagnetic fields, and start from the following Schrödinger equation:

$$\left(\bar{H}_M + \bar{H}_R + \int \phi^* V \phi d\mathbf{x} - c \int \phi^* \boldsymbol{\alpha} \mathbf{A} \phi d\mathbf{x} + \frac{c^2}{2} \int \frac{\phi^*(\mathbf{x}) \phi(\mathbf{x}) \phi^*(\mathbf{x}') \phi(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x} d\mathbf{x}' + \frac{\hbar}{i} \frac{\partial}{\partial t} \right) \psi = 0 \quad (2.1)$$

where

$$\begin{aligned} \bar{H}_M &= \int \phi^* (c \boldsymbol{\alpha} \mathbf{p} + \beta \mu) \phi d\mathbf{x} & \mu &= mc^2 \\ \bar{H}_R &= \frac{1}{8\pi} \int (\mathbf{E}^2 + \mathbf{H}^2) d\mathbf{x} \end{aligned} \quad (2.2)$$

and ϕ , \mathbf{A} are the field variables of the electron- and electromagnetic fields respectively, $\boldsymbol{\alpha}$ Dirac's matrix, and V the external potential (Coulomb field of the nucleus). The energy of interaction between a bound electron and a radiation field consists of two parts, one observed as the additional mass of the electron, and the other as the level shift. In order to make this identification clear, it is convenient to separate the interaction Hamiltonian beforehand, by means of a suitable canonical transformation**, into two parts corresponding respectively to these. For this purpose, we first define a unitary operator U as follows:

$$U = e^{-\frac{i}{\hbar} (\bar{H}_R + \bar{H}_M) t} e^{-\frac{ic}{\hbar} \int_{-\infty}^t \int \tilde{\phi}^* \boldsymbol{\alpha} \tilde{\mathbf{A}} \phi d\mathbf{x} dt} e^{\frac{i}{\hbar} (\bar{H}_M + \bar{H}_R) t} \quad (2.3)$$

where

* We treat the electron system as a many-body problem from the outset, using the method of Iwanenko and Sokolov.⁽⁴⁾

** This canonical transformation was first proposed by S. Tomonaga. (Lecture at the symposium on the theory of elementary particles. Nov. 1947.)

$$\begin{aligned}\tilde{\phi} &= e^{\frac{i}{\hbar} \overline{H}_M t} \cdot \phi \cdot e^{-\frac{i}{\hbar} \overline{H}_M t} \\ \tilde{A} &= e^{\frac{i}{\hbar} \overline{H}_R t} \cdot A \cdot e^{-\frac{i}{\hbar} \overline{H}_R t}\end{aligned}\quad (2.4)$$

Transforming (2.1) with this U as transformation function, we obtain, to the approximations of ϵ^2 , the following equation

$$\begin{aligned}& \left\{ \overline{H}_M + \overline{H}_R + A + \frac{ie^2}{2\hbar} [I, \int \phi^* \alpha A \phi dx] - \frac{ic}{\hbar} [I', A] - \frac{e^2}{2\hbar} [I', [I', A]] \right. \\ & \left. + \frac{e^2}{2} \int \frac{\phi^*(x) \phi(x) \phi^*(x') \phi(x')}{|x - x'|} dx dx' - \frac{\hbar}{i} \frac{\partial}{\partial t} \right\} \phi = 0 \quad (2.5)\end{aligned}$$

with

$$\phi = U \varphi$$

and

$$\begin{aligned}I' &= e^{-\frac{i}{\hbar} (\overline{H}_R + \overline{H}_M) t} \int_{-\infty}^t \int \tilde{\phi}^* \alpha \tilde{A} \tilde{\phi} dx dt \cdot e^{\frac{i}{\hbar} (\overline{H}_M + \overline{H}_R) t} \\ A &= \int \phi^* V \phi dx\end{aligned}\quad (2.6)$$

As may be seen from the definition (2.3) and (2.4) this U can be rewritten in a form which does not involve the time, and consequently the Hamiltonian is not altered by this transformation.

In the expression (2.5) describing the interaction between an electron and a radiation field, the fourth term is closely similar in form to the self-energy of a free electron, and can be shown to be interpretable as the additional mass. Thus, the mass-term and level shift term have been separated by the transformation (2.3).

§ 3. The Additional-mass Term $\frac{ie^2}{2\hbar} [I', \int \phi^* \alpha A \phi dx]$

We shall show that the interaction energy due to this term can actually be regarded as the additional mass. Since ϕ and A are both operators not involving t in the Schrödinger representation, we expand each of these in terms of the wave functions of its free state, thus:

$$\begin{aligned}\phi &= \sum_{\mathbf{k} \cdot \sigma \cdot \rho} (a_{\mathbf{k}\sigma} u_{\mathbf{k}\sigma} + b_{\mathbf{k}\rho}^* v_{\mathbf{k}\rho}) e^{i\mathbf{k}\mathbf{r}} \quad (\sigma, \rho = 1, 2) \\ A &= \sqrt{2\pi\hbar c} \sum_{\ell\mu} \frac{(C_{\ell\mu}^* e^{-i\ell r} + C_{\ell\mu} e^{i\ell r}) e_{\ell\mu}}{\sqrt{\ell}} \quad (\mu = 1, 2)\end{aligned}\quad (3.1)$$

where $u_{k\sigma}$, $v_{k\rho}$ are two ground states of the electron and positron of momentum $\hbar\mathbf{k}$ and $-\hbar\mathbf{k}$ respectively, $e_{l\mu}$ the unit vector in the direction of polarization, and $a_{k\sigma}$, $a_{k\sigma}^*$, $b_{k\rho}$, $b_{k\rho}^*$, $c_{l\mu}$, $c_{l\mu}^*$ are respectively the operators which decrease or increase by one the number of electrons, positrons and photons.

Substituting this expansion and using the relation

$$\int_{-\infty}^{\infty} e^{ikt} dt = 2\pi\delta_-(k) = \pi\delta(k) - \frac{i}{k}$$

derived by Heisenberg, Γ can be rewritten as

$$\begin{aligned} \Gamma = & \sqrt{2\pi\hbar c} i \sum_{\lambda k' \sigma \sigma' \rho \rho' l \mu} \frac{1}{\sqrt{l}} \\ & \cdot \left[a_{k\sigma}^* a_{k'\sigma} (u_{k\sigma}^* a' u_{k'\sigma}) \left\{ \frac{C_{l\mu}^* \delta(\mathbf{k} - \mathbf{k}' + \mathbf{l})}{-K + K' - l} + \frac{C_{l\mu} \delta(\mathbf{k} - \mathbf{k}' - \mathbf{l})}{-K' + K' + l} \right\} \right. \\ & + a_{k\sigma}^* b_{k'\rho} (u_{k\sigma}^* a' v_{k'\rho}) \left\{ \frac{C_{l\mu}^* \delta(\mathbf{k} - \mathbf{k}' + \mathbf{l})}{-K - K' - l} + \frac{C_{l\mu} \delta(\mathbf{k} - \mathbf{k}' - \mathbf{l})}{-K - K' + l} \right\} \\ & + b_{k\rho}^* a_{k'\sigma} (v_{k\rho}^* a' u_{k'\sigma}) \left\{ \frac{C_{l\mu}^* \delta(\mathbf{k} - \mathbf{k}' - \mathbf{l})}{K + K' - l} + \frac{C_{l\mu} \delta(\mathbf{k} - \mathbf{k}' - \mathbf{l})}{K + K' + l} \right\} \\ & \left. + b_{k\rho}^* b_{k'\rho'} (v_{k\rho}^* a' v_{k'\rho'}) \left\{ \frac{C_{l\mu}^* \delta(\mathbf{k} - \mathbf{k}' + \mathbf{l})}{K - K' - l} + \frac{C_{l\mu} \delta(\mathbf{k} - \mathbf{k}' - \mathbf{l})}{K - K' + l} \right\} \right] \end{aligned} \quad (3.3)$$

with

$$K = K(k) = \sqrt{k^2 + (mc/\hbar)^2} \quad K' = K(k') \quad a' = (\mathbf{a} \cdot \mathbf{e}_{l\mu})$$

Taking the diagonal elements of $\frac{ic^2}{2\hbar} [\Gamma, \{\psi \mathbf{A} \psi d\mathbf{x}\}]$ the self-energy due to this interaction is obtained up to the approximation of ϵ^2 , but first calculating the first term in the brackets, we get

$$\begin{aligned} \Gamma \int \psi^* \mathbf{A} \psi d\mathbf{x} = & 2\pi\hbar i \left\{ \sum \frac{N_{k\sigma}(1 - N_{k'\sigma'}) (u_{k\sigma}^* a' u_{k-l\sigma'}) (u_{k-l\sigma'}^* a' u_{k\sigma})}{(-K(k) + K(k-l) + l)l} \right. \\ & + \sum \frac{N_{k\sigma} N_{k-l\rho} (u_{k\sigma}^* a' v_{k-l\rho}) (v_{k-l\rho}^* a' u_{k\sigma})}{(-K(k) - K(k-l) + l)l} \\ & + \sum \frac{(1 - N_{k\rho})(1 - N_{k-l\sigma}) (v_{k\sigma}^* a' u_{k-l\sigma}) (u_{k-l\sigma}^* a' v_{k\sigma})}{(K(k) + K(k-l)l + l)l} \\ & \left. + \sum \frac{(1 - N_{k\rho}) N_{k-l\rho} (v_{k\rho}^* a' v_{k-l\rho'}) (v_{k-l\rho'}^* a' v_{k\rho})}{(K(k) - K(k-l) + l)l} \right\} \end{aligned} \quad (3.4)$$

where $N_{k\sigma}$, $N_{k\rho}$ are respectively the numbers of electrons and positrons in states denoted by $k\sigma$, $k\rho$.

The eigenfunction ϕ^m of (2.5) can be written, in the zeroth approximation disregarding the interaction with the electromagnetic field, as

$$\phi_0^m = \sum_k a_k^{(m)} \delta_{k\sigma} e^{-iE_m t/\hbar} \quad (3.5)$$

where $a_k^{(m)}$ is the Fourier coefficient of the expansion of the eigenfunction denoted by m of the Coulomb field in terms of the eigenfunction of the free states, δ_k the wave functions of a free electron (or positron) when the number of the latter in a given state is represented by a diagonal matrix, and have the forms

$$\delta_k = \delta_{0N_{k1}} \delta_{0N_{k2}} \dots \delta_{1N_k} \dots \quad (3.6)$$

while E_m is the energy eigenvalue of the state considered.

(3.4) has a non-vanishing value even for a state in which not a single electron nor positron exists. This is the vacuum energy, and in order to obtain the interaction energy due to this in a given state the effect of the former must be subtracted. Thus, calculating the self-energy due to (3.4) for the case when an electron exist in the state m , we obtain

$$\begin{aligned} -2\pi i \sum_k \left\{ \sum_{l\mu\sigma} \frac{(u_{k\sigma}^* a' u_{k-l\sigma'}) (u_{k-l\sigma'} a' u_{k\sigma})}{l(K(k-l) + l - K(k))} \right. \\ \left. - \sum_{l\rho\mu} \frac{(u_{k\sigma}^* a' v_{k-l\rho}) (v_{k-l\rho} a' u_{k\sigma})}{l(K(k-l) + l + K(k))} \right\} |a_k^{(m)}|^2 \quad (3.7) \end{aligned}$$

Exactly the same expression results from the second term of the commutator, and we finally have, as the self-energy due to $\frac{ic^2}{2\hbar} [\Gamma, \{\phi^* \alpha A \phi\} dx]$

$$\begin{aligned} 2\pi e^2 \sum_k \left\{ \sum_{l\mu\sigma} \frac{(u_{k\sigma}^* a' u_{k-l\sigma'}) (u_{k-l\sigma'} a' u_{k\sigma})}{(K(k-l) + l - K(k))l} \right. \\ \left. - \sum_{l\rho\mu} \frac{(u_{k\sigma}^* a' v_{k-l\rho}) (v_{k-l\rho} a' u_{k\sigma})}{(K(k-l) + l + K(k))} \right\} |a_k^{(m)}|^2 \quad (3.8) \end{aligned}$$

Each of these terms correspond exactly to the result to be obtained by the customary second order perturbation in computing for the electrodynamic self-energy of a free electron and position of momentum $\pm \hbar k$. The static self-energy in this case can be obtained by a slight extension of Weisskopf's⁽⁵⁾ method of calculation, and the result is as follows:

$$\sum |a_k^{(m)}|^2 \times (\text{static self-energy of the free state with momentum } \hbar k (-\hbar k)) \quad (3.9)$$

From (3.8) and (3.9), the self-energy due to $\frac{e^2}{2\hbar}[I, \int \psi^* \alpha A \psi d\mathbf{x}]$ is, including the static part, reduced to the form

$$\sum_{\mathbf{k}} |a_{\mathbf{k}}^{(m)}|^2 \times (\text{electromagnetic self-energy of a free electron (positron) with momentum } \hbar\mathbf{k} (-\hbar\mathbf{k}))$$

$$\sim \sum_{\mathbf{k}} |a_{\mathbf{k}}^{(m)}|^2 \times \frac{3}{2\pi} \frac{e^2}{\hbar c} \cdot \frac{m^2 c^4}{E(\mathbf{k})} \int \frac{dl}{l}, \quad E(\mathbf{k}) = \sqrt{m^2 c^2 + \hbar^2 k^2} \cdot c. \quad (3.10)$$

Each of these terms is to be regarded as the additional mass of a free electron or positron due to interaction with radiation, so that (3.10) is clearly interpretable as the radiative mass of a bound electron.

§4. Level Shift.

Since, as seen in the previous section, the fourth term in the Hamiltonian (2.5) can be regarded as the additional mass due to interaction with the radiation field, the subsequent terms involving $[I, A]$ are necessarily the ones that contribute to the level shift.

These can, up to the approximation of $e^2/\hbar c$, be obtained by starting from ϕ_0 and performing perturbation calculations, of the second order for the term involving e linearly, and the first for that quadratic in e . But as these contain terms diverging in the infra-red region, it is convenient previously to rewrite them in such forms that have the diverging parts already cancelled. For this purpose, we introduce an operator A defined as follows:

$$A = [I, \int \psi^* (c\alpha p + \beta\mu) \psi d\mathbf{x}] \quad (4.1)$$

Using this we have

$$[I, A] = [I, \int \psi^* (c\alpha p + \beta\mu + V) \psi d\mathbf{x}] - A \quad (4.2)$$

Employing this A , and calculating positron-theoretically for the case when a single electron exists in the state m the respective terms become as follows:

$$(i) \text{ The term linear in } e: -\frac{i e}{\hbar} [I, A]$$

$$\frac{e^2}{\hbar^2} \sum_{r\mu} \frac{[I, A]_{mr}^{r\mu} [I, A]_{rm}^{r\mu}}{E_r - E_m + \varepsilon_r} - \frac{e^2}{\hbar^2} \sum_{s\mu} \frac{[I, A]_{ms}^{s\mu} [I, A]_{sm}^{s\mu}}{E_s + E_m + \varepsilon_s} \quad \varepsilon_i = \hbar c l.$$

$$\begin{aligned}
&= \frac{e^2}{2\hbar^2} \sum_{l\mu r} \frac{(I_{nr}^{\nu\mu} [I, A]_{rnl}^{\nu\mu} - [I, A]_{nr}^{\nu\mu} I_{rnl}^{\nu\mu} - I_{nr}^{\nu\mu} A_{rnl}^{\nu\mu} + A_{nr}^{\nu\mu} I_{rnl}^{\nu\mu})(E_r - E_m) + 2A_{nr}^{\nu\mu} A_{rml}^{\nu\mu}}{E_r - E_m + \varepsilon_l} \\
&+ \frac{e^2}{2\hbar^2} \sum_{l\mu s} \frac{(I_{ms}^{\nu\mu} [I, A]_{sm}^{\nu\mu} - [I, A]_{ms}^{\nu\mu} I_{sm}^{\nu\mu} - I_{ms}^{\nu\mu} A_{sm}^{\nu\mu} + A_{ms}^{\nu\mu} I_{sm}^{\nu\mu})(E_s + E_m) - 2A_{ms}^{\nu\mu} A_{sm}^{\nu\mu}}{E_s + E_m + \varepsilon_l} \quad (4.3)
\end{aligned}$$

here the first summation is extended to all the eigenstates of a bound electron in Coulomb field and the second that of the positron. Here $I_{nr}^{\nu\mu}$ etc. denotes the matrix element composed of the wave function for the case when there exist an electron in the state r and a photon of momentum $\hbar l$, and the wave function of an electron in the state m , and has the following form:

$$I_{nr}^{\nu\mu} = \sum_{k, \sigma, \sigma'} \sqrt{2\pi\hbar c} i \frac{(u_{k\sigma}^* a' u_{k-l\sigma'})}{c \sqrt{l} (K(k-l) + l - K(k))} a_k^{(m)} a_{k-l}^{(r)} \quad (4.4)$$

The first term in (4.3) corresponds to the emission or absorption of one photon, and the second implies that pair cannot be created in a state m already occupied by an electron.

(ii) The term quadratic in e : $-\frac{e^2}{2\hbar^2} [I, [I, A]]$

Taking the diagonal elements of this matrix, those counting in the final result are seen to be

$$-\frac{e^2}{2\hbar^2} \sum_{rs l\mu} N_r (1 - N_s) (I_{rs}^{\nu\mu} [I, A]_{sr}^{\nu\mu} - [I, A]_{sr}^{\nu\mu} I_{rs}^{\nu\mu}) \quad (4.5)$$

This, too, has a non-vanishing value for vacuum, so the energy due to it must be discounted and the resulting interaction energy for the same case as above is as follows.

$$\begin{aligned}
&-\frac{e^2}{2\hbar^2} \left\{ \sum_{rl\mu} \frac{(I_{nr}^{\nu\mu} [I, A]_{nr}^{\nu\mu} - [I, A]_{nr}^{\nu\mu} I_{nr}^{\nu\mu})(E_r - E_m + \varepsilon_l)}{E_r - E_m + \varepsilon_l} \right. \\
&\quad \left. + \sum_{sl\mu} \frac{(I_{ms}^{\nu\mu} [I, A]_{sm}^{\nu\mu} - [I, A]_{sm}^{\nu\mu} I_{ms}^{\nu\mu})(E_s + E_m + \varepsilon_l)}{E_s + E_m + \varepsilon_l} \right\} \quad (4.6)
\end{aligned}$$

If (4.3) and (4.6) are added, the terms involving $[I, A] \cdot I$ which are responsible for the infra-red catastrophe, just cancel each other out, and the following result is obtained for the level shift,

$$\begin{aligned}
L.S. &= \frac{e^2}{2\hbar^2} \sum_{rl\mu} \frac{2\varepsilon_l I_{nr}^{\nu\mu} I_{rm}^{\nu\mu} (E_r - E_m) + 2A_{nr}^{\nu\mu} A_{rm}^{\nu\mu} - (I_{nr}^{\nu\mu} A_{rm}^{\nu\mu} - A_{nr}^{\nu\mu} I_{rm}^{\nu\mu})(E_r - E_m - \varepsilon_l)}{E_r - E_m + \varepsilon_l} \\
&- \frac{e^2}{2\hbar^2} \sum_{sl\mu} \frac{2\varepsilon_l I_{ms}^{\nu\mu} I_{sm}^{\nu\mu} (E_s + E_m) + 2A_{ms}^{\nu\mu} A_{sm}^{\nu\mu} + (I_{ms}^{\nu\mu} A_{sm}^{\nu\mu} - A_{ms}^{\nu\mu} I_{sm}^{\nu\mu})(E_s + E_m - \varepsilon_l)}{E_s + E_m + \varepsilon_l} \quad (4.7)
\end{aligned}$$

Applying the expansion (3.1) to \mathcal{A} , an expression similar to that for I' given in (3.3) is obtained. As these are comparatively simple operators, (4.7) readily be calculated. The calculation is conveniently carried out if we treat the two cases $\epsilon > mc^2$ and $\epsilon < mc^2$ separately.

(i) The case $\epsilon > mc^2$

Using Casimir's annihilation operator and Born approximation in the intermediate state, (4.7) is calculated as follows.

First summation

$$\begin{aligned}
 &= 4\pi \sum_{\mathbf{k}l} \frac{-k^2 \cos^2 \theta + kl \cos \theta + K(k)K(\mathbf{k}-\mathbf{l})}{l(E_r - E_m + \epsilon_l)K(k)K(\mathbf{k}-\mathbf{l})(K(\mathbf{k}) - K(\mathbf{k}-\mathbf{l}) - l)^2 c^2} \\
 &\quad \cdot (\hbar c)^3 \times \{ (K_n - K_m)l - (K(\mathbf{k}-\mathbf{l}) - K(k))^2 \\
 &\quad + (K(\mathbf{k}-\mathbf{l}) - K(k))(K_n - K_m - l) \} \cdot |a_{\mathbf{k}}^{(m)}|^2 \\
 &\sim \frac{\hbar^2 c^2}{2\pi c^2} \sum_{\mathbf{k}} |a_{\mathbf{k}}^{(m)}|^2 \times \int \frac{K(k) - K_m}{l} dl = -\frac{\hbar c}{2\pi c^2} V_{mm} \int \frac{dl}{l} \quad (4.8)
 \end{aligned}$$

(θ is the angle between \mathbf{k} and \mathbf{l} , $K_m = E_m/\hbar c$)

In the same manner the second summation is calculated to be

$$-\frac{\hbar c}{2\pi c^2} V_{mm} \int \frac{dl}{l} \quad (4.8')$$

Therefore

$$\text{Level Shift} \simeq -\frac{1}{2\pi} \cdot \frac{c^{\frac{5}{2}}}{\hbar c} \cdot V_{mm} \int \frac{dl}{l} \quad (4.9)$$

Here V_{mm} is the potential energy in the state m .

Thus contrary to Bethe's prediction, the level shift cannot obtain a finite value even if the calculations are performed in accordance with the positron theory. But as is seen from (4.8'), this diverging term is proportional to V and we can resort to the method of charge renormalization to cope with it, that is, the contribution of the processes involving photons with energy larger than mc^2 to the energy level is to multiply a constant factor to the whole energy level scheme thus bearing no effect on the level shift considered here. Therefore in the calculation of the level shift, we can limit $\epsilon < mc^2$ and correspondingly confine the energy of the electrons to be taken in the intermediate states below mc^2 . This is thought to be the reason why Bethe's calculation had succeeded.

(ii) The case $\epsilon < mc^2$.

Under the restriction noted above, the terms other than the first become all small and (4.7) is reduced to the form

$$L.S. = \frac{c^2}{\hbar^2} \sum_{r\mu} \frac{I'_{mr} I'_{r\mu} (E_r - E_m) \epsilon_l}{E_r - E_m + \epsilon_l} \quad (4.10)$$

Making the Pauli approximation here for the electron by taking $ca=v$ and neglecting the retardation and recoil, we integrate (4.10) for the directions of the virtual photon, whereupon it becomes

$$\frac{2c^2}{3\pi\hbar c^2} \int_0^{mc^2} \sum_r \frac{v_{mr}^2 (E_r - E_m)}{E_r - E_m + \epsilon_l} d(\hbar c l) \quad (4.11)$$

This is none other than the expression used by Bethe as his startingpoint, obtained by subtracting the self-energy of a free electron from that of the bound. From here on therefore, his results can be used, and for the S -level of the hydrogenlike atoms we have the following result:

$$L.S. = \frac{8}{3\pi} \left(\frac{c^2}{\hbar c} \right)^3 R_y \frac{Z^4}{n^3} \log \frac{mc^2}{\langle E_r - E_m \rangle_{AV}} \quad (4.12)$$

which agrees well with experiment.*

§ 5. Conclusion.

In the final analysis, if a renormalization of the charge is performed, and relativistic calculations made, it is seen that processes involving photons of energy above mc^2 do not count in the level shift, just as Bethe predicts. Current quantum theory thus does give, in this problem, a result agreeing well with experiment.

However, as made evident in § 4, the self-energy of a bound electron possesses, besides a divergence of the mass type, a term proportional to the binding energy due to the polarization of vacuum, and which diverges logarithmically. Though these divergences can, at least tentative be eliminated by appropriate renormalizations of the electronic mass and charge, this seems far from being an essential solution.

* Level-shifts due to other causes, such as the polarization of vacuum due to the external potential V , hyperfine structure, etc., are all small compared to this, so we ignore them. The polarization of vacuum in this sense also involves a logarithmically diverging term and necessitate the renormalization of the potential which is of different kind than that considered in this section.

The former can be made to converge, as in the case of a free electron, by introducing the C -meson⁽⁶⁾ and setting the condition $f^2=2e^2$, but this procedure is of no avail against the latter, and it seems that a fundamental revision of current quantum electrodynamics, apart from the introduction of the C -meson, is required in order to find an essential solution against this divergence problem.

In conclusion we wish to express our sincere gratitude to Prof. Sakata and Dr. Tanikawa who continually guided us throughout this work.

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Theory of the Interaction of Elementary Particles, IV.

The Problem of Vacuum Polarisation. (2)

Hiroomi UMEZAWA, Jiro YUKAWA and Eiji YAMADA.

*Institute of Theoretical Physics,
Nagoya University.*

(Received Aug. 11, 1948)

§ 4. Compton Scattering.

In order to see whether the existence of Fermi particles and scalar or pseudoscalar mesons with a relative abundance of 1:2 is effective also against divergences due to vacuum polarizations appearing in problems involving collision processes, we took as a first simple example, the Compton scattering, and made the fourth order perturbation calculations of it. The transition probability w for Compton scattering in fourth order perturbation is

$$w = \frac{2\pi}{\hbar} |e^2 H_2 + e^4 H_4|^2 \rho_F$$

where ρ_F is the number of final states per energy interval dE_F , and $e^2 H_2$, $e^4 H_4$ are the contributions of the second and fourth order perturbations respectively. Hereunder we shall treat only the diverging term in $e^4 H_4$. We first calculate for the case in which only the photon-electron interaction is considered.

The processes contributing to the fourth order perturbation of Compton scattering are: (a) those which can be divided into classes corresponding

(Remark) The unbroken line in the middle denotes the state free from charged particles, the lines above and below it respectively states with negatively and positively charged particles; the processes occur in sequence from the left to the right, each ending at the head of the arrow. Further, charged (pseudo-) scalar particles are involved only in processes shown in broken lines in Fig. 4. (~~~~) and (.....) respectively denote the emission and absorption of a photon.

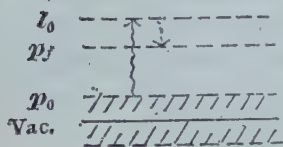


Fig. 2 One of the four processes of second order Compton scattering.

to the four processes in second order perturbation; and (b) those which find no counterpart in second order perturbation. As an example of the former type, we study the divergence corresponding to the following process. (Fig. 2):

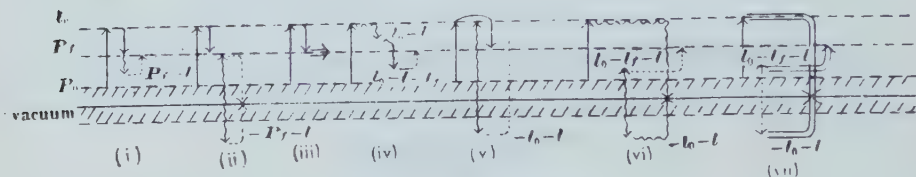


Fig. 3 The seven representative processes (a) of Compton scattering. The unbroken lines denote those corresponding to processes of the second order perturbation (Fig. 2), while the double lines those due to the Coulomb force.

$$\begin{array}{ccccc}
 l_0 & & & l_F \\
 \downarrow & \rightarrow & & \rightarrow \\
 p_0 & & p_0 + l_0 & & p_0 + l_0 - l_F
 \end{array} \quad (I)$$

where l_0 , l , l_F are the momenta of photon in the initial, virtual, and final states respectively while p_0 , p , p_F are those of the electron correspondingly, and p_0 is taken to be zero.

The processes of this group can be subdivided into two types according as the divergence arises from: (i) the photon produced in the intermediate state, or (ii) the electron pair produced in the intermediate state. Furthermore, there are, in this calculation, processes for which the energies in the initial and virtual states are equal, in which case the following formula must be used instead of the usual formula for $e^4 H_1$:

$$\begin{aligned}
 & - \frac{(i | H' | v_1)(v_1 | H' | v_2)(v_2 | H' | v_3)(v_3 | H' | f)}{2(E_i - E_{v_1})^2(E_i - E_{v_3})} \\
 & \quad - \frac{(i | H' | v_1)(v_1 | H' | v_2)(v_2 | H' | v_3)(v_3 | H' | f)}{2(E_i - E_{v_1})(E_i - E_{v_3})^2} \quad (20)
 \end{aligned}$$

Throughout the whole work, calculations are performed positron-theoretically, and as in Weisskopf's calculation of self-energy, they are generalized by cancelling out the vacuum effect (in which vacuum particles caused divergences) in the following way:

$$\begin{aligned}
 & [\text{processes involving vacuum particles}] \\
 & \quad - [\text{processes caused by vacuum particles in vacuum}]
 \end{aligned}$$

For example, the process

$$(\mathbf{p}_0, \mathbf{l}_0) \rightarrow (\mathbf{l}_0, \mathbf{l}, \mathbf{p}_0, \mathbf{p}_0 + \mathbf{l}_0, -\mathbf{p}_0 - \mathbf{l} - \mathbf{l}_0) \rightarrow$$

$$(\mathbf{l}, \mathbf{p}_0 + \mathbf{l}_0, \mathbf{p}_0 + \mathbf{l}_0, -\mathbf{p}_0 - \mathbf{l}_0 - \mathbf{l}) \rightarrow (\mathbf{p}_0, \mathbf{l}_0) \rightarrow (\mathbf{l}_f, \mathbf{p}_f)$$

is forbidden by Pauli's principle, so that it does not enter into the first term in (21) but as a process in vacua, it is merely the ordinary electron-pair creation, and must therefore be included in the second term of (21).

(a1) Processes of this type can again be divided into those the causes of whose divergences line in the differences from vacuum caused by the creation of the photon in the intermediate state due respectively to the existence of the electron, \mathbf{p}_0 , $\mathbf{p}_0 + \mathbf{l}_0$, or $\mathbf{p}_0 + \mathbf{l}_0 - \mathbf{l}_f = \mathbf{p}_f$ in the above process 1. We give here seven processes among those corresponding to \mathbf{p}_f , though it must be remarked that the last two, (vi) and (vii) each imply four other processes of the same type besides those explicitly written :

- (i) $(\mathbf{l}_0, \mathbf{p}_0) \rightarrow (\mathbf{p}_0 + \mathbf{l}_0) \rightarrow (\mathbf{l}_f, \mathbf{p}_f) \rightarrow (\mathbf{l}_f, \mathbf{l}, \mathbf{p}_f - \mathbf{l}) \rightarrow (\mathbf{l}_f, \mathbf{p}_f)$
- (ii) $(\mathbf{l}_0, \mathbf{p}_0) \rightarrow (\mathbf{l}_0 + \mathbf{p}_0) \rightarrow (\mathbf{l}_f, \mathbf{p}_f) \rightarrow (\mathbf{l}_f, \mathbf{l}, \mathbf{p}_f, \mathbf{p}_f - \mathbf{l}) \rightarrow (\mathbf{l}_f, \mathbf{p}_f)$
- (iii) $(\mathbf{l}_0, \mathbf{p}_0) \rightarrow (\mathbf{l}_0 + \mathbf{p}_0) \rightarrow (\mathbf{l}_f, \mathbf{p}_f) = (\mathbf{l}_f, \mathbf{p}_f)$

Here the arrow in bold type indicates a transition due to Coulomb force. (ii) results from the difference from vacuum due to Pauli's principle, and is included in the second term in (21).

These processes are those whose parts contributing to the divergence correspond to the self-energy of the electron, and if we obtain the difference from vacuum in accordance with (21), the contribution of the second term in (20) becomes finite, while the first term becomes, using the interaction H' , for instance in the case of (i), as follow :

$$- \frac{(\mathbf{l}_0, \mathbf{p}_0 | H' | \mathbf{l}_0 + \mathbf{p}_0)(\mathbf{l}_0 + \mathbf{p}_0 | H' | \mathbf{l}_f, \mathbf{p}_f)}{2(\mathbf{l}_0 + \mu - E_{l_0})^2}$$

$$(\mathbf{p}_f | \frac{3e^2}{2\pi} \int \frac{dl}{l} \phi^* \beta \mu \phi | \mathbf{p}_f) \quad (\mathbf{p}_0 = 0)$$

Likewise for the other two, and the final result is that the divergence is of the same type as the caused by the presence of a term

$$H_1 = \frac{3e^2}{2\pi} \int \frac{dl}{l} \phi^* \beta \mu \phi \quad (22)$$

within the Hamiltonian function.

- (iv) $(\mathbf{l}_0, \mathbf{p}_0) \rightarrow (\mathbf{l}_0 + \mathbf{p}_0) \rightarrow (\mathbf{l}, \mathbf{l}_0 + \mathbf{p}_0 - \mathbf{l}) \rightarrow (\mathbf{l}_f, \mathbf{l}, \mathbf{p}_f - \mathbf{l}) \rightarrow (\mathbf{l}_f, \mathbf{p}_f)$.
- (v) $(\mathbf{l}_0, \mathbf{p}_0) \rightarrow (\mathbf{l}_0 + \mathbf{p}_0) \rightarrow (\mathbf{l}, \mathbf{l}_0 + \mathbf{p}_0, \mathbf{p}_0 + \mathbf{l}_0, -\mathbf{p}_0 - \mathbf{l} - \mathbf{l}_0)$
 $\rightarrow (\mathbf{l}, \mathbf{l}_f, \mathbf{p}_f, \mathbf{p}_0 + \mathbf{l}_0, -\mathbf{p}_0 - \mathbf{l} - \mathbf{l}_0) \rightarrow (\mathbf{l}_f, \mathbf{p}_f)$

(v) appears because of the difference from vacuum due to Pauli's principle, and is included in the second term in (21).

$$(vi) \quad (l_0 p_0) \rightarrow (l_0 + p_0) \rightarrow (l_f, p_0 + l_0, -p_0 - l_0 + l, p_f - l) \rightarrow (l_f, l, p_f - l) \rightarrow (l_f, p_f)$$

$$(vii) \quad (l_0 p_0) \rightarrow (l_0 + p_0) \rightarrow (l_f, p_0 + l_0, -p_0 - l_0 + l, p_f - l) \rightarrow (l_f, p_f)$$

Processes falling under these types are characterized by an interchange of positive energy electrons. The process (iv), as seen below, diverges in itself.

$$\frac{(l_0, 0 | H' | l_0) (l_0 | H' | l_0 - l) (l_0 - l | H' | l_0 - l - l_f) (l_0 - l - l_f | H' | p_f)}{(l_0 + \mu - E_{p_0}) (l_0 + \mu - l - E_{l_0 - l}) (l_0 + \mu - l - l_f - E_{l_0 - l - l_f})} \\ = \frac{(l_0, 0 | H' | l_0)}{l_0 + \mu - E_{l_0}} n_{l_0} A_{l_0}^+ \left\{ \frac{H' A_{l_0 - l}^+ H' A_{l_0 - l - l_f}^+ H'}{(l_0 + \mu - l - E_{l_0 - l}) (l_0 + \mu - l - l_f - E_{l_0 - l - l_f})} A_{l_f}^+ n_{l_f} \right\}$$

Where n_p is the wave function of the electron of momentum P , and is Casimir's operator selecting positive states out from n_p . Reducing the expression in brackets into a single operator and integrating, the diverging term becomes as follows:

$$\frac{(l_0, 0 | H' | l_0) (l_0 | -\frac{e^2}{12\pi} \int \frac{dl}{l} a A | p_f)}{l_0 + \mu - E_{l_0}}$$

Thus, the processes of the types (iv), (v) and (vii) processes of the produce various divergences in themselves, but they cancel each other on being added, giving a finite result. Also, it is easy to see that in type (vi), the contributions from the two mutually perpendicular directions of polarization of the photon in the intermediate state cancel each other and produce no divergence.

a2) Processes of this type can be divided into those whose differences from vacuum lie in the existence of the photon l_0 or l_f respectively, in process (ii) in second order perturbation. Hereunder we give only those corresponding to l_f (Fig. 4). However, type (v) embraces other of similar nature.

$$(i) \quad (l_0 p_0) \rightarrow (l_0 + p_0) \rightarrow (l_f, p_f) \rightarrow (p_f, p + l_f, -p) \rightarrow (l_f, p_f)$$

$$(ii) \quad (l_0 p_0) \rightarrow (l_0 + p_0) \rightarrow (l_f, p_f) \rightarrow (l_f, l_f, p_f, p - l_f, -p) \rightarrow (l_f, p_f)$$

These are processes in which the self-energy arises in a manner corresponding to the origin of the self-energy of a photon, and when (20) is applied after adding these two contributions they can be combined into the following expression:

$$-\frac{(l_0 p_0 | H' | l_0 + p_0) (l_0 + p_0 | H' | l_f p_f)}{2(l_0 + \mu - E_{l_0})} (l_f | -\frac{e^2}{3\pi} \int p d p A^2 | l_f) \\ -\frac{(l_0 p_0 | H' | l_0 + p_0) (l_0 + p_0 | H' | l_f p_f)}{2(l_0 + \mu - E_{l_0})}$$

$$\propto \left[\frac{(\mathbf{l}_f | H' | \mathbf{l}_f, \mathbf{l}_f, \mathbf{p} - \mathbf{l}_f - \mathbf{p})(\mathbf{l}_f, \mathbf{l}_f, \mathbf{p} - \mathbf{l}_f - \mathbf{p} | H' | \mathbf{l}_f)}{(-l_f - E_p - E_{\mathbf{p} - \mathbf{l}_f})} + \frac{(\mathbf{l}_f, | H' | \mathbf{p} + \mathbf{l}_f, -\mathbf{p})(\mathbf{p} + \mathbf{l}_f, -\mathbf{p} | H' | \mathbf{l}_f)}{(l_f - E_p - E_{\mathbf{p} + \mathbf{l}_f})} \right] \quad (23)$$

The first term here is a divergence identical with that caused by the presence of a term

$$H_2 = -\frac{e^2}{3\pi^2} \int_0^\infty p dp A^2 \quad (24)$$

with the Hamiltonian function:

$$\begin{aligned} \text{(iii)} \quad (\mathbf{l}_0 \mathbf{p}_0) &\rightarrow (\mathbf{l}_0 + \mathbf{p}_0) \rightarrow (\mathbf{l}_f, \mathbf{l}_0 + \mathbf{p}_0, \mathbf{p} - \mathbf{l}_f, -\mathbf{p}) \\ &\rightarrow (\mathbf{l}_f, \mathbf{l}_f, \mathbf{p}_f, \mathbf{p} - \mathbf{l}_f, -\mathbf{p}) \rightarrow (\mathbf{l}_f \mathbf{p}_f) \\ \text{(iv)} \quad (\mathbf{l}_0 \mathbf{p}_0) &\rightarrow (\mathbf{l}_0, \mathbf{l}_f, \mathbf{p}_0, \mathbf{p} - \mathbf{l}_f, -\mathbf{p}) \rightarrow (\mathbf{l}_f, \mathbf{l}_0 + \mathbf{p}_0, \mathbf{p} - \mathbf{l}_f, -\mathbf{p}) \\ &\rightarrow (\mathbf{l}_f, \mathbf{l}_f, \mathbf{p}_f, \mathbf{p} - \mathbf{l}_f, -\mathbf{p}) \rightarrow (\mathbf{l}_f \mathbf{p}_f) \end{aligned}$$

From process (iii) .

$$\begin{aligned} & \frac{(\mathbf{l}_0 \mathbf{p}_0 | H' | \mathbf{l}_0 + \mathbf{p}_0)(\mathbf{l}_0 + \mathbf{p}_0 | H' | \mathbf{l}_f, \mathbf{l}_0 + \mathbf{p}_0, \mathbf{p} - \mathbf{l}_f, -\mathbf{p})(\mathbf{l}_0 + \mathbf{p}_0 | H' | \mathbf{l}_f, \mathbf{p}_f)(\mathbf{l}_f, \mathbf{l}_f, \mathbf{p} - \mathbf{l}_f, -\mathbf{p} | H' | \mathbf{l}_f)}{(l_0 + \mu - E)(l_0 + \mu - E_{l_0} - l_f - E_p - E_{\mathbf{p} - \mathbf{l}_f})(-l_f - E_p - E_{\mathbf{p} - \mathbf{l}_f})} \\ & \sim \frac{(\mathbf{l}_0 \mathbf{p}_0 | H' | \mathbf{l}_0 + \mathbf{p}_0)(\mathbf{l}_0 + \mathbf{p}_0 | H' | \mathbf{l}_f, \mathbf{p}_f)(\mathbf{l}_f | H' | \mathbf{l}_f, \mathbf{l}_f, \mathbf{p} - \mathbf{l}_f, -\mathbf{p})(\mathbf{l}_f, \mathbf{l}_f, \mathbf{p} - \mathbf{l}_f, -\mathbf{p} | H' | \mathbf{l}_f)}{(l_0 + \mu - E_{l_0})(-l_f - E_p - E_{\mathbf{p} - \mathbf{l}_f})^2} \\ & - \frac{(\mathbf{l}_0, \mathbf{p}_0 | H' | \mathbf{l}_0 + \mathbf{p}_0)(\mathbf{l}_0 + \mathbf{p}_0 | H' | \mathbf{l}_f, \mathbf{p}_f)(\mathbf{l}_f | H' | \mathbf{l}_f, \mathbf{l}_f, \mathbf{p} - \mathbf{l}_f, -\mathbf{p})(\mathbf{l}_f, \mathbf{l}_f, \mathbf{p} - \mathbf{l}_f, -\mathbf{p} | H' | \mathbf{l}_f)}{(-l_f - E_p - E_{\mathbf{p} - \mathbf{l}_f})^3} \end{aligned} \quad (25)$$

The third term of this expression differs only in sign from the diverging term of process (iv), so that when these two contributions are added, only the first term of (25) remains, which, added to the second term of (23), gives

$$\begin{aligned} & - \frac{(\mathbf{l}_0, \mathbf{p}_0 | H' | \mathbf{l}_0 + \mathbf{p}_0)(\mathbf{l}_0 + \mathbf{p}_0 | H' | \mathbf{l}_f, \mathbf{p}_f) \left[\frac{(\mathbf{l}_f | H' | \mathbf{p} + \mathbf{l}_f, -\mathbf{p})(\mathbf{p} + \mathbf{l}_f, -\mathbf{p} | H' | \mathbf{l}_f)}{2(l_f - E_p - E_{\mathbf{p} + \mathbf{l}_f})^2} \right.} \\ & \left. - \frac{(\mathbf{l}_f | H' | \mathbf{l}_f, \mathbf{l}_f, \mathbf{p} - \mathbf{l}_f, -\mathbf{p})(\mathbf{l}_f, \mathbf{l}_f, \mathbf{p} - \mathbf{l}_f, -\mathbf{p} | H' | \mathbf{l}_f)}{2(-l_f - E_p - E_{\mathbf{p} - \mathbf{l}_f})^2} \right] \end{aligned} \quad (26)$$

Substituting (2) for H' , the expression in the brackets becomes $e^2/3\pi \int_0^\infty d\mathbf{p}/p$ and (26) reduces to

$$\frac{(\mathbf{l}_0, \mathbf{p}_0 | H' | \mathbf{l}_0 + \mathbf{p}_0)(\mathbf{l}_0 + \mathbf{p}_0 | -\frac{e^2}{3\pi} \int \frac{d\mathbf{p}}{p} H' | \mathbf{l}_f \mathbf{p}_f)}{l_0 + \mu - E_{l_0}} \quad (27)$$

This expression also arises from another similar process, and can be identified with the divergence due to the presence of a term

$$H_3 = -\frac{e^3}{3\pi} \int \frac{d\mathbf{p}}{p} \phi^* (\mathbf{a} \cdot \mathbf{A}) \phi \quad (28)$$

in the Hamiltonian function.

$$\begin{aligned} (\text{v}) \quad (\mathbf{l}_0, \mathbf{p}_0) &\rightarrow (\mathbf{l}_0, \mathbf{l}_f, \mathbf{p}_0, \mathbf{p} - \mathbf{l}_f, -\mathbf{p}) \rightarrow (\mathbf{l}_0, \mathbf{l}_f, -\mathbf{l}_f, \mathbf{p}_0) \\ &\rightarrow (\mathbf{l}_f, -\mathbf{l}_f, \mathbf{l}_0 + \mathbf{p}_0) \rightarrow (\mathbf{l}_f, \mathbf{p}_f) \end{aligned}$$

This is characterized by the appearance of $-\mathbf{l}_f$ or $-\mathbf{l}_0$ in the intermediate state, but when the many terms of this type are added after suitable transformations, divergences of the orders of $\int d\mathbf{p}$ and $\int d\mathbf{p}/p$ cancel one another out irrespective of the explicit form of the Hamiltonian, and here again the divergence produced by (24), namely that corresponding to the sort caused by the operator c_{ij} , c_{-ij} in \mathbf{A}^2 is the only one to remain.

b) Representative among the processes finding no counterpart in second order perturbation is, for example, something like the following

$$\begin{aligned} (\mathbf{l}_0 \mathbf{p}_0) &\rightarrow (\mathbf{l}_0, \mathbf{l}_f - \mathbf{l}_0, \mathbf{p}_f) \rightarrow (\mathbf{l}_0, \mathbf{p} + \mathbf{l}_f - \mathbf{l}_0, -\mathbf{p}, -\mathbf{p}_f) \\ &\rightarrow (\mathbf{p} + \mathbf{l}_f, -\mathbf{p}, \mathbf{p}_f) \rightarrow (\mathbf{l}_f, \mathbf{p}_f) \end{aligned}$$

This process is characterized by the presence of a photon of momentum $\mathbf{l}_f - \mathbf{l}_0$ or $\mathbf{l}_0 - \mathbf{l}_f$ in the intermediate state, and though each term of it diverges separately, their contributions when added after performing calculation introducing the explicit form of H' , are found to cancel one another and give rise to no divergence.

We thus arrive at the conclusion that the diverging term appearing when only the photon-electron interaction is considered is equivalent to that deducible from an interaction of the following form*:

$$\begin{aligned} H_1 + H_2 + H_3 = & -\frac{e^2}{3\pi^2} \int \mathbf{p} d\mathbf{p} \mathbf{A}^2 + \frac{3e^2}{2\pi} \int \frac{d\mathbf{l}}{l} \phi^* \beta \mu \phi \\ & - \frac{e^2}{3\pi^2} \int \frac{d\mathbf{p}}{p} \phi^* \mathbf{a} \cdot \mathbf{A} \phi \quad (29) \end{aligned}$$

Nextly, if the existence of a C -meson is considered this particle plays the rôle of appearing instead of the photon \mathbf{l} in the intermediate state in processes of type (a 1). Among those, the processes (i), (ii), (iii) give rise, through interaction (14), to a divergence identical with that ascribable to a term.

$$H' = -\frac{3f^2}{4\pi} \int \frac{d\mathbf{l}}{l} \phi^* \beta \mu \phi \quad f^2 = 2e^2 \quad (22')$$

* Koba and Takada⁽⁷⁾ have also calculated the Compton scattering by fourth order perturbation, and obtained the same result as (29). They used the method of self-consistent subtraction, making the 2nd and 3rd terms converge by the renormalization of mass and charge respectively, while the 1st term was still being investigated. We wish to thank them for their many useful advices given us during our calculations.

corresponding to (22), in the Hamiltonian.

Further, the divergences of types (iv), (vii) cancel each other and become finite, so that the only divergence due to a C -meson is that of (22'). This (22') exactly cancels (22) so we see that in this case too, the C -meson performs the function of making the electronic mass term converge.

Further, if two charged scalar or pseudoscalar fields are also introduced, these give rise to the pair creation of the corresponding particles instead of electrons in the above processes a2) and b); and from types (i) to (iv) of the former, calculations on exactly similar lines with (25), (26), (27) using interaction (5), lead to the following, divergences.

$$H_2' = + \frac{e^2}{3\pi} \int p dp A^2 \quad (24')$$

$$H_3' = - \frac{e^2}{6\pi} \int \frac{dp}{p} (\psi^* a A \psi) \quad (28')$$

The divergence involved by process (v) of a2) becomes, as before, only that due to (24'), while the processes of b) cause no divergence.

The divergence of (24') cancels that of (24), while those of (28) and (28') are equal, so that when the whole system

[electromagnetic field + C -meson field]
+ [electron + 2 scalar (or pseudoscalar) mesons]

is taken into account, the resultant divergence is

$$- \frac{e^3}{2\pi} \int \frac{dp}{p} \psi^* a A \psi \quad (30)$$

Further, we should also consider the existence of a proton, or, more generally n Fermi particles and $2n$ (pseudo) scalar mesons corresponding to them, in which case the divergence becomes, instead of (30),

$$- \frac{ne^3}{2\pi} \int \frac{dp}{p} \psi^* (a A) \psi \quad (31)$$

This is characterized by the fact that it is proportional to e , so we tentatively rewrite interaction (2) as follows:

$$\begin{aligned} & e \psi^* a A \psi - \frac{ne^3}{2\pi} \int \frac{dp}{p} \psi^* (a A) \psi + \frac{ne^3}{2\pi} \int \frac{dp}{p} \psi^* (a A) \psi \\ &= \left(1 - \frac{ne^3}{2\pi}\right) e \psi^* a A \psi + \frac{ne^3}{2\pi} \int \frac{dp}{p} \psi^* a A \psi \\ &= e_{0b} \psi^* a A \psi + \frac{ne^3}{2\pi} \int \frac{dp}{p} \psi^* a A \psi \end{aligned} \quad (32)$$

where e_{0b} denotes the observable charge. Thus it is possible to cancel the

divergence of (31) by the second term appearing in (32) through performing the charge renormalization implied in it, regarding the actually observed charge as already including (31).

This is the method of self-consistent subtraction already proposed by Tomonaga and Koba, and Lewis,⁽⁶⁾ and it is being used also against the divergence of (22), in which case it is adapted to a renormalization of mass to take the place of introducing the C -meson in order to remedy the difficulty. However, the concept of mass renormalization presents, as indicated by Pais, certain difficulties against the particle aspect of the electron,⁽⁹⁾ while the expression in (24), too, has the drawback that it necessitates a radical change of the Maxwell field. On our part, we are looking forward to a more substantial means of coping with the expression (31), but have tentatively used the method of charge renormalization, as it does not seem to present any particular difficulty just now.

Incidentally, we would remark that the divergence which had appeared on using the damping theory of Heitler and Wilson⁽¹⁰⁾ correspond to the self-energies of the initial and final states, and can be eliminated by introducing the C -meson and a charged scalar (or pseudoscalar) meson.

In concluding, we express our sincere gratitude to Prof. Sakata, Dr. Taketani and Dr. Tanikawa for their guidance throughout this work.

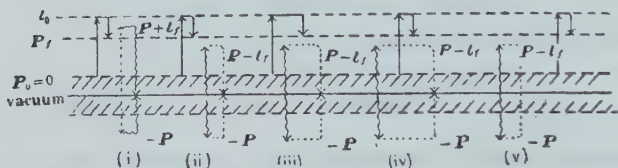


Fig. 4

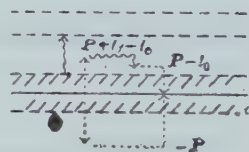


Fig. 5

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A Self-Consistent Subtraction Method in the Quantum Field Theory. II.

Hiroshi FUKUDA, Yoneji MIYAMOTO and Sin-itiirô TOMONAGA,

*Physics Institutes, Tokyo University and
Tokyo Bunrika Daigaku.*

(Received Sep. 23, 1948)

§ 5. Approximate Treatment for the Electron with Non-Relativistic Velocity.*

We now enter into the second stage of the non-relativistic approximation assuming that $p_1/\sqrt{p_1^2+x^2} \ll 1$, $p_2/\sqrt{p_2^2+x^2} \ll 1$. We thus expand the integrands in (V') in power series of $p_1/\sqrt{p_1^2+x^2}$ and $p_2/\sqrt{p_2^2+x^2}$ and retain terms up to the second order. We put namely

$$P_4'^2 = a^2 \pm 2a, \quad P_4''^2 = a^2 \pm 2b \quad (16)$$

with

$$a^2 = K_4^2 + P_{24}^2, \quad a = (kp_2), \quad b = (kp_1) \quad (16')$$

and expand various quantities in powers of a/a and b/a , for instance

$$\left. \begin{aligned} P_4' &= a \left(1 \pm \frac{a}{a^2} - \frac{a^2}{2a^4} \right), \quad \frac{1}{P_4''} = \frac{1}{a} \left(1 \mp \frac{b}{a^2} + \frac{3b^2}{2a^4} \right) \\ \frac{1}{K_4 + P_4' \mp P_{24}} &= \frac{1}{\pm 2(K \cdot P_2)} \left\{ a \left(1 \pm \frac{a}{a^2} - \frac{a^2}{2a^4} \right) - K_4 \pm P_{24} \right\} \end{aligned} \right\} \quad (16'')$$

thereby noticing that in this approximation we may put

$$|\mathbf{p}_1| = |\mathbf{p}_2| = p \quad (16''')$$

because the difference between $|\mathbf{p}_1|$ and $|\mathbf{p}_2|$ occurs only in the higher approximation. Now, if we carry out the integration over $d|\mathbf{k}|$ in (V) with (V'), we find immediately that the integral of (A), (B), (C), (E) and (F) diverge logarithmically on the ultra-violet side, and the integrals of (A), (D), (E) and (F) on the infra-red side. So we first avoid the divergency assuming an upper limit

$$k_{\text{up}} = G \sqrt{p^2 + x^2} \quad (17)$$

and a lower limit

$$k_{\text{low}} = \varepsilon \sqrt{p^2 + x^2} \quad (17')$$

for the integration over $d|\mathbf{k}|$. Putting

* §§ 1, 2, 3 and 4 of this appeared in Prog. Theor. Phys. 4 (1949), 47.

$$\left. \begin{aligned} a' &= \frac{a}{|k| \sqrt{p^2 + x^2}} = \frac{(ep_2)}{\sqrt{p^2 + x^2}}, \quad b' = \frac{b}{|k| \sqrt{p^2 + x^2}} = \frac{(ep_1)}{\sqrt{p^2 + x^2}} \\ c_\lambda &= \frac{K_\lambda}{|k|} = (e, i) \end{aligned} \right\} \quad (18)$$

we find, after the integration over $d|k|$

$$\left. \begin{aligned} \int (A) \quad d|k| &= i c_\nu (\gamma \cdot e) \left\{ \frac{3}{2} \ln (G + \sqrt{G^2 + 1}) - \frac{5}{4} + \frac{3}{4} (a' + b') \right. \\ &\quad + \frac{5}{4} (a'^2 + a' b' + b'^2) \left. \right\} + i \{ -\gamma_\nu P_1 P_2 - 2x \sqrt{p^2 + x^2} \delta_{\nu 4} \} / \sqrt{p^2 + x^2} \\ &\quad \cdot \left\{ -\ln 2\varepsilon \cdot \frac{\sqrt{p^2 + x^2}}{(P_1 \cdot e)(P_2 \cdot e)} + 1 + \frac{a' + b'}{2} + \frac{5}{6} (a'^2 + a' b' + b'^2) \right\} \\ &\quad + i \{ \gamma_\nu (P_1 + P_2) \cdot e - (P_1 + P_2)_\nu \gamma \cdot e + 2x \delta_{\nu 4} + i c_\nu x \} / \sqrt{p^2 + x^2} \\ &\quad \cdot \left\{ 1 + \frac{3}{2} (a' + b') + a'^2 + a' b' + b'^2 \right\} \\ \int (B) \quad d|k| &= i \{ i \gamma_4 (P_1 + P_2)_\nu + 2 \gamma_\nu \sqrt{p^2 + x^2} - x \delta_{\nu 4} \} (a' + b') / \sqrt{p^2 + x^2} \\ &\quad + 2(-i \delta_{\nu 4} \gamma \cdot e - \gamma_\nu - i \gamma_4 e_\nu) i \{ \ln (G + \sqrt{G^2 + 1}) - 1 \\ &\quad + \frac{1}{2} (a'^2 + a' b' + b'^2) \} \\ \int (C) \quad d|k| &= i \left(\frac{\gamma_\nu - 2 \delta_{\nu 4}}{2} \right) \{ 2 \ln (G + \sqrt{G^2 + 1}) - 2 \\ &\quad - a' - b' + a'^2 + a' b' + b'^2 \} \\ \int (D) \quad d|k| &= x \delta_{\nu 4} \left\{ \frac{2 \ln 2\varepsilon \sqrt{p^2 + x^2}}{(P_1 \cdot e)(P_2 \cdot e)} - \frac{a'^2 + a' b' + b'^2}{3} \right\} / \sqrt{p^2 + x^2} \\ \int (E) \quad d|k| &= \frac{1}{2} \left[\gamma_4 \gamma \cdot e \gamma_\nu \left\{ + \frac{1}{2} \ln (G + \sqrt{G^2 + 1}) - \frac{1}{4} + \frac{3}{4} a'^2 \right\} \right. \\ &\quad \left. - i \gamma_4 \gamma_\nu \sqrt{\frac{x a'}{p^2 + x^2}} + x^2 \gamma_\nu i \left\{ \frac{\ln 2\varepsilon \sqrt{p^2 + x^2}}{(P_2 \cdot e)^2} + a' + \frac{a'^2}{2} \right\} \right] / \sqrt{p^2 + x^2} \\ \int (F) \quad d|k| &= \frac{1}{2} \left[\gamma_\nu \gamma_4 \gamma_4 \left\{ + \frac{1}{2} \ln (G + \sqrt{G^2 + 1}) - \frac{1}{4} + \frac{3}{4} b'^2 \right\} \right. \\ &\quad \left. - i \gamma_\nu \gamma_4 \sqrt{\frac{x b'}{p^2 + x^2}} - x^2 \gamma_\nu i \left\{ \frac{\ln 2\varepsilon \sqrt{p^2 + x^2}}{(P_1 \cdot e)^2} + b' + \frac{b'^2}{2} \right\} \right] / \sqrt{p^2 + x^2} \end{aligned} \right\} \quad (VI')$$

Collecting together the terms diverging on the ultra-violet side we have

$$\begin{aligned} i \ln \{ G + \sqrt{G^2 + 1} \} &\left\{ -\frac{3}{2} e_\nu \gamma \cdot e - 2i \delta_{\nu 4} \gamma \cdot e - 2i \gamma_4 e_\nu \right. \\ &\quad \left. - (\gamma_\nu + 2 \delta_{\nu 4} \gamma_4) - \frac{i(\gamma_4 \gamma \cdot e \gamma_\nu + \gamma_\nu \gamma \cdot e \gamma_4)}{4} \right\} \end{aligned} \quad (19)$$

which, integrated over the direction of \mathbf{e} , gives

$$\left. \begin{aligned} i2\pi \ln(G + \sqrt{G^2 + 1})(\gamma_\nu - 2\gamma_\nu + \gamma_\nu) &= 0 && \text{for } \nu \neq 4 \\ i2\pi \ln(G + \sqrt{G^2 + 1})(-3\gamma_4 + 4\gamma_4 + 4\gamma_4 - 6\gamma_4 + \gamma_4) &= 0 && \text{for } \nu = 4. \end{aligned} \right\} (19')$$

This fact shows that, although separately the terms (A), (B), (C), (E) and (F) diverge, the energy density (A) + (B) + + (F) as a whole converges on the ultra-violet side. Collecting together the terms diverging on the infra-red side we obtain, on the other hand.

$$-\frac{i\gamma_\nu}{2} \ln 2\epsilon \cdot \left\{ \frac{P_1}{(\epsilon \cdot P_1)} - \frac{P_2}{(\epsilon \cdot P_2)} \right\}^2. \quad (20)$$

It is to be noticed that the expression (20) has the same form with the expression for the difference between the self-fields of an electron with the momentum \mathbf{p}_1 and \mathbf{p}_2 respectively; the expression found by Bloch and Nordsieck⁽¹⁾ as well as by Pauli and Fierz⁽²⁾ in the non-relativistic approximation. This term diverges logarithmically when we go over to the limit $\epsilon \rightarrow 0$.

Although our expression for the effective interaction energy contains thus a divergency on the infra-red side, this makes no difficulty. In applying our result to the scattering problem, for instance, this divergency can be eliminated if one takes into account the scattering process with the emission of a photon. In the calculation of the energy-level shift it is eliminated by the contribution from the second order effect caused by the term (III) because the latter effect has the same divergency with the opposite sign. This situation will be discussed in the next section.

Now the integration of (VI') over the direction of \mathbf{e} and the further use of the relation (14) gives

$$\begin{aligned} & -\frac{i\omega^2}{2} \int \phi_{P_2}^\dagger \{ (A) + (B) + \dots + (F) \} \phi_{P_1} \frac{d\mathbf{k}}{(2\pi)^3} \\ &= -\frac{e^2}{4\pi^2} \phi_{P_2}^\dagger \left[-\frac{2}{3} + \frac{17}{72} \frac{(\mathbf{p}_1 - \mathbf{p}_2)^2}{p^2 + x^2} - \frac{1}{3} \ln 2\epsilon \cdot \frac{(\mathbf{p}_1 - \mathbf{p}_2)^2}{p^2 + x^2} \right. \\ & \quad \left. - \frac{i\mathbf{x}(\mathbf{P}_1 + \mathbf{P}_2)_\nu}{4(p^2 + x^2)} \right] \phi_{P_1} \text{ for } \nu \neq 4 \\ &= -\frac{e^2}{4\pi^2} \phi_{P_2}^\dagger \left[\gamma_4 \left\{ -\frac{1}{2} + \frac{53}{72} \frac{(\mathbf{p}_1^2 + \mathbf{p}_2^2)}{p^2 + x^2} + \frac{(\mathbf{p}_1 \mathbf{p}_2)}{36(p^2 + x^2)} \right\} \right. \\ & \quad \left. - \frac{1}{3} \ln 2\epsilon \cdot \frac{(\mathbf{p}_1 - \mathbf{p}_2)^2}{p^2 + x^2} + \frac{x}{\sqrt{p^2 + x^2}} \left(\frac{1}{2} - \frac{\mathbf{p}_1^2 + (\mathbf{p}_1 \mathbf{p}_2) + \mathbf{p}_2^2}{3} \right) \right] \phi_{P_1} \text{ for } \nu = 4 \end{aligned} \quad (21)$$

which are correct up to the order of $\mathbf{p}_i^2/(p^2 + x^2)$ and $\mathbf{p}_i^2/(p^2 + x^2)$. Returning back from the momentum representation to the representation in the

coordinate space using the relations

$$\begin{aligned}
 & \int d\mathbf{x} \int \mathfrak{U}_\nu^\circ \phi_{P_2}^\dagger \frac{(\mathbf{p}_1 - \mathbf{p}_2)^2}{p^2 + x^2} \gamma_\nu \phi_{P_1} \exp \{i(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{x}\} d\mathbf{p}_1 d\mathbf{p}_2 \\
 & \quad = -\frac{1}{x^2} \int \phi_{P_1}^\dagger \gamma_\nu \phi_{P_2} \mathfrak{U}_\nu^\circ d\mathbf{x} \\
 & \int d\mathbf{x} \int \mathfrak{U}_\nu^\circ \phi_{P_2}^\dagger i(P_1 + P_2)_\nu \phi_{P_1} \exp \{i(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{x}\} d\mathbf{p}_1 d\mathbf{p}_2 \\
 & \quad = -2x \int \mathfrak{U}_\nu^\circ \phi_{P_2}^\dagger \gamma_\nu \phi_{P_1} d\mathbf{x} + \sum_{\nu \neq \mu} \int \phi_{P_1}^\dagger \gamma_\nu \gamma_\mu \frac{\partial \mathfrak{U}_\nu^\circ}{\partial x_\mu} \phi_{P_2} d\mathbf{x} \\
 & \quad = -2x \int \mathfrak{U}_\nu^\circ \phi_{P_2}^\dagger \gamma_\nu \phi_{P_1} d\mathbf{x} - \int \phi_{P_1}^\dagger \left\{ i\sigma \text{rot } \mathbf{A} - \rho_1 \sigma \frac{\partial \mathbf{A}}{\partial x^2} \right\} \phi_{P_2} d\mathbf{x} \\
 & \int d\mathbf{x} \int \mathfrak{U}_4^\circ \phi_{P_2}^\dagger \left(\frac{\mathbf{x}}{\sqrt{p^2 + x^2}} - \gamma_4 \right) \phi_{P_1} \exp \{i(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{x}\} d\mathbf{p}_1 d\mathbf{p}_2 \\
 & \quad = \int \phi_{P_1}^\dagger \left\{ \gamma_4 \frac{-p_1^2 - p_2^2}{2(p^2 + x^2)} \mathfrak{U}_4^\circ + \frac{ix\rho_1 \sigma}{2(p^2 + x^2)} \text{grad } \mathfrak{U}_4^\circ \right\} \phi_{P_2} d\mathbf{x}
 \end{aligned} \tag{22}$$

and further returning to the ordinary unit system by putting $\epsilon^2 \rightarrow \frac{4\pi e^2}{\hbar c}$, we can express the required effective energy density in the form

$$\begin{aligned}
 H_{eff}^{(1)} & \equiv \frac{ie^2}{2} \mathfrak{U}_\nu^\circ \int \phi_{P_2}^\dagger \{ (A) + (B) + \dots + (F) \} \phi_{P_1} \frac{d\mathbf{K}}{(2\pi)^3} \exp \{i(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{x}\} d\mathbf{p}_1 d\mathbf{p}_2 \\
 & = -\frac{\epsilon^2}{\pi \hbar c} \phi_{P_1}^\dagger \left\{ -\frac{1}{6} \rho_2 \sigma \mathbf{A} - \frac{\hbar^2}{12m^2 c^2} \mathfrak{U}_2^\circ \sigma \mathbf{J} \mathbf{A} \right. \\
 & \quad \left. + \frac{\hbar^2}{m^2 c^2} \left(\frac{1}{3} \ln 2\varepsilon - \frac{11}{72} \right) \gamma_4 \cdot \mathbf{J} \mathbf{A} + \frac{\hbar}{4mic} (i\rho_1 \sigma \mathbf{E} - \sigma \mathbf{H}) \right\} \phi_{P_2}. \tag{VII}
 \end{aligned}$$

Since we have

$$\phi_{P_1}^\dagger \left(\gamma_4 - \frac{\mathbf{x}}{\sqrt{p^2 + x^2}} \right) \phi_{P_2} \approx \left\{ \frac{P_1 \cdot P_2 + i\sigma [p_2 p_1]}{2} + \frac{P_1^2 + P_2^2}{4} \right\} \phi_{P_2}^\dagger \phi_{P_1}$$

in the Pauli approximation, we can express it also in the form

$$\begin{aligned}
 H_{eff}^{(1)} & = -\frac{\epsilon^2}{\pi \hbar c} \int \phi_{P_2}^\dagger \left[\left(\frac{1}{3} \ln 2\varepsilon - \frac{5}{18} \right) \frac{\mathbf{J} \mathfrak{U}_4^\circ}{x^2} + \frac{i\sigma}{4x} [p_2 p_1 \mathfrak{U}_4^\circ] \right] \phi_{P_1} \\
 & \quad \exp \{i(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{x}\} d\mathbf{p}_1 d\mathbf{p}_2 \quad \text{for } \nu=4. \tag{VII'}
 \end{aligned}$$

As was mentioned above it is necessary to take into account also the processes involving the emission of a photon in order to eliminate the divergency on the infra-red side. For such processes the energy density (III) was responsible. The evaluation of (III) is carried out quite easily because it contains no divergency. The result is found to be

$$H_{eff}^{(2)} = \int dX' [H', H'']_{Brems} = -\sqrt{\frac{4\pi}{\hbar c}} \int \frac{d\mathbf{K}}{(2\pi)^3} \frac{1}{|\mathbf{K}|^{\frac{1}{2}}} e^{-i(\mathbf{K} \cdot \mathbf{x})} d_{\mathbf{K}}^\dagger d_{\mathbf{K}}^\circ$$

$$\left[\phi^\dagger \gamma_v \left\{ \frac{P_{1\mu}}{(c \cdot P_2)} - \frac{P_{2\mu}}{(c \cdot P_1)} \right\} \phi + \frac{1}{2} \phi^\dagger \left\{ \frac{\tilde{r}_\mu (\tilde{r} \cdot K) \tilde{r}_\nu}{(c \cdot P_2)} + \frac{\tilde{r}_\nu (\tilde{r} \cdot K) \tilde{r}_\mu}{(c \cdot P_1)} \right\} \phi \right]. \quad (\text{III}')$$

In this way we have obtained the energy densities necessary for the calculation of the field reaction upon electron. We shall now go over to the application our result to some field-reaction problems.

§ 5. Application.

(a) Anomalous Magnetic Moment of an Electron.

The last term of (VII) expresses that the electron interacts with the magnetic field as if it had an extra magnetic moment of the magnitude

$$\delta\mu = \frac{1}{2\pi} \left(\frac{e^2}{\hbar c} \right) \mu. \quad (23)$$

This anomaly in the magnetic moment will be a cause of the deviation of the hyperfine structure of line spectra from the prediction of the ordinary theory.⁽⁶⁾ In this problem also the term $-(1/6)\mu_2 \sigma \mathcal{A}$ (VII) may play a rôle. This effect will be accounted for as a reinterpretation of the nuclear moment, but its appearance is somewhat problematic as will be mentioned in the next section.*

(b) Energy-Level Shift of a Hydrogen-like Atom.

The interaction energy (VII') causes a part of the level shift of the bound electron as the first order effect, which together with the second order effect due to the interaction energy (III') contributes to the total level shift. This part of the level shift can be calculated in the following way:

Transforming back from the momentum representation to the representation in the coordinate space and integrating over the space, the interaction energy corresponding to the second term of (VII') becomes

$$\begin{aligned} \int H_{eff}^{(1)} d\mathbf{x} &= -\frac{e^2}{\pi\hbar c} \int \phi_{x_2}^\dagger \gamma_1 \frac{i\sigma[\mathbf{p}_2 \mathbf{p}_1]}{4x^2} \mathcal{A}_4^\circ \phi_{x_1} \exp \{i(\mathbf{p}_1 - \mathbf{p}_2) \cdot \mathbf{x}\} d\mathbf{x} d\mathbf{p}_1 d\mathbf{p}_2 \\ &= -\frac{e^2}{\pi\hbar c} \int \phi^\dagger \text{grad } \mathcal{A}_4^\circ \left[\frac{i\sigma \text{grad } \phi}{4x^2} \right] d\mathbf{x} \end{aligned}$$

Provided that \mathcal{A}_4° has a spherical symmetry, it is

$$= -\frac{e^2}{\pi\hbar c} \int \frac{1}{r} \frac{d\mathcal{A}_4^\circ}{dr} \phi^\dagger \left\{ \frac{1}{4x^2} \{j(j+1)l(l+1) - s(s+1)\} \right\} \phi d\mathbf{x}. \quad (24)$$

Thus the energy corresponding to (VII') can be expressed as

$$\begin{aligned} H_{eff}^{(1)} &= -\frac{e^2}{\pi\hbar c} \phi^\dagger \gamma_1 \left\{ \left(\frac{1}{3} \ln 2\varepsilon - \frac{5}{18} \right) \frac{d\mathcal{A}_4^\circ}{x^2} \right. \\ &\quad \left. + \frac{1}{4x^2} \{j(j+1) - l(l+1) - s(s+1)\} \frac{1}{r} \frac{d\mathcal{A}_4^\circ}{dr} \right\} \phi, \quad (\text{VIII}) \end{aligned}$$

*See note 2 added at the end of this paper.

or, in case of $\mathcal{U}_4^0 = i \frac{Zc^2}{r}$

$$= \frac{e^2}{\pi \hbar c} \frac{Zc^2}{x^2} \psi^* \left[\left(\frac{1}{3} \ln 2\epsilon - \frac{5}{18} \right) 4\pi \delta(x) + \frac{1}{4x^3} \{ j(j+1)l(l+1) - s(s+1) \} \right] \psi. \quad (\text{VIII}')$$

The part of the level shift under consideration is now calculated as the mean value of this energy. For the $2S_{1/2}$ -level the second term gives no contribution and the shift is given by

$$\Delta E^{(1)}(2S_{1/2}) = -\frac{e^2}{\pi \hbar c} \frac{Zc^2}{x^2} \left(\frac{1}{3} \ln 2\epsilon - \frac{5}{18} \right) \frac{1}{2a^3}, \quad a = \frac{\hbar^2}{Ze^2 m} \quad (25)$$

For the $2P_{1/2}$ -level, on the other hand, it is contributed from the second term only. It is

$$\Delta E^{(1)}(2P_{1/2}) = -\frac{e^2}{\pi \hbar c} \frac{Zc^2}{x^2} \frac{1}{2} \left\langle \frac{1}{r^2} \right\rangle_{Av} = -\frac{e^2}{\pi \hbar c} \frac{Zc^2}{2x^2} \frac{1}{24a^3} \quad (26)$$

As mentioned above the divergency $\ln 2\epsilon$ in (25) can be eliminated by the second part of the level shift arising from the interaction energy (III'). Neglecting the small contribution from the second term in (III') and neglecting the retardation too, the calculation for this part can be carried out in a similar manner as was done by Bethe.⁽⁴⁾ It is found that only $2S_{1/2}$ -term undergoes the shift and the effect diverges on the infra-red side in the same way as in the first part, and, if we assume the same lower limit ϵ for the integration over the momentum k of the virtual photon, we obtain the level shift of the magnitude

$$\Delta E^{(2)}(2S_{1/2}) = \frac{e^2}{\pi \hbar c} \frac{Zc^2}{x^2} \frac{1}{3} \left\{ \ln \frac{x}{\langle E_n - E_m \rangle_{Av}} - \ln \epsilon \right\} \frac{1}{2a^3}. \quad (25')$$

Combining the two effects the total energy-level shift for $2S_{1/2}$ -term is found to be

$$\begin{aligned} \Delta E(2S_{1/2}) &= \Delta E^{(1)}(2S_{1/2}) + \Delta E^{(2)}(2S_{1/2}) \\ &= \frac{e^2}{3\pi \hbar c} \frac{Zc^2}{x^2} \left\{ \ln \frac{x}{\langle E_n - E_m \rangle_{Av}} - \ln 2 + \frac{5}{6} \right\} \frac{1}{2a^3} \end{aligned} \quad (27)$$

If we use Bethe's value

$$\ln \frac{x}{\langle E_n - E_m \rangle_{Av}} = 7.63, \quad \Delta E(2S_{1/2}) = 1040 \text{ Mcycles}, \quad (28)^*$$

the total level shift for the $2S_{1/2}$ term is

$$\Delta E(2S_{1/2}) = 1040 - \frac{5/6 - \ln 2}{7.63} \times 1040 = 1059 \text{ Mcycles}. \quad (27')$$

For the $2P_{1/2}$ term, on the other hand, (26) gives

*See note 1 added at the end of this paper.

$$\Delta E(2P_{1/2}) = -17 \text{ Mcycles.} \quad (26')$$

From (27) and (25') the observable line shift is found to be

$$\Delta E(2E_{1/2}) - \Delta E(2P_{1/2}) = \frac{e^2}{\pi \hbar c} \frac{Zc^2}{6x^2 a^3} \left\{ \ln \frac{x}{\langle E_n - E_m \rangle_{Av}} - \ln 2 + \frac{23}{24} \right\} \quad (29)$$

$$= 1076 \text{ Mcycle} \quad (29')$$

(c) e^2 -Correction to the Rutherford Formula.⁽⁵⁾

A correction to the cross section for the elastic scattering of an electron is caused by the first term of (VII'). The relative correction due to this term is found to be

$$\frac{\delta \sigma_{el}}{\sigma_0} = -\frac{e^2}{\pi \hbar c} \frac{(p_2 - p_1)^2}{x^2} \left(-\frac{2}{3} \ln \frac{x}{2\varepsilon} + \frac{5}{9} \right) \quad (30)$$

As mentioned in the preceding section, the divergency on the infra-red side is eliminated if we add to this correction a further one arising from the possibility of inelastic collisions. The correction to the Rutherford scattering due to this possibility was calculated by Mott⁽⁶⁾ who found the relative correction of the magnitude

$$\frac{\delta \sigma_{in el}}{\sigma_0} = \frac{e^2}{\pi \hbar c} \frac{(p_2 - p_1)^2}{x^2} \left(\frac{2}{3} \ln \frac{x}{\varepsilon} + \frac{4}{3} \ln \frac{p}{m} + \frac{2}{3} \ln 2 - 1 \right) \quad (30')^{(5)}$$

Adding the contributions from both elastic and inelastic collisions, we find the total correction

$$\frac{\delta \sigma_{el} + \delta \sigma_{in el}}{\sigma_0} = \frac{e^2}{\pi \hbar c x^2} (p_2 - p_1)^2 \left(-\frac{4}{3} \ln \frac{2p}{m} - \frac{14}{9} \right) \quad (31)$$

§ 6. Concluding Remarks.

In concluding the paper we shall add the following remarks. In our calculation we have dropped off altogether the term (f) in (IV₁) + (IV₂) which represents the effect of the vacuum polarization due to the external field \mathfrak{H}_1^0 . But it is still problematic whether this procedure corresponds to the correct prescription. It is possible that only the diverging part in this expression is to be subtracted, thereby the separation of the diverging part from the converging remainder being performed by some suitable prescription. For instance, it is possible to set up a prescription in such a way that the converging part separated has the same form as that obtained by Serber⁽⁷⁾ on the ordinary positron theory. Then the effect of this converging remainder will be that which was calculated by Uehling,⁽⁸⁾ who found the level shift of -27 Mcycle for the S -level. When we take account of this effect, the total shift for the $2S_{1/2}$ -level becomes 1032 Mcycles.

and the observable line shift becomes 1049 Mcycles.*

The second fact we must emphasize is again about the provisional character of our calculation because of its imperfectness in view of the relativistic invariance. In evaluating the improper expression of the form $\infty - \infty$ it is necessary to carry through the calculation in obedience to some definite prescription which must be, of course, relativistic invariant and free from any ambiguity arising from the pathological nature of the D -functions occurring in our formalism. As a matter of fact, our method of calculating the integral over dX' , first expanding the D -functions in Fourier integrals and then use the formula (7) of I, gives often results depending on the order of integration, on the choice of the integration variables, on the reference system in which the integrations are performed and so on. Thus, for instance, in the calculation of the electromagnetic mass correction described in I we had to evaluate the integral according to the following disposition: we first transform the reference system into such one in which the electron is at rest and perform the integral in this system then we go back to the original system making use of the relativistically invariant character of the formula. If one carried out here the calculation in a different way, one could not obtain the correct answer. So, it may possibly occur that also the calculation in this paper would not give the correct answer because the calculation was carried out in a non-relativistic approximation from the beginning. In fact, our result has a singular property that it has quite a different form for space and time component, and this seems to show the incorrectness of our way of calculation. As one sees in (21), for instance, the expression for the energy has different form for $\nu \approx 4$ and for $\nu = 4$. The appearance of the term $-(1/6)\rho_3\sigma A$ in (VII) seems also to show the inadequateness of our calculation,** because this term requires that the electron charge must be interpreted when it interacts with a magnetic field whereas no such reinterpretation is required when it interacts with an electric field. In the latest issue of the Physical Review we found that the unpublished result of Bethe and Schwinger of the same problem was cited by Fowler⁽⁹⁾. Comparing our result (29) with theirs, we find that in their formula $1/2$ stands in the place where $23/24$ stands in ours. This difference will probably be due to the difference of their treatment from ours.* It is desirable to develop a scheme of calculation which fits better

*See the note 1 added at the end of this paper.

**See the note 2 added at the end of this paper.

for the use of the D -function than the imperfect scheme used by us, which perhaps will be done along the line suggested lately by Nambu.⁽¹⁰⁾

It will be of much interest to compare our results with those obtained on the cohesive force-field hypothesis. Such a calculation is undertaken by the group of Nagoya University.

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- (9) G. R. Fowler, Phys. Rev. **74** (1948), 219.
- (10) Y. Nambu, to be published in this journal.

Note added in proof. Several papers of American authors treating the same problem appeared meanwhile in the Physical Review (W. M. Kroll and W. E. Lamb, Phys. **75** (1949), 388, J. B. French and V. F. Weisskopf, *ibid.* 1240; J. Schwinger, *ibid.* 651.), so we should like to add some remarks in order to relate our results to theirs.

Note 1. Corresponding to our value 1049 Mcycles for the line shift mentioned in the first line of page 128, they obtained 1051 Mcycles. This difference is due to the different values of

$\ln \frac{\chi}{\langle E_n - E_m \rangle_{av}}$ used. If we use instead of 7.63 Bethe's new value 7.6876 we obtain the same value as theirs. In fact, the analytical expression for this line shift is

$$\Delta E(2S_{\frac{1}{2}}) - \Delta E(2P_{\frac{1}{2}}) = \frac{e^2}{\pi \epsilon c} \frac{Zc^2}{6x^2 a^3} \left\{ \ln \frac{\chi}{\langle E_n - E_m \rangle_{av}} - \ln 2 + \frac{23}{24} - \frac{1}{5} \right\}$$

which is obtained by adding Uehling's result to our (29) agrees with the corresponding expression of the cited authors.

Note 2. The problematic term was occurring also in the work of American authors, which could be eliminated by using Pauli's "regulator"

Radiation Reaction in Collision Process. III.

— *First Radiative Correction for an Arbitrary Process Including
Electrons, Positrons, and Light Quanta* —

Zirô Koba and Gyô Takeda.

*Institute of Physics, Faculty of Science,
Tokyo University.*

(Received Oct. 4, 1948)

§ 5. Divergence of Mass Type.

It is sufficient to investigate the case $N=1$ as long as we restrict ourselves to the first radiative corrections; the case $N=2$ does not bring any new aspect. We denote as before the emission and the absorption of the virtual photon by j^* and j respectively. Those connections which could contribute to divergence are the following:

- (I)
- | | |
|--|-------|
| $(\dots k, j^*, j, l \dots \mid \dots j^*, j \dots)$ | (5.1) |
| $(\dots k, j^*, j, l \dots \mid \dots \text{Coulomb} \dots)$ | (5.2) |
| $(\dots k, j, j^*, l \dots \mid \dots j^*, j \dots)$ | (5.3) |
| $(\dots k, j, j^*, l \dots \mid \dots \text{Coulomb} \dots)$ | (5.4) |
- (II)
- | | |
|---|-------|
| $(\dots k, j^*, g, j, l \dots \mid \dots j^*, j \dots)$ | (5.5) |
| $(\dots k, j^*, g, j, l \dots \mid \dots \text{Coulomb} \dots)$ | (5.6) |
| $(\dots k, j, g, j^*, l \dots \mid \dots j^*, j \dots)$ | (5.7) |
| $(\dots k, j, g, j^*, l \dots \mid \dots \text{Coulomb} \dots)$ | (5.8) |
- (III)
- | | |
|---|--------|
| $(\dots k, j^*, j, l \dots \mid \dots j^*, g, j \dots)$ | (5.9) |
| $(\dots k, j, j^*, l \dots \mid \dots j^*, g, j \dots)$ | (5.10) |
- (IV)
- | | |
|--|--------|
| $(\dots k, j^*, g, j, l \dots \mid \dots j^*, g, j \dots)$ | (5.11) |
| $(\dots k, j, g, j^*, l \dots \mid \dots j^*, g, j \dots)$ | (5.12) |

Here k , l and g are certain elements of the process considered (emission or absorption of real photons); "Coulomb" in the latter sequence means

* §§ 1, 2, 3 and 4 appeared in Prog. Theor. Phys. 4 (1949), 60.

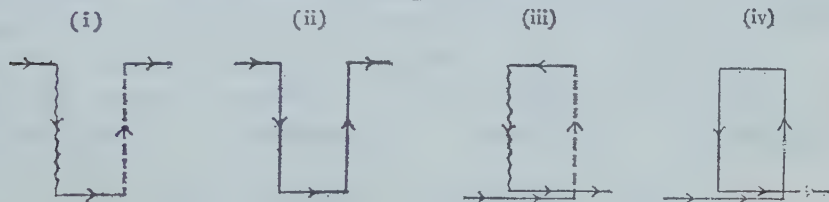
that j^* , j are to be replaced by "longitudinal photons". Other connections do not contribute to divergence because a virtual photon or so stays too long and the energy-denominator includes a high power in $|j|$.

In (1), which would yield the worst divergence, we are to distinguish between two cases according as $\{t(k) - t(j)\} \{t(l) - t(j)\} \geq 0$, if we denote by $t(k)$ the position of the element k in the sequence of the field time:

(a) $t(k) < t(j^*) < t(j) < t(l)$.

In this case (5.1), (5.2), (5.3) and (5.4) are represented by the diagrams of Fig. 8. One will at once notice that this is the very figure of self-energy (Fig. 2). Indeed this process is to be interpreted as the self-energy or the mass increase of the intermediate electron. Denoting by p the height of the electron-line just after the element k in the proper time, and by E_i the total energy of the intermediate state just before $t(j^*)$ (in

Fig. 8



field time) we obtain as correction the following factor to be multiplied into the matrix element of the corresponding uncorrected connection (which can be derived out of the above sequence by striking off j and j^*):

$$\frac{e^2}{4\pi^2} \int d\mathbf{j} \left[\frac{\sum_{\eta} \alpha^*(\mathbf{p}) a_{\eta} \lambda_{p-j}^+ a_{\eta} \alpha(\mathbf{p})}{\{E_0 - E_i + E_p - E_{p-j} - j\} j (E_0 - E_i)} + \frac{\alpha^*(\mathbf{p}) \lambda_{p-j}^+ \alpha(\mathbf{p})}{j^2 (E_0 - E_i)} \right. \\ \left. - \frac{\sum_{\eta} \alpha^*(\mathbf{p}) a_{\eta} \lambda_{p-j}^- a_{\eta} \alpha(\mathbf{p})}{\{E_0 - E_i - E_p - E_{p-j} - j\} j (E_0 - E_i)} - \frac{\alpha^*(\mathbf{p}) \lambda_{p-j}^- \alpha(\mathbf{p})}{j^2 (E_0 - E_i)} \right] \quad (5.13)$$

the diverging part of which turns out

$$\left(\frac{3e^2 m}{2\pi} \int \frac{dj}{j} \right) \frac{\alpha^*(\mathbf{p}) \beta \alpha(\mathbf{p})}{E_0 - E_i} \quad (5.14)$$

In (5.13) and (5.14) a_{η} means the component of a in the direction of the polarization unit vector of the photon j ; otherwise the same notations are used as in II.

(a') $t(l) < t(j^*) < t(j) < t(k)$.

This case is similar to (a); only the direction of the flow of proper time is inverted, corresponding to the self-energy of an intermediate posi-

tron $(-p)^+$. The diverging part of the correction factor is

$$\left(\frac{3e^2m}{2\pi}\int\frac{dj}{j}\right)\frac{b^*(-p)\beta b(-p)}{E_0-E_i} \quad (5.15)$$

(b) $t(k), t(l) < t(j^*) < t(j)$.

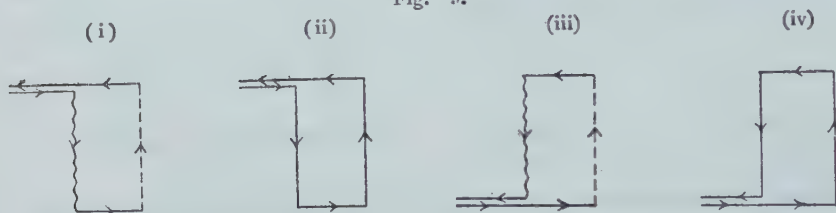
The corresponding diagrams are given in the Fig. 9.

The correction factor with its diverging part is found to be

$$\begin{aligned} \frac{e^2}{4\pi^2}\int dj \left[\frac{\sum_{\eta} b^*(-p) a_{\eta} \lambda_{p-j}^+ a_{\eta} \alpha(p)}{(E_0-E_i-2E_p)(E_0-E_i-E_p-E_{p-j}-j)j} + \frac{b^*(-p) \lambda_{p-j}^+ \alpha(p)}{(E_0-E_i-2E_p)j^2} \right. \\ \left. - \frac{\sum_{\eta} b^*(-p) a_{\eta} \lambda_{p-j}^- a_{\eta} \alpha(p)}{(E_0-E_i-2E_p)(E_0-E_i-E_p-E_{p-j}-j)j} - \frac{b^*(-p) \lambda_{p-j}^- \alpha(p)}{(E_0-E_i-2E_p)j^2} \right] \\ \sim \left(\frac{3e^2m}{2\pi}\int\frac{dj}{j}\right)\frac{b^*(-p)\beta\alpha(p)}{E_0-E_i-2E_p} \quad (5.16) \end{aligned}$$

This form is what we have called the non-diagonal part of the mass-correction.

Fig. 9.



(b') $t(j^*) < t(j) < t(k), t(l)$.

This case is related to (b) as (a') is to (a). The diverging part of the correction factor is

$$\left(\frac{3e^2m}{2\pi}\int\frac{dj}{j}\right)\frac{\alpha^*(p)\beta b(-p)}{(E_0-E_i-2E_p)} \quad (5.17)$$

It is clear that the diverging corrections (5.14), (5.15), (5.16) and (5.17) are equivalent to those caused by a single perturbation term

$$\delta m u^* \beta u, \quad (5.18)$$

with

$$\delta m = \frac{3e^2m}{2\pi}\int\frac{dj}{j} \quad (5.19)$$

therefore all of them can be eliminated by introducing the counter-mass-term.

$$-\delta m u^* \beta u \quad (5.20)$$

into the interaction Hamiltonian.

When j and j^* stand in both sequences at the (left or right) end, we have a "renormalization correction" with factor $1/2$; but in this case also it is cancelled by the corresponding correction of the counter-term.

Having disposed of (I), we now go over to the discussion of (II), (III) and (IV). Each connection (5.5)–(5.12) yields a matrix element that diverges logarithmically, but when we combine (5.5) and (5.7), (5.6) and (5.8), (5.9) and (5.10), (5.11) and (5.12) respectively, the leading terms are cancelled out in every pair because of the opposite sign which originates from the one inversion $(j, j^* \leftrightarrow j^*, j)$, and we are left with convergent corrections.

Summing up we can state that the counter-self-energy term in the self-consistent subtraction method

$$-\delta m u^* \beta u$$

is sufficient to get rid of the mass-type divergence, though it has its counter part only for (I). The hypothesis of the cohesive force field,⁽⁸⁾ on the other hand introduces also the counterparts for (II), (III), (IV) and for other converging connections, too. The difference of the both methods will thus be found in the finite part of the mass-type correction.⁽⁹⁾

§ 6. Divergence of Polarization Type.

First we shall treat $[P, I]$ corrections for the case $N=1$ in some detail. Take an arbitrary connection of the original (uncorrected) process and fix your eyes upon its certain step, the emission of a photon k , say. By virtue of the diagram method we can easily construct $[P, I]$ corrections with regard to this element in the following manner.

(a) Put an island-like connection consisting of two elements k^* and $(-k)^*$ into any position before the original $k(k^*)$ and at the same time

Fig. 10 (i)

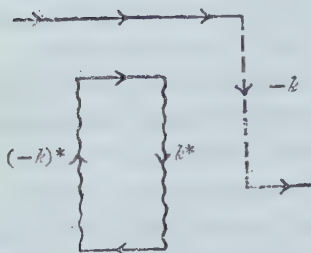


i) Uncorrected connection

replace the element k^* on the main electron-line by $(-k)$, which of course does not alter the configuration of the original diagram for the uncorrected connection. There are two ways of building the closed line according to the direction of the proper time, as will be seen from Fig. 10 (ii) and (iii).

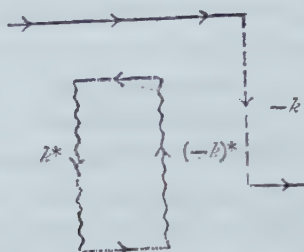
(b) A similar island-connection is put into such a position that its left element lies before the original $k(k^*)$ but its right element after the latter (Fig. 11). When the proper time of the island-connection

Fig. 10 (ii)



ii) Corrected Connection No. 1

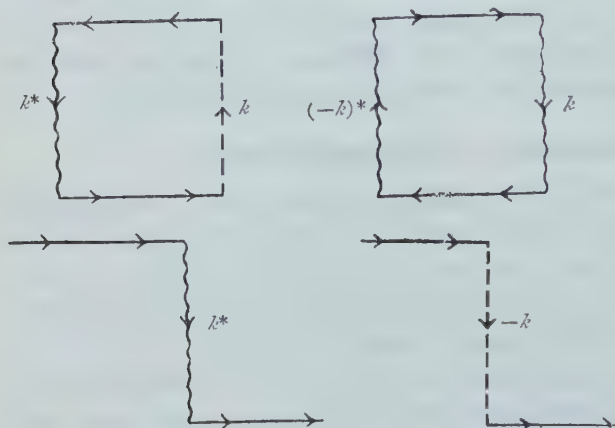
Fig. 10 (iii)



iii) Corrected Connection No. 2

flows counter-clockwise in this figure its elements are different from those of (a) (Fig. 11 (i)). In this case it is the effect of induced emission of k that makes the distinction from the pure vacuum.

Fig. 11



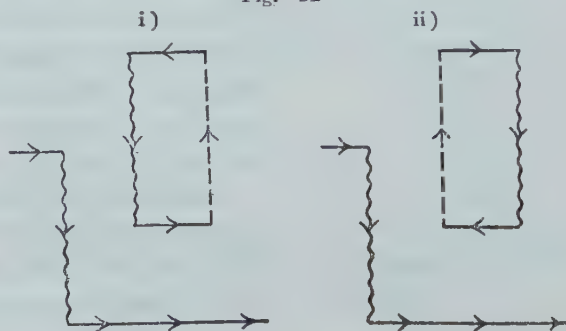
(c) A similar island-like connection is put into a position entirely after the original $i(k^*)$. Two cases are possible as before, beginning with induced emission or absorption of the emitted photon (Fig. 12). These processes are identical with those considered in the calculation of the self-

energy of a photon.

It is also possible that in some cases the emission and reabsorption of the virtual photon is replaced by Coulomb interaction. But such a process does not contribute at all since its matrix element vanishes.

In the above mentioned

Fig. 12



island-like figures, the vertical distance of the two horizontal segments are fixed, to be sure, but the figure as a whole can be removed in a vertical direction into any position since ν is arbitrary; and so the total effect is obtained by integrating with respect to ν .

The configurations of the $[P, I]$ corrections having been thus clarified, the next task is to decide where to put these two elements of the island-connection into the sequence in field time. The worst divergence will appear when we put the two elements close together in the sequence. Then the virtual pair stays during only one intermediate state, and the energy denominator of the transition matrix element contains $|\nu|$ linearly and the integration with respect to ν yields a quadratic divergence or, if we expand the integrand into the descending powers of $|\nu|^{-1}$, quadratic, linear and logarithmic divergence successively. These we may denote by $[S]$, $\{S\}$ and (S) respectively, when the island-connection is situated between the $(S-1)$ -th and the S -th elements of the uncorrected field-time-sequence, that is to say, when the polarization process begins at the S -th intermediate state of the original connection and is immediately over at the same stage. So we have $Q+1$ quadratic divergences $[0], [1], \dots, [Q]$ and $Q+1$ linear divergences $\{0\}, \{1\}, \dots, \{Q\}$ and $Q+1$ logarithmic divergences $(0), (1), \dots, (Q)$.

Next we consider the cases where one element of the original process intervenes between those of the island-connection in the sequence in field time. For example, the polarization process (island-connection) starts at the S -th intermediate state of the original connection, but is not completed before the main system goes over to its (original) $(S+1)$ -th intermediate state, where the pair annihilation is carried through. Such a correction affords a linear and a logarithmic divergence when the integrand is expanded into the power series of $|\nu|^{-1}$, which we denote by $\{S, S+1\}$ and $(S, S+1)$ respectively.

Further the case is to be considered when the polarization process starts at the S -th intermediate state and finishes at the $(S+2)$ -th. This will yield a logarithmic divergence, denoted by $(S, S+1, S+2)$.

When the polarization process is extended over more than three intermediate states, the integral gives only a converging correction and for the present purpose we may disregard all these cases.

Now we shall investigate the explicit forms of these diverging corrections.

As is suggested also by the closed configuration, the polarization cor-

rection makes up a separate "Spür" in the numerator of the transition matrix element and this is common to all the connections above described. The only difference between them lies in the energy-denominator to which the order in the field time is essential.

With regard to the quadratic divergence, elementary calculation yields the following results.

$$\begin{aligned}
 [0] &= -\left(\frac{4c^2}{3\pi k} \int r dr\right) \frac{H_0}{(-2k)(E_0 - E_1 - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})} \\
 [1] &= -\left(\frac{4c^2}{3\pi k} \int r dr\right) \frac{H_Q}{(E_0 - E_1)(E_0 - E_1 - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})} \\
 &\dots\dots\dots \\
 [n-1] &= -\left(\frac{4c^2}{3\pi k} \int r dr\right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-1})(E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})} \\
 [n] &= -\left(\frac{4c^2}{3\pi k} \int r dr\right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-1})(E_0 - E_n)^2(E_0 - E_{n+1}) \dots (E_0 - E_{Q-1})} \\
 &\dots\dots\dots \\
 [Q-1] &= -\left(\frac{4c^2}{3\pi k} \int r dr\right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-1})(E_0 - E_n) \dots (E_0 - E_{Q-2})(E_0 - E_{Q-1})^2} \\
 [Q] &= -\left(\frac{4c^2}{3\pi k} \int r dr\right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-1})(E_0 - E_n) \dots (E_0 - E_{Q-1})} \\
 &\quad \left\{ \frac{1}{E_0 - E_1} + \frac{1}{E_0 - E_2} + \dots + \frac{1}{E_0 - E_{Q-1}} \right\} \quad (6.1)
 \end{aligned}$$

Here the emission of real photon k under consideration is assumed to take place at the n -th step in the uncorrected connection; H_Q in the numerator means the same expression as in the numerator of the uncorrected case. $[Q]$ has a particular form, because it belongs to a case of so-called renormalization. (When absorption, not emission, of a real photon is concerned, the correction $[0]$ belongs to renormalization, while $[Q]$ has an ordinary from.) Anyhow, these terms are just cancelled out if we add to the inter-

action Hamiltonian a new term of the form

$$\frac{e^2}{3\pi^2} \int r dr \int A^2(\mathbf{X}) dV$$

because the latter implies processes just equivalent to the above mentioned ones.

There are two kinds of linear divergences as we have seen. They are given respectively by

$$\begin{aligned} \{0\} &= \left(\frac{2e^2}{3\pi} \int dr \right) \frac{II_0}{(-2k)(E_0 - E_1 - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})}, \\ \{1\} &= \left(\frac{2e^2}{3\pi} \int dr \right) \frac{\{k - (E_0 - E_1)\} II_0}{(E_0 - E_1)k(E_0 - E_1 - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})}, \\ \{2\} &= \left(\frac{2e^2}{3\pi} \int dr \right) \frac{\{k - (E_0 - E_2)\} H_Q}{(E_0 - E_1)(E_0 - E_2k)(E_0 - E_2 - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})}, \\ &\dots\dots\dots \\ \{n-1\} &= \left(\frac{2e^2}{3\pi} \int dr \right) \frac{\{k - (E_0 - E_{n-1})\} H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-1})k(E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})}, \\ \{n\} &= - \left(\frac{2e^2}{3\pi} \int dr \right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-1})(E_0 - E_n) \dots (E_0 - E_{Q-1})} \cdot \frac{1}{k}, \\ \{n+1\} &= \{n+2\} = \dots\dots\dots = \{Q-1\} = \{n\} \\ \{Q\} &= - \left(\frac{e^2}{3\pi} \int dr \right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-1})(E_0 - E_n) \dots (E_0 - E_{Q-1})} \cdot \frac{1}{k}; \end{aligned}$$

and

$$\begin{aligned} \{0, 1\} &= \left(\frac{2e^2}{3\pi} \int dr \right) \frac{II_0}{k(E_0 - E_1 - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})}, \\ \{1, 2\} &= \left(\frac{2e^2}{3\pi} \int dr \right) \frac{H_Q}{(E_0 - E_1)k(E_0 - E_2 - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})}, \\ &\dots\dots\dots \\ \{n-2, n-1\} &= \left(\frac{2e^2}{3\pi} \int dr \right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-1})k(E_0 - E_n) \dots (E_0 - E_{Q-1})}, \\ \{Q-1, Q\} &= \left(\frac{2e^2}{3\pi} \int dr \right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{Q-1})k}. \end{aligned}$$

Between these terms we find the following relations :

$$\begin{aligned} \{0\} + \frac{1}{2}\{0, 1\} &= 0 \\ \frac{1}{2}\{0, 1\} + \{1\} + \frac{1}{2}\{1, 2\} &= 0 \\ &\dots\dots\dots \end{aligned}$$

$$\frac{1}{2}\{n-1, n\} + \{n\} + \frac{1}{2}\{n, n+1\} = 0$$

.....

$$\frac{1}{2}\{Q-1, Q\} + \{Q\} = 0$$

Therefore, the linear divergences entirely disappear when summed up.

Finally we turn to the logarithmic divergences. By the origin they can be classified into three groups as stated before.

$$(i) \quad (0) = 0$$

$$(1) = -\left(\frac{c^2}{3\pi} \int \frac{dr}{r}\right) \frac{H_Q}{k(E_0 - E_Q - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})}$$

$$(2) = -\left(\frac{c^2}{3\pi} \int \frac{dr}{r^2}\right) \frac{H_Q}{(E_0 - E_1)k(E_0 - E_2 - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})}$$

.....

$$(n-1) = -\left(\frac{c^2}{3\pi} \int \frac{dr}{r}\right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-1})k(E_0 - E_{n+1}) \dots (E_0 - E_{Q-1})}$$

.....

$$(Q-1) = -\left(\frac{c^2}{3\pi} \int \frac{dr}{r}\right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{Q-2})k}; \quad (Q) = 0$$

$$(ii)$$

$$(0, 1) = \left(\frac{c^2}{3\pi} \int \frac{dr}{r}\right) \frac{H_Q}{k(E_0 - E_2 - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})}$$

$$(1, 2) = -\left(\frac{c^2}{3\pi} \int \frac{dr}{r}\right)$$

$$\times \frac{(2k - 2E_0 + E_1 + E_2)H_Q}{(E_0 - E_1)k(E_0 - E_2 - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})} \dots$$

$$(n-2, n-1) = -\left(\frac{c^2}{3\pi} \int \frac{dr}{r}\right)$$

$$\times \frac{(2k - 2E_0 + E_{n-2} + E_{n-1})H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-2})(E_0 - E_{n-1} - 2k)k(E_0 - E_n) \dots (E_0 - E_{Q-1})}$$

$$(n-1, n) = -\left(\frac{c^2}{3\pi} \int \frac{dr}{r}\right) \frac{(k - 2E_0 + E_{n-1} + E_n)H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-1})(E_0 - E_n) \dots (E_0 - E_{Q-1})k}$$

$$(n, n+1) = -\left(\frac{c^2}{3\pi} \int \frac{dr}{r}\right) \frac{(-2E_0 + E_n + E_{n-1})H_Q}{(E_0 - E_1) \dots (E_0 - E_n)(E_0 - E_{n-1}) \dots (E_0 - E_{Q-1})k}$$

.....

$$(Q-1, Q) = -\left(\frac{c^2}{3\pi} \int \frac{dr}{r}\right) \frac{(-E_0 + E_{Q-1})H_Q}{(E_0 - E_1) \dots (E_0 - E_{Q-1})k}$$

$$(iii)$$

$$(0, 1, 2) = -\left(\frac{c^2}{3\pi} \int \frac{dr}{r}\right) \frac{H_Q}{k(E_0 - E_2 - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})}$$

$$\begin{aligned}
 (1, 2, 3) &= -\left(\frac{e^2}{3\pi} \int \frac{dr}{r}\right) \frac{H_Q}{(E_0 - E_1)k(E_0 - E_{n-1} - 2k) \dots (E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})} \\
 &\dots\dots\dots \\
 (n-3, n-2, n-1) &= -\left(\frac{e^2}{3\pi} \int \frac{dr}{r}\right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-1})k(E_0 - E_{n-1} - 2k)(E_0 - E_n) \dots (E_0 - E_{Q-1})} \\
 (n-2, n-1, n) &= -\left(\frac{e^2}{3\pi} \int \frac{dr}{r}\right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{n-2})k(E_0 - E_n) \dots (E_0 - E_{Q-1})} \\
 &\dots\dots\dots \\
 (Q-2, Q-1, Q) &= -\left(\frac{e^2}{3\pi} \int \frac{dr}{r}\right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{Q-2})k}
 \end{aligned}$$

In summing up these terms we can make use of the following relations.

$$\begin{aligned}
 (0) &= 0 \\
 (1) + (0, 1) &= 0 \\
 (2) + (1, 2) + (0, 1, 2) &= 0 \\
 &\dots\dots\dots \\
 (n-1) + (n-2, n-1) + (n-3, n-2, n-1) &= 0 \\
 (n) + (n-1, n) + (n-2, n-1, n) &= -\left(\frac{e^2}{3\pi} \int \frac{dr}{r}\right) \frac{H_Q}{(E_0 - E_1) \dots (E_0 - E_{Q-1})} \\
 (n+1) + (n, n+1) + (n-1, n, n+1) &= 0 \\
 &\dots\dots\dots \\
 (Q) + (Q-1, Q) + (Q-2, Q-1, Q) &= 0
 \end{aligned}$$

The logarithmic divergence owing to the polarization effect with respect to the element k^* can be in this way reduced to the correction factor

$$-\frac{e^2}{3\pi} \int \frac{dr}{r}.$$

When the process under consideration includes Q photons — the case of absorption can be treated in quite an analogous way — we have for each of them the above derived correction factor, so that the total logarithmic divergence can be expressed in the factor

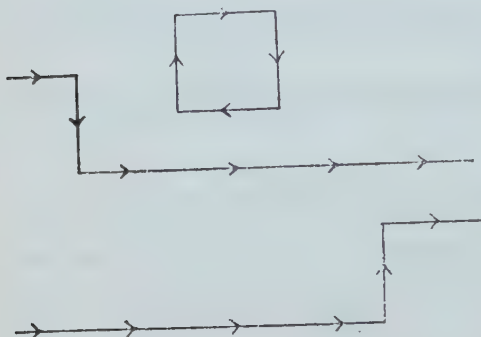
$$-\frac{e^2 Q}{3\pi} \int \frac{dr}{r},$$

Consequently the cross section σ_0 is corrected by an amount $\delta\sigma$:

$$\delta\sigma \sim -\frac{2e^2 Q}{3\pi} \int \frac{dr}{r} \cdot \sigma_0$$

The same correction will be obtained if we vary the coupling constant by an amount $\delta e \sim -(e^2/3\pi) \int (dr/r)$, since $\sigma_0 \sim e^{\alpha Q}$. But this is just what we have to prove: effect of the charge renormalization.

Fig. 13

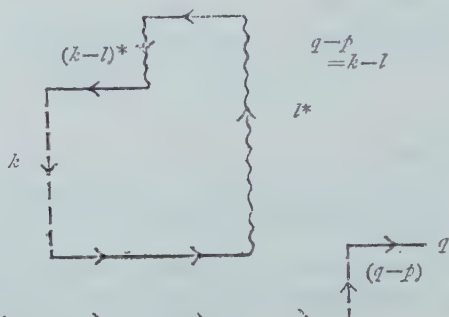


The general case $N \geq 2$ can be treated along the same line. The only difference from the case $N=1$ consists in the polarization correction with regard to the original Coulomb interaction. Here the polarization effect through a virtual transverse photon vanishes while the same effect through Coulomb interaction becomes effective (Fig. 13). This results in a logarithmic divergence,

too, which can be cancelled by the counter term to the modified Coulomb interaction.

Now that we have fully discussed $[P, I]$ corrections, we proceed to the investigation of $[P, II]$'s. As an example of the latter a correction to the Compton scattering: $p, \tilde{k} \rightarrow q, \tilde{l}$ is illustrated in Fig. 14. (The uncorrected connection is given in Fig. 2).

Fig. 14



In the cases of $[P, S]$ with $S \geq 2$ a corrected connection

no longer resembles the uncorrected one, the path of the original electron-line is entirely changed. From this diagram one will at once recognize that the island-connection represents a process of splitting up of a photon \tilde{k} into two, \tilde{l} and $\tilde{k}-\tilde{l}$. Any connection of $[P, II]$ implies a process of this kind or its inverse. But such a transition is entirely forbidden as can be shown by Furry's symmetry theorem.⁽¹⁰⁾ Strictly speaking the matrix element is not exactly zero in our case, because the presence of other

electrons and positrons with certain momenta impairs the symmetry between virtual electron and virtual position on account of exclusion principle. At all events, however, this kind of correction does not diverge and can be neglected for our present purpose.

As for the $[P, III]$ corrections some of them have logarithmically divergent form, but when all possible connections are taken into account the leading terms drop off and one is left with a finite correction. We may remark by the way that $[P, III]$ corrections, when separately considered, should not only converge but also vanish; otherwise a photon could spontaneously split up into three photons. But here we shall not enter into this problem.

It is from the outset obvious that all further corrections $[P, IV]$, $[P, V]$... $[P, Q]$ cannot contribute to divergence since the energy-denominator will contain more than 4 $|\boldsymbol{p}|$'s. ($E \sim |\boldsymbol{p}|$ is the energy of a virtual electron, and one has to integrate with respect to $|\boldsymbol{p}|$). Moreover all $[P, S]$ corrections with even S do vanish, if the presence of other electrons is not taken into account, because of symmetry property.

After all we can state that the polarization corrections for an arbitrary process yield quadratic and logarithmic divergence, which are just set off by our counter-terms prescribed in (1.1), (1.2).

Concluding, we wish to express our heartiest thanks to Prof. S. Tomonaga and Dr. T. Miyazima for their interest taken in our work and for their discussions and encouragement.

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Note added in proof. Recently we read two works treating diagram methods similar to ours: F. J. Dyson, Phys. Rev. **75** (1949) 486; and A. Houriet and A. Kind. (We are very much obliged to Dr. Houriet who sent the MS to Prof. Tomonaga before publication). Both these papers introduce diagrams in ordinary space-time, and we hope our momentum-diagram will act as an intermediary between the formalism of these authors and the conventional perturbation calculation. Judging from Dyson's work, Feynman's radiation theory, of which we know very little, seems also to employ some diagram method.

Pri la Teorio de Kolektivaj Elektronoj de Feromagnetismo.

Syôhei MIYAHARA.

Instituto de Fiziko, Nagoya Universitato.

(Received Dec. 11, 1948)

Synopsis.

An extension and an atmost generalization of the collective electron theory of ferromagnetism is developed on the basis of Slater's theory. From this we can get general relations between magnetism and temperature. The relations obtained naturally depend on the form of the state density function. The conditions for the appearance of Langevin-function are also considered and the case of iron is specially discussed.

§ 1. Enkonduko.

La teorio de feromagnetismo estas konstruita el du fame konataj tipoj: tipo de Heisenberg⁽¹⁾ (aŭ de atomo-funkcio) kaj tio de Slater⁽²⁾ (aŭ de latiso-funkcio), tiuj ĉi teorioj havas specialajn fortajojn; sed oni volas konstrui pli superan kaj sintezigitan teorion. Tiu ĉi volo montras sin multfoje de kelkaj studadoj.⁽³⁾ Tamen la provadoj estas ankoraŭ ne sufiĉe sukcesintaj.

Mi pensas, ke sintezo de ambaŭaj teorioj necesas la sufiĉan ekzamenon de la teorioj kaj klarigon de rilato inter ili. De tiu ĉi vidpunkto la rilaton inter la teorio de Heisenberg kaj la alojteorimaniere mi ekzamenis kaj tie montris la unua estas esence ne malsama de la alia.⁽⁴⁾

Ĝeneraligita transformigo de teorio de Slater estas unue per Stoner⁽⁵⁾ publikita kiel la teorio de kolektivaj elektronoj. El lia teorio la potenciala energio de elektrono en molekulokampo de Weiss estas enkonduita kun ĝia nivela energio en Fermi-statistikon, kie la formo de energizono estas supozita kiel parabolo. Pli ĝeneralan rilaton derivis Hirone kaj Miyahara,⁽⁶⁾ kaj aplikis tion al arbitra formo de energizono precipe senlarĝeca zono (aŭ niveloj) de duon-kondukilo.

En sekŭanto mi volas provi sufiĉan ĝeneraligon de la teorio de kolektivaj elektronoj, por ke oni obtenu la finofaran formon de la teorio de tipo de Slater (aŭ ĝenerale la teorio de latiso-funkcio).

§ 2. Ĝenerala teorio.

Per duakvantumigo la energio de tuta sistemo de elektronoj estas ĝenerale esprimata per la nombroj de elektronoj kaj la variantoj kanonike konjugaj al la nombroj. Nun ni supozas, ke la energiesprimo enhavas nur la nombrojn. Sekve oni povas skribi

$$U = U_0 + U_1(n_1, n_2, \dots, n_i, \dots), \quad (1)$$

$$U_0 = \sum_i E_i n_i, \quad (2)$$

kie U_0 signifas la linian parton de energio rilate al la nombroj n_i de elektronoj de i -a sfereto de stato-spaco kaj U_1 estas la nelinia parto. Tiam la tuta nombro de elektronoj N estas

$$N = \sum_i n_i. \quad (3)$$

La nombro W de komplekso de disdonoj de n_1, n_2, \dots elektronoj respektive al unua, dua, ..., i -asferetoj estas evidente

$$W = \prod_i z_i / n_i (z_i - n_i), \quad (4)$$

aŭ

$$\log W = \sum_i \{ z_i \log z_i - n_i \log n_i - (z_i - n_i) \log (z_i - n_i) \}, \quad (5)$$

kie z_i signifas la nombron de permesitaj statoj de la i -a sfereto. Esprimojn (2) kaj (3) povas oni transskribi en integralan formon,

$$U_0 = \int E(x) dn(x),$$

$$N = \int dn(x),$$

kie x estas la varianto kiu difinas sfereton. Tiuj ĉi esprimoj superas kompare kun (2) kaj (3), ĉar ili povas esprimi ne nur la kazon de nekontinua energispekto sed ankaŭ la kazon de kontinua. Ankoraŭ ni transskribas ilin en

$$U_0 = \int E(x) f(x) dz(x), \quad (6)$$

$$N = \int f(x) dz(x), \quad (7)$$

kie f estas la okupadogrado n_i/z_i aŭ dn/dz . Simile de (5)

$$\log W = - \int \{ (1-f) \log (1-f) + f \log f \} dz(x). \quad (8)$$

Do ni volas decidi de funkcia formon de $f(x)$ por ke $\log W$ fariĝu maksimumo. De esprimo (8)

$$\delta \log W = \int \{\log(1-f) - \log f\} \delta f dz(x) = 0 \quad (9)$$

kaj de (1), (6) kaj (7)

$$\delta N = \int \delta f dz(x) = 0, \quad (10)$$

$$\delta U = \int E \delta f dz(x) + \int \frac{\delta U_1}{\delta f} \delta f dz(x) = 0. \quad (11)$$

kie $\delta U_1/\delta f$ estas funkcionala derivajo, kies konkretan formon ni donos poste. De (9), (10) kaj (11) kun multiplikantoj λ kaj x de Lagrange

$$\int \{\log(1-f) - \log f\} + \lambda + xE + x \frac{\delta U_1}{\delta f} \delta f dz(x) = 0$$

Pro arbitreco de funkcio δf oni obtenas

$$\log(1-f) - \log f + \lambda + xE + x \frac{\delta U_1}{\delta f} = 0,$$

aŭ

$$f = 1 / \{1 + \exp(\lambda + xE + x \frac{\delta U_1}{\delta f})\}, \quad (12)$$

kaj per termodinamika konsidero $\lambda = -\zeta/kT$ kaj $x = 1/kT$, kie T , k kaj ζ signifas respektive la temperaturon, konstanton de Boltzmann kaj Fermi-energion. De tiu ĉi funkcionala ekvacio oni povas decidi funkcion f .

Do $\log W$ fariĝas

$$\begin{aligned} \log W = & \int \log \{1 + \exp(\zeta - E - \frac{\delta U_1}{\delta f})/kT\} dz(x) \\ & + (1/kT) \{-\zeta N + U_0 + \int \frac{\delta U_1}{\delta f} f dz(x)\} \end{aligned}$$

La libera energio F estas

$$F = U - kT \log W = \zeta N + \Omega + U_1 - \int \frac{\delta U_1}{\delta f} f dz(x), \quad (13)$$

kie

$$\Omega = -kT \int \log \{1 + \exp(\zeta - E - \frac{\delta U_1}{\delta f})/kT\} dz(x). \quad (14)$$

Do de la rilatoj

$$M = -\frac{\partial F}{\partial H} = -\frac{\partial \Omega}{\partial H} \text{ kaj } N = -\frac{\partial \Omega}{\partial \zeta} \quad (15)$$

ni povas obteni transcendaĵojn rilate al ζ kaj M , de kiuj oni povas decidi magnetismon.

Por antaŭenpuŝi la teorion, mi devas asigni la konkretan formon de funkcionalo U_1 de funkcio f . Koncerne al problemo de feromagnetismo, min interesas la interŝanĝintegralo de du elektronoj kiel U_1 .

Se U_1 povas esti esprimata en kvadrata formo rilate al n_i aŭ $n(x)$, t.e.

$$U_1 = \iint_{x \geq y} j(x, y) f(x) f(y) dz(x) dz(y) = \frac{1}{2} \iint j(x, y) f(x) f(y) dz(x) dz(y), \quad (16)$$

kaj $j(x, y)$ estas simetra, t.e.

$$j(x, y) = j(y, x).$$

Tiam

$$\delta U_1 = \int j(x, y) f(y) dz(y) \delta f(x) dz(x).$$

Kunkonsiderante kun (11)

$$\frac{\delta U_1}{\delta f} = \int j(x, y) f(y) dz(y),$$

sekve

$$\int \frac{\delta U_1}{\delta f} f(x) dz(x) = 2U_1 \text{ (etendita Euler'a teoremo),}$$

kaj

$$F = \zeta N + Q - U_1. \quad (17)$$

§ 3. Proksimuma solvo.

Ĉar ni ne povas solvi la integralan ekvacion (12) praktike, ni ne konas la konkretan formon de funkcio f ; tial mi devas uzi konvenan proksimumon. Unue mi kalkulos U_1 , kiu aperas en (13).

En la esprimo (16) x kaj y enhavas kune spino-variantojn, sekve la signo de integro kunsignifas la sumon pri spino-variantoj σ kaj σ' . Kiam $j(x, y)$ signifas la interŝanĝintegralon, evidente

$$j(x, y; \sigma, \sigma') = 0 \text{ por } \sigma \neq \sigma' \\ j(x, y; \sigma, \sigma) \neq 0.$$

Do, transskribante j en $-j$, (16) fariĝas

$$U_1 = -\frac{1}{2} \sum_{\sigma} \iint j(x, y) f_{\sigma}(x) f_{\sigma}(y) dz(x) dz(y).$$

Se oni uzas mesvaloron de $j(x, y)$

$$\iint j(x, y) f_{\sigma}(x) f_{\sigma}(y) dz(x) dz(y) = N_{\sigma}^2 j_{\sigma},$$

kie

$$J_o = \iint j(x, y) f_o(x) f_o(y) dz(x) dz(y) / \iint f_o(x) f_o(y) dz(x) dz(y),$$

kaj

$$N_o = \int f_o(x) dz(x).$$

Do

$$U_1 = -\frac{1}{2} \sum_o J_o (N_o)^2.$$

Kiam oni enkondukas la magnetokvantumnombron S

$$N_o = \frac{1}{2} N + \sigma S; \quad \text{kie } \sigma \pm 1$$

t.e.

$$U_1 = -\frac{1}{2} \sum_o J_o (\frac{1}{4} N^2 + S^2 + \sigma NS).$$

Se oni hipotezas $J_1 = J_{-1}$, do

$$U_1 = -J (\frac{1}{4} N^2 + S^2). \quad (18)$$

Simile

$$\frac{\partial U_1}{\partial f} = - \int j(x, y) f_o(y) dz(y) = -\bar{j}(x) N_o = -\bar{j}(x) (\frac{1}{2} N + \sigma S). \quad (19)$$

Kiam oni skribas $E = E_o + \mu H$ kaj ŝovas konvene la nulpunkton de energio, de (14) kaj (19) mi obtenas

$$\begin{aligned} \Omega = & -kT \int \log \{ 1 + \exp(\zeta - E_o - \mu H - \bar{j}(x) S / kT) \} dz(x) \\ & - kT \int \log \{ 1 + \exp(\zeta - E_o + \mu H + \bar{j}(x) S / kT) \} dz(x). \end{aligned} \quad (20)$$

Se $j(x)$ ne dependas de x (kaj egalas al J/L , kie J kaj L signifas respektive intersangintegralo ĉe atomo kaj tutan nombron de atomĉeloj), tiu ĉi esprimo koincidas perfekte kun Hirone kaj Miyahara antaŭe obtenis. Se oni hipotezus ankoraŭ

$$dz/dE \propto E^{1/2} \quad (21)$$

ĝi koincidas kun tio de Stoner:

§ 4. Temperaturo-dependado de feromagnetigo spontanea.

La feromagnetigo spontanea estas kalkulita ĝenerale per

$$\lim_{H \rightarrow 0} M = - \lim_{H \rightarrow 0} \frac{\partial \Omega}{\partial H} \quad (22)$$

kun (20). Kiel Stoner montris, sub la hipotezo (21) la temperaturo-dependado de feromagnetigo spontanea estas simila al tio de Weiss-Heisenberg. Tamen ĝenerale dirante, la temperaturo-dependado el (22) estas neĉiam

simila al Weiss-Heisenberg. En fakto, kiam oni supozas, ke dz/dE estas nedependa de E , la temperaturo-dependado rimarkinde diferencas de la formo fame konata.

Aliflanke el teorio de Weiss-Heisenberg temperaturo-dependado de spontanea magnetigo estas esprimita de tipa Langevin-funkcio,* t.e.

$$M = M_{\infty} \tanh \frac{M\theta}{M_{\infty} T}$$

Do mi montras la kondiĉon, por ke la esprimon (22) fariĝu Langevin-funkcio. Se oni skribas

$$y = M/L\mu, \quad x = (E - E_0)/(E_1 - E_0), \quad \eta = (\zeta - E_0)/(E_1 - E_0), \quad \tau = kT/(E_1 - E_0), \\ h = \mu H/(E_1 - E_0), \quad L\nu(x) = dx(x)/dx, \quad w = \frac{1}{2}J/(E_1 - E_0), \quad n = N/L;$$

kie E_0 kaj E_1 estas konvenaj energioj. La tie difinita x estas nensio alia ol unu specialigo de kion mi antaŭe uzis. Por simpligo ankoraŭ ni enkondukas $u = (x - \eta)/\tau$, $v = wy/\tau$, kaj $h = 0$, sekve de (15)

$$y = \int \frac{\sinh v}{\cosh u + \cosh v} \nu(x) dx \quad (23)$$

kaj

$$n = \int \frac{e^{-u} + \cos v}{\cosh u + \cosh v} \nu(x) dx. \quad (24)$$

(A) Kiam w estas nedependa de x , t.e. $\bar{j}(x)$ en (20) nedependa de x .

La esprimon (23) oni povas transformigi en sekvantan:

$$y = \tanh pv \left[n - \int \frac{e^{-u} - \sinh(1-p)v/\sinh pv}{\cosh u + \cosh v} \nu(x) dx \right], \quad (25)$$

kie p estas ia konstanto. Se la integralo de (25) estus nulo, povos ni skribi $y = n \tanh pv$, kiu perfekte koincidas tion de Weiss-Heisenberg. Sed ĝenerale oni ne povas elekti tian konstanton p , ke la integralo malaperu en iu ajn temperaturo.

i) En la kazo kiam $\nu(x)$ estas tre mallarĝa aŭ senlarĝeca, t.e. ĝi diferencas nulton nur ĉirkaŭ $x = x_1$, kun la natura rilato $x_1 = \eta$, oni obtenas

$$y = \tanh pv \left[n - \frac{1 - \sinh(1-p)v/\sinh pv}{1 + \cosh v} \int_{x_1} \nu(x) dx \right].$$

Metu $p = 1/2$, tiam

$$y = n \tanh \frac{1}{2}v, \quad (26)$$

* Per "Langevin-funkcio" oni signifas ĝenerale $L(x) = \frac{1}{2}(2j+1)^{-1} j \cdot \coth \frac{1}{2}(2j+1)x / j - \frac{1}{2} / j \cdot \coth \frac{1}{2}x / j$, sed por elektronspino $j = \frac{1}{2}$, t.e. $L(x) = \tanh x$.

kaj la Curie-punkto estas

$$\tau_c = \frac{1}{2} n v \quad \text{aŭ} \quad T_c = J n / 4k$$

ii) En la kazo de duon-kondukilo, kie la energizono $\nu(x)$ konsistas el du aŭ pli multaj niveloj, la integro fariĝas nura adicio. Sed tie ĉi la Fermi-energio ζ ne ĉiam koincidas kun unu de niveloj, t.e. $n(x_i) \neq 0$, sekve oni ne povas skribi kiel (26) aŭ $y = n' \tanh p'v$ kun iuj ajn konstantaj n' kaj p' .

Kiel la plej simpla, mi konsideros duon-kondukilon kun du niveloj, nur unu de tiuj rilatas al magnetigo. Do de (23)

$$y = \frac{\sinh v}{\cosh n_1 + \cosh v} \int_1 \nu(x) dx, \text{ kie } 1 \text{ signifas } x = x_1,$$

$$n_1 = \frac{e^{-u_1} + \cosh v}{\cosh n_1 + \cosh v} \int_1 \nu(x) dx,$$

$$n_2 = \frac{e^{-u_2} + 1}{\cosh n_2 + 1} \int_2 \nu(x) dx;$$

aŭ

$$y = \tanh p'v \left[n_1 - \frac{e^{-u_1} - \sinh(1-p)v / \sinh p'v}{\cosh n_1 + \cosh v} \int_1 \nu(x) dx \right].$$

En supra esprimo la dua termo en angulaj klampoj ĝenerale diferencas de nulo kaj dependas de v (aŭ y). Tial oni ne povas skribi $y = n_1 \tanh p'v y / \tau$. Sed, kiam la radikoj de ekvacio pri p :

$$e^{-u_1} - \frac{\sinh(1-p)v}{\sinh p'v} = 0$$

estas ĉirkaŭ $1/2$ aŭ 1 , la termoj en angulaj klampoj estas preskaŭ nedependa de v (aŭ y). Ĝi signifas la ekstremajn kazojn, kie la nivelo en demando estas koincidanta kun, aŭ tre apartega de Fermi-energio.

iii) Nun ni konsideros la kazon de metaloj.

En la kazo de nikelo, kiel Slater klarigis, $\nu(x)$ estas mallarĝa kaj alta, kaj la Fermi-energio estas proksima al la rando de energizono, la proksimumo de Stoner aŭ la hipotezo (21) estas trafa, kaj la proksimumo, ke $\nu(x)$ estas senlarĝeca, estas ankaŭ uzebla. Tial la temperaturo-dependado de magnetigo spontanea el teorio de kolektivaj elektronoj preskaŭ koincidas kun tio de Weiss-Heisenberg.

Konsideri la kazon de fero ekstreme interesas min. Tiun ĉi kazon, kiel Manning kaj Greene⁽⁷⁾ montris, Fermi-energio estas ne proksima al la rando de energizono, tiu ĉi mem estas ne mallarĝa. Tial nenju kondiĉo,

kun kiu la temperaturo-dependado povas esprimi sin per Langevin-funkcio, estas kontenta.

Tamen la temperaturo-dependado de magnetigo spontanea de fero estas eksperimente tre bone esprimita per Langevin-funkcio. Tial ni devas ekzameni alian kazon.

(B) Kiam $w(x)$ aŭ $\tilde{j}(x)$ estas δ -funkcio-simila kun malgrandega larĝeco Δx ĉirkaŭ $x=x_1$.

Do

$$y = \frac{\sinh v_1}{\cosh u_1 + \cosh v_1} \nu(x_1) \Delta x,$$

kaj

$$n = \int_{\text{krom } \Delta x} \frac{2}{e^u + 1} \nu(x) dx + \frac{e^{-u_1} + \cosh v_1}{\cosh u_1 + \cosh v_1} \nu(x_1) \Delta x = \bar{n} + n''.$$

Se oni skribas $y = n' \tanh v$, n' estas klare pli malgranda ol n'' , kiu ĉi mem estas tre malgranda kompare al \bar{n} , krom ke $\nu(x)$ estas ankaŭ δ -funkcio-simila. Sekve oni ne povas obteni grandan magnetigon spontanen.

§ 5. Konkludo.

De supre obtenitaj rezultoj la temperaturo-dependado de magnetigo spontanea diferencas ĝenerale de tio de Weiss-Heisenberg. Precipe en la kazo de fero el ĝia energizono oni ne povas trovi teorie nenium bazon por apero de Langevin-funkcio kontraŭ eksperimenta fakto. Tiu ĉi estas fatala por la teorio de kolektivaj elektronoj, ĉar la obtenitaj estas finofaraj kiel la teorio de tiu ĉi tipo. La kontraŭdiro, ke ĝi alprenas nur la mesvaloron por la interŝanĝenergio sub donata S , estas ne esenca. Ĉar la Langevin-funkcio jam aperis en teorio de Heisenberg sub la sama proksimumo (proksimumo de Weiss). Kunkonsideri la larĝigon de la energio sub donata S donas, kiel Heisenberg montris, malgranda devio de Langevin-funkcio. Ankaŭ, kiel ni montris antaŭe,⁽⁸⁾ precizigo de proksimumo per alojteorio ne ĉiam estas signifa. Tial, krom negravaj punktoj (ekz. specifa varmo super Curie-punkto), proksimumo de Weiss, t.e. la alpreno de mesvaloro, estas trafa.

Ĉu ni devas reveni al la teorio de Weiss-Heisenberg? Antaŭ respondi tiun ĉi demandon, ni devas klarigi la rilaton inter ambaŭ per sama modelo. Tiu ĉi estas interesa sed malfacila problemo.

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Progress of Theoretical Physics Vol. IV, No. 2, Apr.~Jun., 1949.

On the Relation between Heisenberg's Theory of Ferromagnetism and that of Slater.

Kei YOSIDA.

Physical Institute, Osaka University.

(Received Dec. 11, 1948)

§ 1. Introduction.

Ferromagnetism arises from the fact that pairs of electrons with parallel spins have an extra energy of the exchange integral $-J$ besides the energy of the Coulomb integral, while pairs with antiparallel spins do not. Thus a system with more electrons having parallel spins has an energy lower than that with less such electrons.

Generally, the interaction energy of a system of electrons consists of the Coulomb energy e^2/r_{ik} between two electrons. According to quantum mechanics this energy must be averaged over the positions of electrons.

Such an average value is

$$\int \frac{e^2}{r_{ik}} P(r_i, r_k) dr_i dr_k = I(i, k), \quad (1)$$

where $P(r_i, r_k)$ is the probability that the electron i exists in the range $r_i, r_i + dr_i$ and the electron k in the range $r_k, r_k + dr_k$. Then the total energy is given by $\sum_{i>k} I(i, k)$. If the probability with which one electron exists at the position r is independent of the positions of other electrons, $P(r_i, r_k)$ has the form $P(r_i) \cdot P(r_k)$, where $P(r)$ is the probability with which an electron is in the position r . If the electron i is in the state φ_i , $P(r)$ is equal to $|\varphi_i(r)|^2$. Thus if two electrons are in the states φ_i and φ_k respectively and there exists no correlation between them, the interaction energy between them becomes

$$\int \frac{e^2}{r_{ik}} |\varphi_i(r_i)|^2 |\varphi_k(r_k)|^2 dr_i dr_k$$

which is the usual Coulomb integral.

In the case that there exists, however, some correlation between two electrons, $P(r_i, r_k)$ cannot be expressed in the form $P(r_i) \cdot P(r_k)$. It is convenient in such a case to express $P(r_i, r_k)$ in the form $P(r_i) \cdot P(r_k) - Q(r_i, r_k)$, where $Q(r_i, r_k)$ is regarded as a correction term due to the

correlation between two electrons. Then we have as the interaction energy,

$$\int \frac{e^2}{r_{ik}} P(r_i) \cdot P(r_k) dr_i dr_k - \int \frac{e^2}{r_{ik}} Q(r_i, r_k) dr_i dr_k \quad (2)$$

In this expression the first term is the Coulomb integral and the second the correction integral.

Since, according to quantum statistics, electrons obey Fermi-Dirac statistics, we must adopt, as the wave function of the system of electrons, a function antisymmetrical for the exchange of coordinates and spin variables of electrons, i.e. the determinantal form

$$\begin{vmatrix} \varphi_1(r_1) & \varphi_1(r_2) & \dots & \varphi_1(r_N) \\ \varphi_2(r_1) & \varphi_2(r_2) & \dots & \\ \vdots & & & \\ \varphi_N(r_1) & \dots & \dots & \varphi_N(r_N) \end{vmatrix} \quad (3)$$

where r_i represents coordinate and spin variable of the electron i . And N is the number of electrons. With this determinantal form we have the following results;

$$\begin{aligned} P(r_i, r_k) &= |\varphi_i(r_i)|^2 |\varphi_k(r_k)|^2 \quad \text{if two electrons have antiparallel spins,} \\ &= \frac{1}{2} |\varphi_i(r_i)\varphi_k(r_k) - \varphi_k(r_i)\varphi_i(r_k)|^2 \quad \text{two electrons have parallel spins.} \end{aligned} \quad (4)$$

Hence, there exists no correlation between electrons with antiparallel spins but this is not the case for electrons with parallel spins. For the latter we obtain from (4)

$$Q(r_i, r_k) = \varphi_i(r_i)\varphi_k(r_k) - \varphi_k(r_i)\varphi_i(r_k). \quad (5)$$

The integral with this Q expressing the correction for parallel spins is precisely the exchange integral.

Thus the interaction energy for an electron-pair with parallel spins becomes smaller by the value of the exchange integral than that for a pair with antiparallel spins. This results from the antisymmetry of the wave function of the system of electrons.

Slater's theory⁽¹⁾ shows that the theoretical value of the exchange integral is larger than the experimental value as obtained from the Curie temperature of ferromagnetic substances. But the correlation between antiparallel spins existing in a higher approximation would probably lower the effective value of the exchange integral.

§2. General Considerations.

Ferromagnetism is, as mentioned above, based on the fact that an an-

antisymmetrical wave function is adopted for the system of electrons. Hence, the two typical theories of ferromagnetism, that of Heisenberg⁽²⁾ and that of Slater,⁽³⁾ both start from a determinantal form as the wave function of the total system of electrons.

The difference between the two theories consists in the point that Heisenberg's theory adopts an atomic function as the wave function of an electron, while Slater's theory a lattice function. That is to say, in the former, one of the wave functions of the total system of electrons in the zero-th order approximation has the form

$$(N!)^{-1/2} \begin{vmatrix} v_1(r_1) & v_1(r_2) & \dots \\ v_2(r_1) & v_2(r_2) & \dots \\ \vdots & \vdots & \ddots \\ v_N(r_1) & & \end{vmatrix} \quad (5)$$

where $v_i(r)$ is an atomic function for the i -th lattice point and one lattice point does not appear more than twice in one column, while in the latter

$$(N!)^{-1/2} \begin{vmatrix} u_{\mathbf{k}_1}(r_1) & u_{\mathbf{k}_1}(r_2) & \dots \\ u_{\mathbf{k}_2}(r_1) & u_{\mathbf{k}_2}(r_2) & \dots \\ \vdots & \vdots & \ddots \\ u_{\mathbf{k}_N}(r_1) & & \end{vmatrix} \quad (6)$$

where $u_{\mathbf{k}_i}(r_i)$ is the lattice function with the wave number vector \mathbf{k}_i .

Having once established the basic functions in the zero-th approximation in both theories, we must solve the eigenvalue problem of the perturbation energy consisting of the interactions between all the electrons. In either theory since, this problem can not be completely solved on account of mathematical difficulties, we must be content with the mean value of the eigenvalues under the condition that the total spin magnetic quantum number m is constant.

The perturbation energy consists of the interaction energy between all the electrons and between electrons and ions in Heisenberg's theory, but in Slater's the interaction between electrons alone, because it is considered that the lattice functions have already been perturbed by the interaction between electrons and ions.

In Slater's theory one of the basic functions $\phi(\mathbf{k}_1, \mathbf{k}_2, \dots)$ is expressed as

$$\phi(\mathbf{k}_1, \mathbf{k}_2, \dots) = (N!)^{-1/2} \sum_{\mathbf{p}} \partial_{\mathbf{p}} \mathbf{P} u_{\mathbf{k}_1}(r_1) u(\sigma_1) u_{\mathbf{k}_2}(r_2) u(\sigma_2) \dots u_{\mathbf{k}_i}(r_i) \beta(\sigma_i) \dots \quad (7)$$

Where \mathbf{P} is the substitution operator of electrons, $\partial_{\mathbf{p}}$ is $+1$ if \mathbf{P} is even and -1 if odd, and $u(\sigma)$ and $\beta(\sigma)$ are positive and negative spin eigen-

functions respectively. In the tight binding approximation $u_{\mathbf{k}_i}$ is expressed as follows;

$$u_{\mathbf{k}_i}(r_i) = N^{-1/2} \sum_{\mathbf{n}} e^{i\mathbf{k}_i \cdot \mathbf{R}_n} v_n(r_i), \quad (8)$$

where \mathbf{R}_n represents the position of a lattice point \mathbf{n} and v_n is the atomic function at the lattice point \mathbf{n} . By using (8) for $u_{\mathbf{k}_i}(r_i)$ in (7), $\phi(\mathbf{k}_1, \mathbf{k}_2, \dots)$ is given by

$$\phi(\mathbf{k}_1, \mathbf{k}_2, \dots) = (N!)^{-1/2} (N^{-1/2})^N \sum_{\mathbf{p}} \delta_{\mathbf{p}} \mathbf{P} \cdot \sum_{n_1, n_2, \dots} e^{i\mathbf{k}_1 \cdot \mathbf{R}_{n_1}} v_{n_1}(r_1) a(\sigma_1) e^{i\mathbf{k}_2 \cdot \mathbf{R}_{n_2}} v_{n_2}(r_2) a(\sigma_2) \dots \dots \dots (9)$$

Since coordinates of electrons are not included in $e^{i\mathbf{k}_i \cdot \mathbf{R}_i}$ etc., by exchanging the order of summations (9) becomes

$$\phi(\mathbf{k}_1, \mathbf{k}_2, \dots) = (N!)^{-1/2} (N^{-1/2})^N \sum_{n_1, n_2, \dots} \exp i(\mathbf{k}_1 \cdot \mathbf{R}_{n_1} + \mathbf{k}_2 \cdot \mathbf{R}_{n_2} + \dots) \cdot \sum_{\mathbf{p}} \delta_{\mathbf{p}} \mathbf{P} v_{n_1}(r_1) v_{n_2}(r_2) \dots \dots a(\sigma_1) a(\sigma_2) \dots \dots \dots (10)$$

In (10) the last factor has the determinantal form, because of which it vanishes if there are any equal functions among $v_{n_1} \cdot a$, $v_{n_2} \cdot a$, $v_{n_3} \cdot \beta$, ... i.e. if there are more than three electrons in one atom or there are two electrons with parallel spins. The terms remaining are the basic functions in Heisenberg's theory generalized so as to include not only states which have only one electron on one lattice point, but also states which have two electrons on one lattice point with antiparallel spins. If these basic functions are designated by $\Psi(n_1, n_2, \dots)$, Eq. (10) is expressed in the following form,

$$\phi(\mathbf{k}_1, \mathbf{k}_2, \dots) = (N-1/2)^N \sum_{n_1, n_2, \dots} \exp i(\mathbf{k}_1 \cdot \mathbf{R}_{n_1} + \mathbf{k}_2 \cdot \mathbf{R}_{n_2} + \dots) \Psi(n_1, n_2, \dots). \quad (11)$$

It is seen from this equation that the basic functions in Slater's theory are expressed by a linear combination of the basic functions in the generalized Heisenberg theory, with the linear form of the type $N^{-N/2} \exp i(\mathbf{k}_1 \cdot \mathbf{R}_{n_1} + \dots)$ as the coefficients as Slater frequently emphasized⁽⁴⁾. From this fact, the conclusion can be derived that Heisenberg's theory must be generalized in order to correspond equivalently to Slater's theory.

In Slater's theory the mean value E_s of the perturbation energy with respect to constant m satisfies the following relation

$$G(m) \cdot E_s = \text{Trace} \left(\sum_{i>k} \frac{e^2}{r_{ik}} \right), \quad (12)$$

where $G(m)$ is the number of states with the same m . Since the number of states with the same m in the generalized Heisenberg theory is obviously equal to that in Slater's theory, and the basic functions of the two

theories are transformed by a linear transformation between each other, which makes the trace invariant, the mean value in Heisenberg's theory E_H must be equal to E_S .

§ 3. Energy in Slater's Theory.

As a concrete example, we try to calculate the mean energy of the interactions between electrons with constant m in each theory. For the sake of simplicity, let us adopt such a model that the crystal contains N atoms and N electrons, and in Slater's theory the considered energy band includes N states, and corresponding to this in the generalized Heisenberg theory there exists only one state per atom except for two different spins.

Firstly we consider the case of Slater's theory. The Coulomb integral ϵ_c between two electrons which are in the states u_{k_i} and u_{k_j} respectively is given by

$$\epsilon_c = \iint u_{k_i}^*(r_1) u_{k_j}^*(r_2) \frac{e^2}{r_{12}} u_{k_i}(r_1) u_{k_j}(r_2) dr_1 dr_2, \quad (13)$$

where r_1 and r_2 represent the coordinates of two electrons. If we divide the volume into N polyhedral cells and integrate separately over each pair of cells, the integral becomes the sum of the integrals in which dr_1 ranges over the interior of the p -th cell and dr_2 over the q -th cell. According to Bloch's theory, the values of the wave functions $u_{k_i}(r_1)$ and $u_{k_i}^*(r_1)$ in the p -th cell are equal to the values $u_0(r_1)$ and $u_0^*(r_1)$ at the corresponding point in the central cell, multiplied by the factor $\exp(i\mathbf{k}_i \cdot \mathbf{R}_p)$ and $\exp(-i\mathbf{k}_i \cdot \mathbf{R}_p)$ respectively. Since the two exponential factors give 1, the integral ϵ_c becomes

$$\begin{aligned} \epsilon_c &= \sum_{p,q} \iint u_0^*(r_1) u_0^*(r_2) \frac{e^2}{r_{12}} u_0(r_1) u_0(r_2) dr_1 dr_2 \\ &= \sum_p \iint u_0^*(r_1) u_0^*(r_2) \frac{e^2}{r_{12}} u_0(r_1) u_0(r_2) dr_1 dr_2 \\ &\quad + \sum_{p \neq q} \iint u_0^*(r_1) u_0^*(r_2) \frac{e^2}{r_{12}} u_0(r_1) u_0(r_2) dr_1 dr_2. \end{aligned} \quad (14)$$

Here the first integral is equal to the Coulomb integral I_0 between two electrons in the same atom, divided by N^2 on account of the normalization, because u_0 is very much like the atomic function in the central cell and the second equal to the Coulomb integral between two electrons which are in the atom p and in the atom q respectively, divided by N^2 . If we neglect all the other terms except the ones I_N between nearest neighbors,

we obtain

$$\epsilon_e = \sum_p \frac{I_0}{N^2} + \sum_n \frac{I_N}{N^2} z = \frac{I_0}{N} + \frac{I_N}{N} z, \quad (15)$$

where I_N is the Coulomb integral between nearest neighbors and z the number of nearest neighbors.

Next, we calculate the exchange integral between two electrons which are in the states \mathbf{k}_i and \mathbf{k}_j in the same way as the above calculations. Then the exchange integral ϵ_e is given by

$$\begin{aligned} \epsilon_e = & \sum_{p,p'} \iint u_0^*(r_1) u_0^*(r_2) \frac{e^2}{r_{12}} u_0(r_1) u_0(r_2) dr_1 dr_2 \\ & + \sum_{n \neq q} \iint \exp\{i(\mathbf{k}_j - \mathbf{k}_i) \cdot (\mathbf{R}_p - \mathbf{R}_q)\} u_0^*(r_1) u_0^*(r_2) \frac{e^2}{r_{12}} u_0(r_1) u_0(r_2) dr_1 dr_2. \end{aligned} \quad (16)$$

Again considering only nearest neighbors we obtain

$$\epsilon_e = \frac{I_0}{N} + \frac{I_N}{N} \cdot 2\{\cos(\mathbf{k}_j - \mathbf{k}_i)_x \cdot a + \cos(\mathbf{k}_i - \mathbf{k}_i)_y \cdot a + \cos(\mathbf{k}_j - \mathbf{k}_i)_z \cdot a\},$$

where the lattice is assumed to be a simple cubic and a is the distance between atoms. By averaging over all the \mathbf{k}_i and \mathbf{k}_j , ϵ_e becomes

$$\epsilon_e = \frac{I_0}{N}, \quad (17)$$

The mean value of the energy of the system of electrons arising from the Coulomb integral is immediately obtained, since all the pairs of electrons have this energy irrespective of their spins.

$$E_c = \frac{N(N-1)}{2} \epsilon_e. \quad (18)$$

On the other hand, the mean value of the energy arising from the exchange integral is obtained by summing over all the energies of the states with the same m and dividing by the number $G(m)$ of such states.

To get the sum of the energies of all the states with same m , we have only to select any pair of electrons with parallel positive spins, multiply the energy $-\epsilon_e$ of this pair by the number of all the configurations of other electrons, and further by the number of all possible ways of selecting the pairs and add to this the same quantity arising from the negative parallel spins. Since the number of positive spins is $(N/2) + m$, the sum we seek for is given by

$$G(m)E_e = \sum -\epsilon_e \cdot \left\{ \binom{N-2}{(N/2)+m-2} \binom{N}{(N/2)-m} + \binom{N}{(N/2)+m} \binom{N-2}{(N/2)-m-2} \right\}.$$

$$= -\epsilon_0 \frac{N(N-1)}{2} \left\{ \binom{N-2}{(N/2)+m-2} \binom{N}{(N/2)-m} + \binom{N}{(N/2)+m} \binom{N-2}{(N/2)-m-2} \right\} \quad (19)$$

Therefore by dividing by $G(m) = \binom{N}{(N/2)+m} \binom{N}{(N/2)-m}$, the mean energy relating to the exchange integral becomes

$$E_0 = -\epsilon_0 \left\{ \frac{N^2}{4} + m^2 \right\}. \quad (20)$$

Adding (20) and (18) and considering (15) and (17) we obtain as the final form of the mean energy with respect to the same m

$$E_s = \frac{NI_0}{4} + \frac{Nz}{2} I_N - \frac{I_0 m^2}{N}. \quad (21)$$

§ 4. Energy in the Generalized Heisenberg Theory.

Here, we try to calculate the mean energy with the same m arising from the interaction energy of all the electrons $\sum_{i>k} \frac{e^2}{r_{ik}}$ in the generalized Heisenberg theory as in the case of Slater's theory. As the model for calculations we adopt, as mentioned before, one in which the crystal contains N atoms and N electrons and one atom has only one atomic function ψ_i . In this case too, we neglect all terms except those representing the actions between electrons in the same atom and in the nearest-neighboring atoms.

Firstly, we calculate the mean energy arising from the Coulomb integral I_0 between electrons in the same atom. For this value, selecting one atom out of N atoms, multiplying the energy I_0 (since there exist two electrons with spins antiparallel to each other in this atom) by the number of configurations of other electrons $\binom{N-1}{(N/2)+m-1} \binom{N-1}{(N/2)-m-1}$, and by the number of ways N of selecting one atom and dividing by $G(m)$, we obtain

$$E_{I_0} = I_0 N \binom{N-1}{(N/2)+m-1} \binom{N-1}{(N/2)-m-1} / \left(\binom{N}{(N/2)+m} \binom{N}{(N/2)-m} \right) \\ = \frac{N^2 - 4m^2}{4N} I_0. \quad (22)$$

Similarly we try again to calculate the mean energy relating to the Coulomb integral I_N and the exchange integral J between electrons in the nearest neighboring atom. In this case too, we calculate this value by summing over the energies of all the states with the same m and dividing this sum by the number $G(m)$ of such states.

Here we have five cases because of the configurations of electrons in the atom-pair selected out of $Nz/2$ pairs, which are shown in the following table.

	I configurations of electrons	II energy	III number of configurations of other electrons
1	\rightarrow, \rightarrow	$I_N - J$	$\binom{N-2}{(N/2)+m-2} \binom{N-2}{(N/2)-m}$ $+ \binom{N-2}{(N/2)+m} \binom{N-2}{(N/2)-m-2}$
2	\rightarrow, \leftarrow	I_N	$2 \binom{N-2}{(N/2)+m-1} \binom{N-2}{(N/2)-m-1}$
3	$\rightleftarrows, \leftarrow$	$2I_N - J$	$2 \binom{N-2}{(N/2)+m-1} \binom{N-2}{(N/2)-m-2}$
4	$\rightleftarrows, \rightarrow$	$2I_N - J$	$2 \binom{N-2}{(N/2)+m-2} \binom{N-2}{(N/2)-m-1}$
5	$\rightleftarrows, \rightleftarrows$	$4I_N - 2J$	$\binom{N-2}{(N/2)+m-2} \binom{N-2}{(N/2)-m-2}$

From this table, the mean energy relating to I_N and J is obtained by summing the products of II and III from 1 to 5, multiplying this sum by the number of ways of selecting pairs $Nz/2$ and dividing by $G(m)$. Thus the mean values relating to I_N and J respectively are given by .

$$E_{I_N} = \frac{Nz}{2} I_N, \quad (23)$$

$$E_J = -J \left(\frac{Nz}{4} + \frac{m^2}{N} z \right). \quad (24)$$

Summing over (22), (23), and (24), E_H is given as the final result by

$$E_H = \frac{NI_0}{4} + \frac{Nz}{2} \left(I_N - \frac{J}{2} \right) - \frac{m^2}{N} (I_0 + Jz). \quad (25)$$

Neglecting J against I_0 and I_N the result is obtained that E_H is equal to E_S , which is naturally expected from the general considerations.

§ 5. On the Condition under which Ferromagnetism appears.

According to (21) and (25), the more electrons there are with parallel spins, the lower becomes the energy of the system of electrons arising from the interactions between electrons. In Slater's theory, in order that the number of electrons with parallel spins can increase, the Fermi energy must

increase because of the breadth of the energy band. Hence, if the decrease of the interaction energy exceeds the increase of the Fermi energy, we can expect ferromagnetism.

On the other hand, in Heisenberg's theory we must consider the energy arising from interactions between electrons and ions besides interactions of electrons between each other. The interaction energy between electrons and ions is expressed by $-\sum_{k, \kappa} \frac{e^2}{r_{k\kappa}}$, which is negative in sign, where k and κ represent electrons and ions respectively. The mean value of this energy would also take a form like (25) owing to the non-orthogonality of the wave functions of nearest-neighboring atoms. In this case, therefore, if the part of the mean energy of this interaction dependent on m which is opposite in sign to that of (25) and hence positive, is exceeded by the interaction energy between electrons $\sum_{kk} \frac{e^2}{r_{kk}}$ dependent on m , then ferromagnetism takes place.

Since the interactions $-\sum_{k, \kappa} \frac{e^2}{r_{k\kappa}}$ between electrons and ions cause a broadening of the energy level, this energy in Heisenberg's theory can be considered as corresponding to the Fermi energy in Slater's theory.

Finally, in the generalized Heisenberg theory, as is seen from the equation (25), it is not the exchange integral, but the Coulomb integral between two electrons in the same atom, i.e. I_0 , that is important for ferromagnetism.

In more general cases, however, where one atom has more eigenfunctions than two, also exchange integrals in the same atom comes to play an appreciable rôle.

In conclusion the author would like to express his deep thanks to Professor Dr. K. Ariyama and Assistant Professor Dr. S. Miyahara for their valuable discussion and kind encouragements during this work.

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A Note on the Theory of Ferromagnetism.

Haruo SHIMAZU.

Physical Institute, Nagoya University.

(Received Dec. 11, 1948)

Two typical theories of ferromagnetism, one of Slater⁽¹⁾ and the other of Heisenberg,⁽²⁾ are compared by using simple wave functions; and a method to obtain a better approximation to express ferromagnetism is considered in a rather rough way.

The exchange integral, which plays an essential rôle in Heisenberg's theory of ferromagnetism, is

$$J_H = e^2 \int \psi_f^*(1) \psi_{f'}^*(2) \left(\frac{1}{r_{12}} - \frac{1}{r_{f2}} - \frac{1}{r_{f'1}} \right) \psi_{f'}(1) \psi_f(2) d\tau_1 d\tau_2,$$

where $\psi_f(1)$ is the wave function of electron 1 belonging to the isolated atom f . This is also written,

$$\begin{aligned} J_H = & e^2 \int \psi_f^*(1) \psi_{f'}^*(2) \frac{1}{r_{12}} \psi_{f'}(1) \psi_f(2) d\tau_1 d\tau_2 \\ & - e^2 \int \psi_f^*(1) \psi_{f'}(1) d\tau_1 \int \psi_{f'}^*(2) \frac{1}{r_{f2}} \psi_f(2) d\tau_2 \\ & - e^2 \int \psi_{f'}^*(2) \psi_f(2) d\tau_2 \int \psi_f^*(1) \frac{1}{r_{f'1}} \psi_{f'}(1) d\tau_1. \end{aligned} \quad (1)$$

The first term (which we denote by J_1) is due to potential between electrons belonging respectively to two nearest neighbouring atoms. The second and third terms are the products of two integrals, and if we denote them by Δ and γ , J_H is simply

$$J_H = J_1 - 2\Delta\gamma. \quad (2)$$

Slater's exchange integral is

$$J_S = E_C / N, \quad (3)$$

where N is the number of atoms, and E_C is the Coulomb integral between electrons within the same atom and is expressed by

$$E_C = e^2 \int \psi_f^*(1) \psi_f^*(2) \frac{1}{r_{12}} \psi_f(1) \psi_f(2) d\tau_1 d\tau_2 \quad (4)$$

Nextly, we consider the energy increase at the absolute zero of temperature when one spin changes its sign at a state in which all spins are parallel. According to Heisenberg's theory, this is

$$E_H = s(J_1 - 2A\gamma). \quad (5)$$

Here we considered simply the mean value of energy change (departing from the idea of Bloch's spin wave⁽³⁾), which is not accurate in this case of low temperature, but convenient in treating ferromagnetism by the lattice model.

In Slater's theory the spreading of the energy band must be taken into account. He considered the $3d$ band of Ni, but here for the sake of facilitating comparison with Heisenberg's theory, the one-electronic wave functions are expressed as linear combinations of the atomic functions, and the perturbation terms in the Hamiltonian include all interactions between electrons and ions not considered in the unperturbed state. Then the wave functions and corresponding energies are

$$\begin{aligned} \phi_k &= \frac{1}{\sqrt{N!}} \sum_j e^{ikR_j} \phi_j(r), \\ E &= E_0 - 2\gamma(\cos k_x a + \cos k_y a + \cos k_z a), \\ \frac{\pi}{a} &\geq k_x, k_y, k_z \geq -\frac{\pi}{a}. \end{aligned} \quad (6)$$

(simple cubic)

γ is the same as that appearing in the case of Heisenberg. The energy increase corresponding to (5) is

$$E_C - 2s\gamma \quad (7)$$

The second term is due to the fact that the electron which occupies the highest energy state of the singly occupied band drops to the lowest of unoccupied one by reversal of its spin.

Then the necessary condition that all spins are parallel at the absolute zero of temperature is

$$\begin{aligned} \text{Heisenberg:} \quad & J_1 > 2A\gamma. \\ \text{Slater:} \quad & E_C > 2s\gamma. \end{aligned} \quad (8)$$

J_1 is very small compared with E_C and is neglected in the calculation of Slater's exchange integral. A is small compared with γ , and vanishes if the atomic functions are orthogonal. That is to say, the necessary condition for all spins to be parallel is related to the difference of two quantities each of which is much larger in the case of Slater than of Heisenberg.

That Slater's exchange integral J_s is equal to E_c/N is concretely understood as follows. As the one-electronic wave functions are spread over the whole crystal, the probability that a particular electron is found in a particular cell is $1/N$. The probability that two electrons are found in a particular cell is $1/N^2$, and if the energy decreases by E_c because of the exchange effect when they meet within a cell, then as there are N cells, the exchange integral between electrons becomes $E_c/N^2 \times N = E_c/N$. Obviously, it compensates the energy due to the Coulomb integral which is gained when electrons meet within a cell. In other words, the probability of finding two electrons with parallel spins in the same cell is *perfectly* zero; *perfectly*, of course, as the result of our approximation of tight binding. Then we can picture a model in which all the electrons are running through the whole crystal with those of parallel spins keeping away from one another. And in fact, it is ascertained by actual calculations⁽⁴⁾ that the interaction energy is identical with that given when the interaction energy between all electrons distributed in every crystal cell is averaged over all permissible configurations, viz. all except those which place two electrons with parallel spins in the same cell.

These configurations do not belong to the same energy even when the number of positive spins is given, but this fact is overlooked in Slater's treatment, which amounts to assuming equal probabilities for the occurrences of space-configurations with different energies. It is desirable to make calculations with the different probabilities taken into account, which, however, would be extremely complicated in the exact quantum mechanical method. We therefore take a simple lattice model in which all the electrons are distributed in every cell in such a way that no two electrons with parallel spins appear in any single cell, though it has the drawback that the spreading of the energy band cannot be taken into account satisfactorily.

Let the number of lattice points be $2n$, the number of positive and negative spins, $n+m$ and $n-m$ respectively, and the number of doubly occupied lattice points s . Then the total energy corresponding to certain values of m and s is given, in Bragg-Williams' approximation, as

$$\bar{E} = s\bar{E}_c + \frac{n^2 - m^2}{2n} z J_H, \quad (9)$$

where the zero point of energy corresponds to the state with all spins parallel. The reason that the Coulomb energies between electrons which belong respectively to nearest neighbouring lattices, next nearest neighbouring ones, and so on, do not appear in (9) is that they do not change

with m and s within Bragg-Williams' approximation.

The number of ways to distribute $(n+m)$ positive spins on $2n$ lattice points is $\binom{2n}{n+m}$, that to distribute s of the $(n-m)$ negative spins on the $(n+m)$ lattice points which are already occupied by positive spins is $\binom{n+m}{s}$, and the number of ways to distribute s holes on $\{2n-(n+m)\}$ lattice points is $\binom{n-m}{s}$.

Then the partition function of a whole crystal is

$$Z(m, s) = \sum_{m, s} \binom{2n}{n+m} \binom{n+m}{s} \binom{n-m}{s} e^{-\frac{sE_C + \frac{n^2-m^2}{2n} zJ_H}{kT}} \quad (10)$$

The free energy, by using Stirling's formula, is

$$\begin{aligned} f(m, s) = -kT \log Z(m, s) = & \left(sE_C + \frac{n^2-m^2}{2n} zJ_H \right) - kT \left[2n \log 2n \right. \\ & - (n+m-s) \log (n+m-s) \\ & \left. - (n-m-s) \log (n-m-s) - 2s \log s \right] \end{aligned} \quad (11)$$

The most probable values of m and s are determined from the two relations, $\partial f / \partial s = 0$ and $\partial f / \partial m = 0$, i.e.,

$$\frac{(1-s)^2 - m^2}{s^2} = e^{\frac{E_C}{kT}}, \quad (12)$$

$$m = (1-s) \tanh \frac{m}{2} \frac{zJ_H}{kT}, \quad (13)$$

where m/n and s/n are replaced by m and s respectively. From (12),

$$s = \frac{1}{\xi - 1} \left\{ \sqrt{\xi \left(1 - m^2 \left(1 - \frac{1}{\xi} \right) \right)} - 1 \right\}, \quad \xi = e^{\frac{E_C}{kT}}. \quad (14)$$

At low temperatures ($\xi \gg 1$),

$$s = \frac{1}{\xi^{1/2}} \sqrt{1 - m^2} - \frac{1}{\xi} \approx 0.$$

Then from (13),

$$m = \tanh \frac{m}{2} \frac{zJ_H}{kT}. \quad (15)$$

This is identical with the result of Heisenberg's theory.

At high temperatures,

$$s = \frac{1}{\xi^{1/2} + 1} - \frac{m^2}{2} \xi^{-1/2}.$$

Inserting this in (13),

$$m = \left(1 - \frac{1}{\xi^{1/2} + 1} + \frac{m^2}{2} \xi^{-1/2} \right) \tanh \frac{m}{2} \frac{zJ_H}{kT}. \quad (16)$$

Then follows the next equation to determine the Curie point.

$$\frac{zJ_H}{kT_0} = 2 \left(1 + e^{-\frac{E_C}{2kT_0}} \right) \quad (17)$$

If we consider $E_C \gg J_H$,

$$T_0 \approx \frac{zJ_H}{2k}.$$

This is also the same result as obtained from Heisenberg's theory.

If all configurations with the same m occur with equal probability (Slater), the partition function becomes

$$Z(m) = \sum_n \binom{2n}{n+m} \binom{2n}{n-m} e^{-\frac{n^2 - m^2}{2n kT} E_C}, \quad (18)$$

where J_H is neglected against E_C .

Then by the same procedure,

$$m = \tanh \frac{m}{4} \frac{E_C}{kT}, \quad (19)$$

$$T_0 = \frac{E_C}{4k}.$$

The Curie point is very high, and this is just to be expected from our approximation of neglecting the spreading of the energy band. In spite of this, the more accurate calculation described above again gives us results similar to those of the original Heisenberg theory.

In conclusion the writer wishes to express his cordial thanks to Prof. K. Ariyama, Dr. S. Miyahara and other members of the Laboratory of Ferromagnetism for their kind discussions.

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Pri la Kalkulado de Elektromagnetaj Fenomenoj per Kampo de Neŭtra Vektor-Mezonoj kun Neglektebla Maso.

F. J. BELINFANTE

Dept. of Physics, Purdue University, Lafayette, Indiana (U.S.A.)

(Received Feb. 10, 1949.)

Summary.

Electromagnetic interactions between matter can be described, for quantum-mechanical as well as for classical phenomena, as caused by a neutral vector-meson field, if the mass of the mesons is assumed to be so small as to be negligible.

One might perhaps think that such a description of an electromagnetic field would lead to difficulties with the derivation of Planck's radiation law, as this law depends essentially on the number of degrees of freedom of the field, which seems to be 3 instead of 2 for each plane wave in a meson field.

It is shown, however, that one can and should consider the longitudinal meson field in this case as a sum of a direct Coulomb field, which does not present a degree of freedom of the meson field, and a field of longitudinal "surplus"-mesons. True enough, the latter corresponds to a longitudinal degree of freedom in the meson field, but the coupling between these surplus-mesons and matter disappears in the limit of vanishing meson mass. The surplus-mesons therefore are not observable in this limit; moreover, they will not be emitted and are not in thermal equilibrium with the surrounding. Thence only the transverse mesons are to be taken into account in the derivation of a radiation law for mesons with vanishing mass, and Planck's law results.

Neŭtra vektormezon-kampo interaganta kun elektronoj estas priskribata de unua-orda Lagrangiana funkcio

$$L = (1/4\pi) \left\{ \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - F^{\mu\nu} \Delta_\mu A_\nu - \frac{1}{2} x^2 A^\nu A_\nu \right\} \\ + A_\nu j^\nu - \psi^\dagger (mc^2 \beta - i\hbar c \alpha^\nu \nabla_\nu) \psi, \quad (1)$$

kie $j^\nu \equiv -c\psi^\dagger \alpha^\nu \psi$, kun $a^0 = 1$, ($j^0 = -c\psi^\dagger \psi = \rho$), kaj kun $(-c) = \text{interagada konstanto}$, kaj $\hbar/c = \text{mezonmaso}$. De tiu ĉi Lagrangiana funkcio sekvas kampaj ekvacioj per sendependa variado de la kontraŭ-simetria tensoro $F^{\mu\nu}$, de la kvar-vektoro A_ν kaj de la undoroj (kvar-spinoroj) ψ^\dagger kaj ψ en $\iiint L dx dy dz dt = 0$. De (1) ankaŭ sekvas la Hamiltoniano

$$\mathfrak{H} = \mathfrak{H}_f + \mathfrak{H}_m + \mathfrak{H}_e, \quad (2)$$

kun

$$8\pi\mathfrak{H}_f = \iiint dx dy dz \{ \mathbf{E}^2 + x^{-2}(\text{div } \mathbf{E})^2 + x^2 \mathbf{A}^2 + (\text{rot } \mathbf{A})^2 \}, \quad (2a)$$

$$\mathfrak{H}_m = \iiint dx dy dz \psi^\dagger (mc^2\beta - i\hbar c \boldsymbol{\alpha} \cdot \nabla) \psi, \quad (2b)$$

$$\mathfrak{H}_e = \iiint dx dy dz \{ -\mathbf{j} \cdot \mathbf{A} - x^{-2} \rho \text{div } \mathbf{E} + 2\pi x^{-2} \rho^2 \}. \quad (2c)$$

La kanonikaj variabloj de ĉi tiu kampo estas la paroj $\{\mathbf{A}, -\mathbf{E}/4\pi c\}$ kaj $\{\psi, i\hbar\psi^\dagger\}$, dum ke $A^0 = \Phi$ kaj $F_{23} = \mathbf{H}_x$ ktp. estas derivitaj variabloj donitaj de la "identajtoj"

$$x^2 \Phi = 4\pi \rho - \text{div } \mathbf{E}; \quad \mathbf{H} = \text{rot } \mathbf{A}. \quad (3)$$

En la limito de nuliganta mezon-maso (t.e. por $x \rightarrow 0$) la kampaj ekvacioj sekvantaj de la variada principo kun L donas la *Maxwell*-ajn ekvaciojn por $\mathbf{E}, \mathbf{H}, \mathbf{A}$ kaj Φ , dum ke ψ obeas *Dirac*-ekvacion. Tial oni povas konjekti ke fizikaj fenomenoj, kalkulitaj por elektronoj interagantaj per tia neŭtra mezonkampo, por ekstreme malgrandaj valoroj de x aŭ en la limito $x \rightarrow 0$ donos simple la *electromagnetajn* fenomenojn de tiuj elektronoj. Efektive eksplicita kalkulo de la matrica elemento por disĵetado de liberaj elektronoj fare de liberaj elektronoj per ĉi tiu metodo donas la *Møller*-an esprimon. Ankaŭ la perturbiĝo de plur-elektrona daŭreca stato de atomo sekve de tia elektrona interagado per ĉi tiu metodo donas simple la *Breit*-an esprimon por la perturba energio. Avantaĝo de la uzado de mezon-kampo estas la fakto, ke por tia kampo la "*Lorentz*-kondiĉo" ne estas altrudita "kondiĉo", sed sekvas de la kampaj ekvacioj kiel q-nombra rilato.

Tamen en unu rilato la mezon-kampo ŝajnas esti tute diferenca de la elektromagneta kampo. En mezona kampo ĉiu ebena ondo havas tri eblajn polariziĝojn: du transversajn kaj unu laŭlongan. En kvantum-elektrodinamiko, la q-nombroj reprezentantaj la kampon principe povas esti polarizitaj en kvar "direktoj": Krom la tri polariziĝoj supre menciitaj ankaŭ "polariziĝo en tempeca direkto" estas ebla, donanta "skalaran ondon", pro la fakto ke la *Lorentz*-a kondiĉo inter Φ kaj \mathbf{A} en kvantum-elektrodinamiko ne estas q-nombra ekvacio kaj tial ne malpligrandigas la nombron da eblaj polariziĝoj. Tamen la *Lorentz*-a kondiĉo nun estas trudita al la stato de la sistemo. El kanonika vidpunkto tio signifas *du* kromajn kon-

diĉojn, ĉar por la nuligo de $S = \nabla_\mu A^\mu$ tra ĉiuj tempoj oni devas postuli ĉe $t=0$ ne nur la nuligon de S , sed ankaŭ de $\partial S / \partial t = 4\pi\rho \frac{1}{c} \text{div } \mathbf{E}$. Tio reduktas la nombron de polarizeblecoj efektive ekzistantaj por elektromagneta ondokampo de kvar al nur du por ĉiu ebena ondo.

Tial en la denombrado de la nombro de la gradoj de libereco de tia ondokampo envenas faktoro 2, kiu estas grava por la kalkulado de la energia denso en radiado de nigra korpo. Se oni anstataŭus tiun faktoron 2 en la nombro de gradoj de libereco per faktoro 3 (por mezonoj anstataŭ por fotonoj), ŝajnas ke la *Planck*'a formulo por la radiada energidensoj kaj por la konstanto de *Stefan-Boltzmann* estus multiplikata per faktoro $3/2$ kontraŭa al la eksperimentaj faktoj.

Tamen mi nun montros, ke en la limito $\hbar \rightarrow 0$ tiu faktoro $3/2$ malaperas, eĉ kvankam neniu grado de libereco malaperas en tiu limito. Por montri tion estas rekomendinde, unue apliki kanonikan transformadon de tipo tute simila al la *Fermi*-transformado uzata en kvantum-elektrodinamiko por la separado de la laŭlonga kampo. Mi diskutis tiun transformadon antaŭ kelkaj jaroj¹. Efektive ni povas uzi senŝange formulojn (42)–(52) de tiu artikolo¹, nur ellasante la lastan membron de ekvacio (48). Tiamaniere ni difinas novajn q-nombrojn $q'(\mathbf{E}', \mathbf{A}', \phi' \text{ ktp.})$, kiuj obeas la samajn komutajn rilatojn kiel la originaj q-nombroj $(\mathbf{E}, \mathbf{A}, \phi \text{ ktp.})$. Post tio ni esprimas la Hamiltonianon (2) per la novaj q-nombroj q' , substituante la esprimojn de ekvacio¹ (52) por la originaj q-nombroj q aperantaj en (2). Tiamaniere ni trovas

$$\mathfrak{H} = \mathfrak{H}' + \mathfrak{H}_m' + \mathfrak{W}'. \quad (4)$$

Ĉi tie \mathfrak{H}' kaj \mathfrak{H}_m' estas la samaj funkcionaloj de q' kiel \mathfrak{H} kaj \mathfrak{H}_m estas de q (vidu formulojn (2a) kaj (2b)). Tamen \mathfrak{W}' tute ne estas donita de (2c). Anstataŭe ni nun trovas

$$\mathfrak{W}' = \mathfrak{C} + \mathfrak{W}'_{\parallel} + \mathfrak{W}'_{\perp}, \quad (5)$$

kie

$$\mathfrak{C} = \frac{1}{2} \iiint \iiint dx dy dz dx' dy' dz' \frac{\rho(\mathbf{x}) \rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (5a)$$

estas *Coulomb*'a interagada energio inter la elektronoj;

$$\mathfrak{W}'_{\parallel} = - \iiint dx dy dz \rho \left(\frac{1}{\nabla} \cdot \mathbf{E}'_{\parallel} \right) \quad (5b)$$

(en la kutima skribmaniero¹) estas interagada energio inter la elektronoj kaj la laŭlognaj mezonaj ondoj \mathbf{E}'_{\parallel} post la *Fermi*-transformado; kaj

$$\mathfrak{W}'_{\perp} = - \iiint dx dy dz (\mathbf{j} \cdot \mathbf{A}'_{\perp}) \quad (5c)$$

estas la radiada interagado kun la transversaj ondoj, kiu ne nur kuplas la elektronan spinon kun magnetaj kampoj, sed ankaŭ kaŭzas absorbadon kaj elsendadon de transversaj kvantumoj fare de atomoj, kaj la kreadon (aŭ neniigadon) de negaton-poziton-paroj dum samtempa neniigo (kreigo) de transversa kvantumo en ekstera kampo.

Estas atentinde ke \mathbf{E}'_{\parallel} en (5b) ne estas la tuta origina kampo \mathbf{E}_{\parallel} , sed laŭ¹ (52) oni havas la rilaton

$$\mathbf{E}_{\parallel} = \mathbf{E}'_{\parallel} + \frac{4\pi}{\text{div}} \rho. \quad (6)$$

Ĉi tie la lasta termo evidente estas la senpera elektrostatika kampa intenso (*Coulomb*'a kampo) kaŭzata de la ŝarĝdenso ρ . Tial \mathbf{E}'_{\parallel} reprezentas nur la pliaĵon de la laŭlonga kampintenso krom tiu *Coulomb*'a kampo. Alivorte: stato kun elektrono, sed sen mezonoj en la origina reprezentaĵo estus stato *sen Coulomb*'a kampo ĉirkaŭ la elektronoj; en la nova reprezentaĵo, stato kun elektronoj kaj sen "kromaj" mezonoj priskribas elektronojn *kun Coulomb*'a kampo sed sen kroma radiada kampo. Evidente tio estas pli bona kiel proksimumo de nula ordo, ol stato kun elektronoj *sen Coulomb*'aj kampoj.

Por utiligi ekvaciojn (5), (5a-c), oni devas esprimi la kampojn \mathbf{E}' , \mathbf{A}' en la kutima maniero per kreadaj kaj neniigaj operatoroj por la mezonoj (*Jordan-Klein*-matricoj). Oni trovas

$$\left(\frac{1}{\nabla} \cdot \mathbf{E}' \right) = -i \sum_{\mathbf{k}} \sqrt{\frac{\hbar c}{\Omega \epsilon_{\mathbf{k}}}} \cdot \frac{\mathbf{x}}{\hbar} a'_{\mathbf{k}0} e^{i\mathbf{k} \cdot \mathbf{x}} + \text{konj.} \quad (7)$$

$$\mathbf{A}'_{\perp} = -i \sum_{\mathbf{k}} \sum_{\mu=\pm 1} \sqrt{\frac{\hbar c}{\Omega \epsilon_{\mathbf{k}}}} \cdot a'_{\mathbf{k}\mu} \mathbf{n}_{\mathbf{k}\mu} e^{i\mathbf{k} \cdot \mathbf{x}} + \text{konj.} \quad (8)$$

Ĉi tie $\mathbf{n}_{\mathbf{k}+}$ kaj $\mathbf{n}_{\mathbf{k}-}$ estas unuecaj vektoroj perpendiklaj al la onda vektoro \mathbf{k} ; $\epsilon_{\mathbf{k}} = +(\mathbf{x}^2 + \mathbf{k}^2)^{1/2}$; Ω estas la volumo de la fundamenta kubo. La koeficientoj en la elvolvaĵoj cetere estas elektitaj en tia maniero, ke la q -nombroj $a'_{\mathbf{k}\mu}$ (kun $\mu = -1, 0, +1$) obeas komutajn rilatojn

$$[a'_{\mathbf{k}'\mu'}; a'_{\mathbf{k}\mu}] = 0, \quad [a'_{\mathbf{k}'\mu'}; a'^*_{\mathbf{k}\mu}] = \delta_{\mathbf{k}',\mathbf{k}} \delta_{\mu',\mu} \quad (9)$$

dank'al la kanonikaj komutaj rilatoj de \mathbf{A}' kaj \mathbf{E}' , kaj tiel ke \mathfrak{H}' estas esprimebla per

$$\mathfrak{H}' = \sum_{\mathbf{k}} \sum_{\mu=-1}^{+1} \left(N'_{\mathbf{k}\mu} + \frac{1}{2} \right) E_{\mathbf{k}\mu}, \quad (N' = a'^* a'). \quad (10)$$

Tiam aŭtomate $E_{\mathbf{k}\mu} = \hbar c \epsilon_{\mathbf{k}}$. Evidente $N'_{\mathbf{k}\mu}$ estas la nombro de mezonoj (krom la *Coulomb*'a kampo) kun movkvanto $\hbar \mathbf{k}$ kaj kun polariziĝo μ ($=0$ por

laŭlonga kaj $= \pm 1$ por transversa ondo). De (10) oni vidas ke la laŭlonga kampo tute ne estas eliminata kaj ke la kampo enhavas energion proporciajn je la nombro de laŭlongaj mezonoj.

La avantaĝo de la nova reprezentaĵo de la kampo tamen tuj evidentiĝas, se oni komparas \mathfrak{W}' laŭ (5) kaj (7)–(8) kun \mathfrak{W} laŭ (2c) kun elvolvaĵo simila al (8) por \mathbf{A}_\perp (nur kun $u_{k\mu}$ anstataŭ $u'_{k\mu}$), kaj kun

$$\mathbf{A}_\parallel = -i \sum_{\mathbf{k}} \sqrt{\frac{\hbar c}{2\epsilon_k}} \cdot \frac{\epsilon_k}{x} \cdot a_{k0} \mathbf{n}_{k0} e^{i\mathbf{k} \cdot \mathbf{x}} + \text{konj.}, \quad (11)$$

$$x^{-2} \text{div } \mathbf{E} = i \sum_{\mathbf{k}} \sqrt{\frac{\hbar c}{2\epsilon_k}} \cdot \frac{k}{x} \cdot u_{k0} e^{i\mathbf{k} \cdot \mathbf{x}} + \text{konj.}, \quad (12)$$

kie $\mathbf{n}_{k0} = \mathbf{k}/k$. Ni vidas ke por $x \rightarrow 0$ la matricaj elementoj de \mathfrak{W} tute infinitiĝas, tiel ke en stato kun elektronoj sen *Coulomb*'a kampo ekzistus infinita probablo de elsendado de laŭlongaj kvantumoj. Aliflanke la matricaj elementoj de \mathfrak{W}' kaj de \mathfrak{C} restas finitaj por $x \rightarrow 0$, kaj la matricaj elementoj de \mathfrak{W}' eĉ nuligigas por $x \rightarrow 0$, pro la faktoro x en la numeratoro.

Tio signifas ke la kuplo inter la laŭlongaj “kromaj” mezonoj kaj la elektronoj malaperas en la limito de nuliganta mezon-maso. La konsekvencoj estas multaj kaj gravaj, se oni rigardas fo'onan kampon kiel kampon de mezonoj kun neglektebla maso ($x \approx 0$):

(1) La probablo de elsendado de laŭlongaj “kromaj” fotonoj estas (praktike) nulo.

(2) La probablo de absorbiĝo de ekzistantaj laŭlongaj “kromaj” fotonoj (praktike) estas nulo. Alivorte: nek la okulo nek la fotografa plako absorbus tiajn, kaj ili ne estas observeblaj. (Iom kiel la neŭtrinoj!)

(3) La kampo de tiuj laŭlongaj “kromaj”-fotonoj kun energio $\sum (N'_{k0} + 1/2) E_{k0}$ neniam estus en temperatura ekvilibro kun la ĉirkaŭanta materio, pro manko de sufiĉa kupliĝo. Fakte ĝi estus tute ne-dependa de materio.

(4) Tial la gradoj de libereco de la laŭlonga (krom)fotona kampo, kvankam ekzistantaj, pro tiu manko de kuplo kun la materio ne kontribuas al la *Planck*'a radiado de nigra korpo, kiu ja konsistas nur el la observebla parto de la radiada kampo *temperature en ekvilibro* kun la korpo.

Ni konkludas ke ankaŭ en ĉi tiu rilato mezona kampo kun neglektebla mezon-maso povas komplete esprimi la proprecojn de fotona kampo.

Estas aldoninde ke ankaŭ en la limito de grandaj nombroj da kvantumoj, kiam oni povas konsideri kaj trakti la mezon-kampon kiel klasikan kampon, ĉi tiu kampo por $x \rightarrow 0$ ekzakte priskribas la fenomenojn de elektrodinamiko, kie la kampaj intensoj estas donitaj de

$$\left. \begin{aligned} \mathbf{E} &= -\nabla \phi_{\text{stat}} - \partial \mathbf{A}'_{\perp} / c \partial t, \\ \mathbf{H} &= \text{rot } \mathbf{A}'_{\perp}, \end{aligned} \right\} \quad (13)$$

kun $\phi_{\text{stat}}(\mathbf{x}, t) \equiv \iiint dx' dy' dz' \rho(\mathbf{x}', t) / |\mathbf{x}' - \mathbf{x}|$. Ĉi tie ϕ_{stat} kaj \mathbf{A}'_{\perp} evidente respondas, kun ajŝo de la potencialoj, kiuj igas la vektor-potencialon sen-divergenta.

Plej facile oni deduktas (13) de la *Ehrenfest*'a teoremo, ke ondopakajo moviĝas kiel korpusklo en la kampa intenso derivita de la potencialoj aperantaj en la ond-ekvaĉio por la elektrona ψ . Tiuj potencialoj sekvas de (5a)–(5c). En (13) ni ellasis la parton de la potencialo devenantan de (b), ĉar tiu parto nuligas por $\mathbf{x} \rightarrow 0$ ankaŭ kiam la kampo estas traktata kiel klasika kampo.

Por demontri la nuligon de \mathbf{E}'_{\parallel} ankaŭ en ĉi tiu kazo, ni unue rimarku ke ϕ estas solvo de la ond-ekvacio

$$\{\square - x^2\} \phi = -4\pi\rho. \quad (14)$$

La solvoj de ĉi tiu ekvacio restas finitaj por $\mathbf{x} \rightarrow 0$. Ĉar laŭ (6) kaj (3) ni trovas ke

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} - \frac{4\pi}{\text{div}} \rho = -\frac{x^2}{\text{div}} \phi, \quad (15)$$

sekvas ke $\mathbf{E}'_{\parallel} \rightarrow 0$ por $\mathbf{x} \rightarrow 0$ ankaŭ en la limo de klasikaj kampoj.

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On the Thermal Reactions in Gas Phase. II.

Eizô KANAI.

Institute of Chemistry, Kyoto University.

(Received Dec. 11, 1948)

§ 4. Second order reactions.

(A). The exchange reactions.

We shall consider the reactions of the exchange type ;



and treat this reaction problem from the collisional standpoint. Here A , B , C and D are atoms or molecules or radicals, but the cases where A and B are both atoms are excluded.

At first, we make the wave equation of the system which contains all valence electrons and atomic cores concerned in the elementary reaction process (of course, the energy of the translational motion as a whole being put outside).

$$H\Psi = W\Psi.$$

Ψ can be expanded as follows :

$$\begin{aligned}\Psi &= \sum_{v,r} \chi_{v,r} f_{v,r}(r) / r \\ &= \sum_{v',r'} \chi_{v',r'} f_{v',r'}(r') / r' \\ &= \dots\dots\dots\end{aligned}$$

where

$$\chi_{v,r} = \chi_v \cdot \chi_r.$$

χ_r is the real rotational eigenfunction which is characterized by the rotational quantum numbers S of A , s of B , their resultant R , the relative orbital quantum number l of A and B and J , i.e. the resultant of R and l (the total angular momentum) and M specifying an independent one among the states with the same J . χ_v is the real vibrational eigenfunction which is characterized by the vibrational quantum numbers A and B . $\chi_{v,r}$'s are all normalized to 1. $\chi_{v',r'}$'s refer to the separation of the system to $C+D$ as like as $\chi_{v,r}$'s to $A+B$. In the limit of $r \rightarrow \infty$ (r being the relative distance of A and B). $f_{v,r}$ satisfies the next equation :

$$\left(-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + W_{v,r} - W\right) f_{v,r} = 0$$

where μ is the reduced mass and $W_{v,r}$ is the energy of the state (v, r) which is only dependent upon v, S and s . Therefore

$$f_{v,r}(r) \rightarrow \exp(\pm i k_{v,r} r)$$

where

$$\frac{\hbar^2}{2\mu} k_{v,r}^2 = W - W_{v,r} = \frac{1}{2} \mu u_{v,r}^2.$$

We note that $k_{v,r}$ or $u_{v,r}$ is the function of v, S and s .

Now let us determine $\Psi_{v,r}$ by the following boundary conditions:

$$f_{v',r'}(r') \rightarrow \delta_{v',r'}^{v,r} e^{-ik_{v',r'} r'} + a_{v',r'}^{v,r}(W) e^{+ik_{v',r'} r'}$$

The normal boundary condition that the plane wave of A collides with B under the inner condition (v, S, S_z, s, s_z) determine Ψ which can be expressed as a linear combination of $\Psi_{v,r}$'s. Noting the relation⁽¹³⁾

$$\chi_{S,S_z} \chi_{s,s_z} e^{ik_{v,r} r} = \frac{i\sqrt{\pi}}{k_{v,r}} \sum_{l=0}^{\infty} \sqrt{2l+1} (-1)^l \sum_r \mathbf{T}_r^{S,S_z,s,s_z,l,0} (e^{-ik_{v,r} r} - (-1)^l e^{+ik_{v,r} r})$$

where χ_{S,S_z} is the inner eigenfunction of A molecule with the quantum number S and S_z , χ_{s,s_z} refers to B molecule and \mathbf{T} is the transformation matrix from the base (S, S_z, s, s_z, l, m) to the base (S, s, R, l, J, M) , we can easily obtain

$$\Psi = \frac{i\sqrt{\pi}}{k_{v,r}} \sum_{l=0}^{\infty} \sqrt{2l+1} (-1)^l \sum_r \mathbf{T}_{S,S_z,R,l,J,M}^{S,S_z,s,s_z,l,0} \Psi_{v,r}.$$

From the expansion of $\Psi_{v,r}$'s by the eigenfunctions corresponding to $C+D$ separation, the cross-section of the transition:

$$\begin{aligned} &\text{incident wave } A+B \quad (v, S, S_z, s, s_z) \\ &\rightarrow \text{scattered wave } C+D \quad (v', S', S', s', R', l', J', M') \end{aligned} \quad (4.1)$$

becomes as follows,

$$\sigma_{v',r'}^{v,S,S_z,s,s_z}(W) = \frac{u_{v',r'}}{u_{v,r}} \frac{\pi}{k_{v,r}^2} \left| \sum_l \sqrt{2l+1} (-1)^l \sum_r \mathbf{T}_r^{S,S_z,s,s_z,l,0} a_{v',r'}^{v,r}(W) \right|^2$$

Or, if we use the scattering matrix $\mathbf{S}^{(14)}$ defined by

$$\sqrt{u_{v',r'}/u_{v,r}} a_{v',r'}^{v,r}(W) \equiv \mathbf{S}_{v',r'}^{v,r}(W) = \mathbf{S}_{v',S',s',R',l',J',M'}^{v,S,S_z,s,s_z,R,l,J,M}(W, J, M) \delta_{J',M'}^{J,M}$$

σ can be written as follows:

$$\sigma_{v',r'}^{v,S,S_z,s,s_z}(W) = \frac{\pi}{k_{v,r}^2} \left| \sum_l \sqrt{2l+1} (-1)^l \sum_r \mathbf{T}_r \cdot \mathbf{S}_{v',r'}^{v,r}(W) \right|^2. \quad (4.2)$$

It can be showed that S is symmetric and unitary.

Already we saw in § 3 that the boundary conditions

$$f_{v',r'}(r') \rightarrow a_{v',r'} e^{+ik_{v',r'} r'}$$

entailed the eigenvalue problem. We shall designate the eigenfunctions by $\Psi^{n,J,M}$'s and the eigenvalues by $E^{n,J,M}$'s.

$$E^{n,J,M} = W^{n,J,M} - \frac{i}{2} \Gamma^{n,J,M}$$

Since $\Psi^{n,J,H}$ can also be expressed by $\Psi_{v,r}$'s, a or S should have the poles at each $W = E^{n,J,M}$. Therefore

$$S_{v',r'}^{v,r}(W) = \frac{C_{v',r'}^{v,r}}{W - E^{n,J,M}} + Q_{v',r'}^{v,r}, \quad W \approx E^{n,J,M}$$

Naturally (v, r) and (v', r') have the same J and M . From the symmetrical character of S , it can be proved that

$$C_{v',r'}^{v,r} = \frac{K^{n,J,M}}{\hbar} C_{v,r}^{n,J,M} \cdot C_{v',r'}^{n,J,M}$$

where

$$C_{v,r}^{n,J,M} \equiv [\hbar u_{v,r}(E^{n,J})]^{1/2} a_{v,r}^{n,J,M}$$

and $K^{n,J,M}$ is the undetermined constant originally contained in $\Psi^{n,J,M}$. To decide this constant, the next normalization of $\Psi^{n,J,M}$ is convenient,

$$\Gamma^{n,J,M} = \sum_{v,r} |C_{v,r}^{n,J,M}|^2 + \sum_{v',r'} |C_{v',r'}^{n,J,M}|^2 + \dots \quad (4.3)$$

where the summation is taken over all modes of disintegration of the system. Thus $|C_{v,r}^{n,J,M}|^2$ can be interpreted as the so-called partial width referring to the disintegration into the state (v, r) .

Next we shall try to compute σ by using the knowledges about the activated complexes which are defined as a kind of the radio-active molecules. We consider the two limiting cases separately.

(a). The case of $P \ll d$.

In this case

$$S_{v',r'}^{v,r} \approx \frac{C_{v,r}^{n,J,M} \cdot C_{v',r'}^{n,J,M}}{W - E^{n,J}}, \quad W \approx W^{n,J}$$

for we can show that $K^{n,J,M} = \hbar$ by (4.3) and the unitary character of S . If we substitute this expression of S into (4.2) and sum up over all possible final states,

$$\sigma_j^{v_j, s_{z_j}, S_j, s_{z_j}}(W) = \sum_{J,M} \frac{\pi}{k_{v_j, s_{z_j}}^2} \frac{\Gamma_i^{n,J,M}(v_j, s_{z_j}, S_j, s_{z_j}) \Gamma_f^{n,J,M}}{(W - W^{n,J})^2 + (\Gamma^{n,J}/2)^2}, \quad W \approx W^{n,J} \quad (4.4)$$

where

$$\Gamma_{i(v, S, S_z, s, s_z)}^{n, J, M} = \left| \sum_l \sqrt{2l+1} (-1)^l \sum_R T_{S, S_z, R, l, J, M}^{S, S_z, s, s_z, l, 0} C_{v, S, s, R, l, J, M}^{n, J, M} \right|^2$$

is the width for the initial state (v, S, S_z, s, s_z) to make up the activated complex (n, J, M) and

$$\Gamma_f^{n, J, M} = \sum_{v', S', s', R', l', J', M'} |C_{v', S', s', R', l', J', M'}^{n, J, M}|^2$$

is the width for the activated complex (n, J, M) to decay towards the final side $C+D$. (4.4) frequently appears in the theory of the nuclear reactions⁽¹⁵⁾

(b). The case of $l \gg d$.

σ in this case can also be seen in the theory of nuclear reactions⁽¹⁶⁾ and can be understood as follows: the collision crosssection due to the relative translational motion is $\pi/k_{v, s}^2$, the relative probability of making up the activated complex (n, J, M) is $\Gamma_{i(v, S, S_z, s, s_z)}^{J, M}(W)/\Gamma^J(W)$ and number of activated complexes concerned in the present process is given by $\Delta W/d^J(W)$, any of which may be made up, where $d^J(W)$ is the level spacing and because of the uncertainty principle $\Delta W = \frac{\hbar}{\Delta t} = \frac{\hbar}{\tau^J} = 2\pi\Gamma^J(W)$. On the other hand the relative probability with which the activated complexes decay to the final products is $\Gamma_f^{J, M}(W)/\Gamma^J(W)$ and hence

$$\sigma_{J, v, S, S_z, s, s_z}^{v, S, S_z, s, s_z}(W) = \sum_{J, M} \frac{2\pi^2}{k_{v, s}^2} \frac{\Gamma_{i(v, S, S_z, s, s_z)}^{J, M}(W) \Gamma_f^{J, M}(W)}{\Gamma^J(W) d^J(W)}, \quad W \geq W_0. \quad (4.5)$$

Of course, σ practically vanishes for $W < W_0$.

There remains the step in which we superpose the elementary 'processes' statistically and compute the specific reaction rate k . Since we assume the thermal equilibrium among the initial systems, the number of the initial systems per cm^3 specified by (v, S, S_z, s, s_z) and the relative momentum $\vec{p}_{v, s}$ (the whole energy is W) is given, at the absolute temperature T , by

$$\begin{aligned} & \frac{N_A N_B}{Z_A Z_B} \int k^{-3} d\vec{p}_0 d\vec{p}_{v, s} \exp\left(-\left[W_{v, s} + \frac{1}{2M} p_0^2 + \frac{1}{2\mu} p_{v, s}^2\right]/kT\right) \\ &= \frac{N_A N_B}{Z_i} Z_i^* k^{-3} d\vec{p}_{v, s} \exp\left(-\left[W_{v, s} + \frac{1}{2\mu} p_{v, s}^2\right]/kT\right) \end{aligned}$$

where N_A, N_B are the numbers of molecules A and B per cm^3 respectively and $Z_i = Z_A Z_B$ is the partition function of the initial states per cm^3 and Z_i^* is the translational partition function per cm^3 ,

$$Z_i^* = \int h^{-3} \vec{dp}_0 \exp(-p_0^2/2MkT) = h^{-3}(2\pi MkT)^{3/2}.$$

M being the sum of the masses of A and B . With above weights we superpose the elementary reactions statistically and get finally

$$k = \frac{Z_i^*}{Z_i} \sum_{v, S, S_z, s_z} \int h^{-3} \vec{dp}_{v, s, s_z} \exp(-W/KT) u_{v, s, s_z}^{v, S, S_z, s_z}(W). \quad (4.6)$$

(a). The case of $\Gamma \ll d$.

We substitute (4.4) in (4.6) and perform the summations and integrations. The computations are straightforward and we shall only write down the result.

$$k = \frac{Z_i^*}{Z_i} \sum_{n, J, M} \left(\frac{\Gamma_i^{n, J, M} \Gamma_f^{n, J, M}}{\hbar \Gamma^{n, J}} \right) \exp(-W^{n, J}/KT)$$

where

$$\Gamma_i^{n, J, M} = \sum_{v, S, S_z, s_z}^{W_{n, S_z} \leq W^{n, J, M}} \Gamma_i^{n, J, M}(v, S, S_z, s_z)$$

is the width for all possible states of the reactants to make up the activated complex (n, J, M) . Above k can be written in another form, when we take only the practically effective activated complexes (with the appreciable Γ 's) and neglect the contributions from the other complexes, if they existed. Namely, if we designate the energy of the lowest level of these effective complexes by W_0 (measured from the lowest level of the initial energy spectrum) and take this energy state as the origin of n and $W^{n, J}$, k becomes as follows:

$$k = \nu \frac{Z^*}{Z_i} \exp(-W_0/KT)$$

where W_0 is interpreted as the (theoretical) activation energy and

$$\nu = \left(\frac{\Gamma_i^{n, J, M} \Gamma_f^{n, J, M}}{\hbar \Gamma^{n, J}} \right)_{av.}$$

is the average frequency with which the systems accomplish the elementary reaction processes: the initial state \rightarrow the activated state \rightarrow the final state.

And

$$Z^* = Z_i^* \sum_{n, J, M} \exp(-W^{n, J, M}/KT)$$

is the partition function per cm^3 of the activated states.

(b). The case of $\Gamma \gg d$.

The calculation is similar to above one. And the result is as follows.

With

$$\Gamma_i^{J,M}(W) = \sum_{\substack{W_{v,S,g} \leq W \\ v, S, g, S_z, g_z}} \Gamma_i^{J,M}(v, S, g, S_z, g_z) (W)$$

k can be written in the next form :

$$k = \nu \frac{Z^*}{Z_i} \exp(-W_0/KT)$$

where

$$\nu = \left(\frac{\Gamma_i^{J,M}(W) \Gamma_f^{J,M}(W)}{\hbar \Gamma^J(W)} \right)_{av.}$$

$$Z^* = Z_i^* \sum_{J,M} \int_0^\infty \rho^J(W) dW \exp(-W/KT).$$

Thus above two limiting cases give us the same formula. Only difference is that the summations over the states in the former case are replaced by the integrations in the latter case. Therefore, it is reasonable to expect the same formula in the general case too,

$$k = \nu \frac{Z^*}{Z_i} \exp(-W_0/KT) \quad (II)$$

where

$$\nu = \left(\frac{\Gamma_i^{n,J,M} \Gamma_f^{n,J,M}}{\hbar \Gamma^{n,J}} \right)_{av.} \quad (4.7)$$

$$Z^* = Z_i^* \sum_{n,J,M} \exp(-W^{n,J,M}/KT). \quad (4.8)$$

It is the well-known fact that the experimental data of the exchange reactions of the second order are expressed in the form :

$$k = F \exp(-W_{ex}/KT) \quad (4.9)$$

except some atom-reactions, where F and W_{ex} may be considered as constant within the small range of temperatures. And the theoretical formula (II) has also the similar form.

As an example of the application of (II), we shall here treat the exchange reactions between H and H_2 and derive the values of the rate constant in the rough approximation. For that purpose, it is necessary to solve the Schrödinger's equation of the system ($H+H_2$) :

$$(\mathfrak{H} - E)\Psi = 0$$

as the eigenvalue problem already explained and under the allowable approximation that the reaction proceeds adiabatically.

If we are contented with the order estimation of k , the following

procedures may be used. For the present, we neglect the exchange of energies among the various normal freedoms in the pseudo-stable H_3 molecule and take as \mathfrak{S} the next approximate form,

$$\mathfrak{S} = \left(\frac{1}{2\mu_s} p_s^2 + \frac{1}{2} \mu_s \omega_s^2 x_s^2 \right) + \left(\frac{1}{2\mu_\theta} p_\theta^2 + \frac{1}{2} \mu_\theta \omega_\theta^2 x_\theta^2 \right) \\ + \left(\frac{1}{2\mu_\theta} p_\theta'^2 + \frac{1}{2} \mu_\theta \omega_\theta^2 x_\theta'^2 \right) + \left(\frac{1}{2\mu_i} p_i^2 + \frac{1}{2} \mu_i \omega_i^2 x_i^2 \right) + W_c',$$

where W_c' is the classical activation energy corresponding to the 'rotational state with the total angular momentum $J\hbar$. Then it is easily proved that the eigenvalues in question are given as follows,

$$E = W - \frac{i}{2} \Gamma$$

$$\left. \begin{aligned} W^J(n_s n_\theta n_\theta') &= (W_c' + \frac{1}{2} \hbar \omega_s + \hbar \omega_\theta) + n_s \hbar \omega_s + (n_\theta + n_\theta') \hbar \omega_\theta \\ \Gamma &= \hbar \omega_i. \end{aligned} \right\}$$

This result corresponds to the fact that three inner degrees of freedom of H_3 perform the quantized vibrations respectively, while the remaining one does the unstable vibration (decay motion) and the life of H_3 is equal to the virtual period of this last vibration, provided that the transitions of energy from one degree of freedom to another were entirely neglected. Since the consideration of the energy exchange between the various inner degrees of freedom is inevitable, we take it into account roughly by the factor $1/4$ multiplied to Γ , for there is only one among four degrees of freedom of H_3 to have to do with the decay motion. Thus we obtain

$$\Gamma'/\hbar \approx \frac{1}{4} \omega_i, \quad (\Gamma_i^n \Gamma_j^n)_{av.} \approx \frac{1}{2} \Gamma \cdot \frac{1}{2} \Gamma.$$

On the other hand, $W^J(n_s, n_\theta, n_\theta')$ enables us to calculate Z^* approximately and in the present case, its value is nearly equal to Eyring's Z^\dagger with the quantum corrections. Hence, finally

$$k \approx \frac{1}{16} \omega_i \frac{Z^\dagger}{Z_i} e^{-W_0/kT}$$

where

$$W_0 = W_c + \left(\frac{1}{2} \hbar \omega_s + \hbar \omega_\theta \right) - \frac{1}{2} \hbar \omega_{H_3}.$$

On the basis of the data given by Hirschfelder, Eyring and Topley⁽¹⁷⁾, the numerical values become as follows,

$$T=300^{\circ}K, k=17 \cdot 10^7 \text{ (} k_{\text{obs.}}=9.0 \cdot 10^7 \text{)}$$

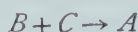
$$T=1000^{\circ}K, k=1.1 \cdot 10^{12} \text{ (} k_{\text{obs.}}=2.2 \cdot 10^{12} \text{) sec}^{-1} \text{ cm}^3 \text{ mol}^{-1}.$$

In the general examples, it is difficult to decide the ratio Γ/d unambiguously by the analogous procedure to the one in §3. But it is clear that the known examples rather belong to the class (i) than to the class (ii) in §3—that is, the non-adiabatic reactions are rare (the rough estimations show that $\Gamma/d < 1 \sim 10$ order as a rule).

These circumstances enable us to adopt the Eyring's approximation as the practical method of calculating k . The approximate assumptions and the behaviors when going to his formula are altogether the same as in §3. Further, it is also unchanged that the two corrections are not only useful but also necessary.

(B). The association reactions.

Next we shall consider the reactions of the association type



which are the reverses of the decomposition reactions. At present, it is supposed that B and C collide with each other and once make up an unstable molecule A^* , which becomes stabilized by the gas-kinematical collisions with other molecules because of its long life ($\gg \tau$), so that the reactions proceed in the second order. Hence the rate-determining step is the one of the formation of A^* . So the theoretical treatment is similar to the case (A) and the detailed explanations of deriving the final expression of k will be unnecessary.

$$k = \nu \frac{Z^*}{Z_i} \exp(-W_0/KT) \quad (\text{III})$$

where

$$\nu = \left(\frac{I_i^{n,J,M}}{\hbar} \right)_{av.} \quad (4.10)$$

$$Z^* = Z_i^* \sum_{n,J,M} (-W^{n,J}/KT). \quad (4.11)$$

Again the experimental formula are given by (4.9), which should enable us to estimate Γ/d of the present reactions. However, the direct estimations are difficult and we are only able to expect that the reactions probably belong to the class (i) in §3. The discussions about the Eyring's approximation are also the same as in §3 and in the above case (A).

§ 5. Conclusions.

At present, we know two famous classes of collisions in the physics, namely, the collisions between the electron and the atom and the collisions between the nucleon and the nucleus. In the former collisions, the electron is acted by the average field of force due to the atom as a whole, so that the elastic collisions are more frequent compared with the inelastic collisions (including the exchange collisions). On the other hand, in the latter collisions, the incident nucleon is acted successively by each nucleon in the nucleus and at last loses its energetical individuality in the nucleus, or in other words, incident nucleon and the emitted nucleon are not directly correlated, so that the exchange collisions are generally more frequent than the elastic collisions. That is, the former are of the surface reaction type and the latter are of the compound reaction type.

In the previous sections, we emphasized that the elementary collisions of the chemical reactions were rather of the compound reaction type. In fact, the frequency factors of the exchange reactions of the second order between simple molecules are of the same order with the frequencies of the gas-kinematical collisions. From the various reasons, we conclude that the elementary reactions in question are rather similar to the nuclear reactions. Of course, this parallelism must not be taken literally. First, the effective inner degrees of freedom of the activated complexes concerned in the elementary chemical reactions are of the order <10 , while the corresponding numbers of the compound nuclei in the nuclear reactions are of the order 100. Second, the nuclear reactions are commonly in the quantum mechanical region, while the chemical reactions lie halfway between the classical region and the purely quantum mechanical one. If these conclusions are admitted, we can understand the situation where the chemical reactions should be put among the collision problems.

At last, we shall touch on the limit of the applicability of the present considerations in brief. The fundamental assumptions 1, 2 and 3 in § 2 will be satisfied for the thermal reactions in gas phase except the rapid reactions and the reactions performed under the extremely high pressures. The fourth assumption in § 2 excludes some atom-reactions, third order reactions and the reactions between ions (in fact, these reactions are not subjected to the experimental formula of the Arrhenius type $k = F \exp(-W_{ex}/KT)$). Further, the reactions of the order between one and two are not treated in this paper, whose solutions have been sought since the works of Rice, Ramsperger and Kassel,

In conclusion, I should like to express my deep gratitude to Professors S. Tomonaga, H. Yukawa and M. Kobayasi for their kind interest in this work. Further the author is indebted to Mr. H. Narumi for his valuable discussions.

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Progress of Theoretical Physics Vol. IV, No. 2, Apr.~Jun., 1949.

Theory of the F-centers of Coloured Alkali Halide Crystals.

Part I. Structure of F-Absorption Bands.*

Toshinosuke MUTO.

Institute of Science and Technology, University of Tokyo.

(Received Jan. 15, 1949).

§ 1. Introduction.

As is well-known, the various physical properties of semiconductors are essentially determined by the existence of the so-called quasi-atomic states in the forbidden region of energy band system characteristic for the body crystal. Since the lattice imperfections due to the inclusion of foreign atoms and stoichiometric excess-atoms, the internal cracks within the crystals or other mechanism may be considered to become possible origins which give rise to the mentioned quasi-atomic states, the consistent, theoretical attack of semiconductors will be necessarily confronted with a difficult problem of treating mathematically the imperfect crystals, while in the perfect crystals the method of solution of Schrödinger wave equation is well known to be much simplified owing to the periodicity and symmetry valid within the perfect crystals. In addition to the mentioned theoretical difficulty in solving the wave equation of the imperfect crystal, the vast experimental investigations of the physical properties of semiconductors reveal that even the experimental data of the same kind of semiconductors are always at variance with each other, showing the sensitive dependencies of the materials upon the experimental method of preparations. Such experimental difficulties to get the precise reproduction of the materials with consistent physical properties will become serious obstacles for proceeding the reliable physical considerations of the semiconductors.

In view of the above situations, both theoretical and experimental, therefore, it seems to be more enlightened at the present stage of the theory that the particularly simple semiconductors with reliable and systematic experimental data shall be thoroughly worked out theoretically in order to

* Read in the annual meeting of Physical Society of Japan, held at Kyoto University, on May 21, 1948.

find a suitable method of approximation for solving the quasi-atomic states characteristic for semiconductors and then the theoretical method obtained shall be extended to those of the more complicated semiconductors accessible in both pure and applied fields of atomic physics. From the standpoint mentioned above, we shall here take the coloured alkali halide crystals which have been worked out experimentally by the systematic observations of Pohl and his coworkers, von Hippel and others.⁽¹⁾ According to the experimental investigations the colour-center, i.e., F-center responsible for the selective absorption of light and the colouration of the crystals may be considered to be typical ones of the mentioned quasi-atomic states within semiconductors. In Part I we shall work theoretically the detailed structure of F-absorption band peculiar to the alkali-halide crystals in connection with its temperature effect, and in Part II the general theory of trapped electrons within the quasi-atomic states of semiconductors shall be developed. The quantitative application of the general theory to the F-centers of NaCl crystal shall be attempted to do in comparison with the experimental observations in Part III.

§ 2. A Model for F-center.

According to the detailed theoretical considerations⁽²⁾ of all the available experimental results on coloured alkali halides, the F-centers, formed by heat treatment in alkali vapour and X-rays or cathode rays bombardments, should be considered to be trapped electrons attached to negative ion vacancies within alkali halides: which model was first proposed by de Boer⁽³⁾ who suggested that when the alkali halides are heated in the presence of alkali metal vapour, halogen ions diffuse to the surface of the crystal and combine with the atoms of the vapour to form new layers of lattice. The valence electron initially attached to the alkali metal atoms wander into the crystal and become attached to the negative ion vacancies. According to this picture, therefore, the F-centers responsible for the longer wave length absorption of the crystal consist of vacancies to which two

electrons are attached. We shall represent the essential features of the mentioned model schematically in Fig. 1, in which one dimensional variation of the potential energy is drawn in the neighbourhood of negative ion missing point. Although such a model has been

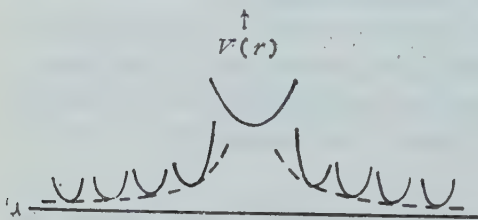


Fig. 1

theoretically worked by Tibbs⁽⁴⁾, his method of approximation to the F-center seems to be oversimplified and disagreeable in comparison with the reliable experimental results, the criticism and improvement of which shall be carried out in Part II and III of the present paper.

§ 3. Stationary States of Trapped Electron System within F-center together with Thermal Vibrations of Lattice.

In the following computation it has been assumed that the F-centers are dilutely distributed within the crystal, the mutual interactions between two or more F-centers being neglected approximately.* In such case each of the F-centers involved may be considered to behave independently with each other. For simplicity, therefore, it will be permissible to take the system consisting of one F-center within the crystal and sum up the contributions from F-center in order to get the total effect of the phenomena concerned.

Now the wave equation of such a system representing the F-center is written as

$$-\frac{\hbar^2}{2m}\Delta\Psi - \frac{\hbar^2}{2}\sum_{\lambda}'\frac{1}{M_{\lambda}}\Delta_{\lambda}\Psi + \sum_{\lambda}'V_{\lambda}(\vec{R}_{\lambda}-\vec{r})\Psi + \sum_{\lambda\mu}'V_{\lambda\mu}(\vec{R}_{\lambda}-\vec{R}_{\mu})\Psi = E\Psi, \quad (1)$$

in which m denotes the electron mass, M_{λ} the ion mass situated at a lattice point specified by radius vector \vec{R}_{λ} , \hbar Planck's constant divided by 2π , Δ and Δ_{λ} represent Laplacian operators with respect to electronic and ionic coordinates respectively. The first and second terms express kinetic energy operators of electron and ions, the third the interaction energies between electron and ions and the fourth the interaction energies between ions specified by the suffices. The dash attached to the summation means the elimination of the contribution from the missing lattice point in the neighbourhood of the F-center involved.

According to the adiabatic approximation⁽⁵⁾ in the theory of molecular structure, the wave equation (1) may be separated into the electronic wave

* Complex centers consisted of two or more F-centers and lattice vacancies have been discussed theoretically by F. Seitz with a special reference to Molnar's new absorption band in KCl coloured by X-ray illumination, the electronic structure of which has been worked quantum-theoretically by the author from the modern view of solid.

equation with nuclear parameter \vec{R}_λ and that of lattice vibration system as follows.

Putting,

$$\psi(\vec{r}, \vec{R}_1, \vec{R}_2, \dots) = \psi(\vec{r}; \vec{R}_1, \dots) \cdot \psi(\vec{R}_1, \dots) = \psi(\vec{r}; \vec{R}_\lambda) v(\vec{R}_\lambda), \quad (2)$$

we obtain, after a conventional rearrangement,

$$-\frac{\hbar^2}{2m} \Delta \psi + \sum' V_\lambda(\vec{R}_\lambda - \vec{r}) \psi = (E^e(\vec{R}_\lambda) - \sum' V_{\lambda\mu}(\vec{R}_\lambda - \vec{R}_\mu)) \psi \quad (3)^*$$

and

$$-\frac{\hbar^2}{2} \sum_\lambda \frac{1}{M_\lambda} \Delta_\lambda v + E^e(\vec{R}_\lambda) v = E v + \frac{\hbar^2}{2} \sum_\lambda \frac{1}{M_\lambda} \int \psi \Delta_\lambda \psi d\vec{r} \cdot v \quad (4)$$

for electronic and ionic motions respectively. The second term on the right hand side of (4) represents the direct coupling between the electronic motion and lattice vibration which may be approximately neglected except for the high excitation of lattice vibrations.

The equilibrium configuration of the lattice may be determined, as usual, by the following minimum conditions,

$$\frac{\partial E^e}{\partial R_{ix}} = \frac{\partial E^e}{\partial R_{iy}} = \frac{\partial E^e}{\partial R_{iz}} = 0 \quad i=1, 2, \dots, \lambda, \dots, \quad (5)$$

which may be considered to give rise to some distortion of lattice configuration in the neighbourhood of the vacancy. The expansion of the electronic energy $E^e(\vec{R})$ near the equilibrium configuration leads to

$$E^e(\vec{R}_\lambda) = E^e(\vec{R}_{\lambda 0}) + \frac{1}{2} \sum_{\lambda x, \mu y} f_{\lambda x, \mu y} u_{\lambda x} u_{\mu y} + \delta E^e, \quad (6)$$

in which $\vec{R}_{\lambda 0}$ specifies the equilibrium configuration of lattice, $u_\lambda = R_{\lambda x} - R_{\lambda 0}$, etc. the displacements of the lattice points from the equilibrium position and $f_{\lambda x, \mu y} = (\partial^2 E^e / \partial R_{\lambda x} \partial R_{\mu y})_{\vec{R}_\lambda = \vec{R}_{\lambda 0}}$.

Neglecting the higher order term δE^e with respect to displacement vectors \vec{u}_λ for low excitation of lattice vibration, the Hamiltonian of (4) leads to

$$H = -\frac{\hbar^2}{2} \sum_\lambda \frac{1}{M_\lambda} \Delta_\lambda + \frac{1}{2} \sum_{\lambda x, \mu y} f_{\lambda x, \mu y} u_{\lambda x} u_{\mu y} + E^e(\vec{R}_{\lambda 0}), \quad (7)$$

which may be written as usual⁽⁶⁾

* The theoretical discussion of the electronic motion near the vacancy shall be carried out in Part II.

$$H = \sum_q \left(\frac{1}{2M} p_q^2 + \frac{M}{2} \omega_q^2 x_q^2 \right) + E^e(\vec{R}_{\lambda 0}), \quad (7')$$

through the linear transformation to the normal coordinates x_q . q specifies each mode of normal vibrations with a wave number vector and a polarization, ω_q its circular frequency dependent upon the curvature near the equilibrium of hypersurface of energy $E^e(\vec{R}_\lambda)$ in configurational space, M represents approximately the total mass or reduced mass of ion pairs according to whether it belongs to elastic branch or optical one of lattice vibration.

Thus, we have, for the approximate solutions of the total system (1),

$$E = E(\vec{R}_{\lambda 0}) + \sum (N_q + 1/2) \hbar \omega_q + \bar{\delta} E^e \quad (8)$$

and

$$\Psi(\vec{r}, \vec{R}_\lambda) = \phi(\vec{r}, \vec{R}_{\lambda 0}) \cdot \prod_q v_q(x_q) + \delta \Psi \quad (9)$$

respectively, where N_q expresses the number of lattice quanta, $\bar{\delta} E^e$ and $\delta \Psi$ the correction terms of higher order, $v_q(x_q)$ stands for the eigenfunction of linear oscillator, being expressed by Hermite polynomials.

§ 4. Intensity Distribution of F-Absorption Bands.

According to the general theory in quantum mechanics,⁽⁷⁾ we get, for the intensity of light absorption due to dipole radiation,

$$S(\nu) = \frac{4\pi^2}{3} \left(\frac{c^2}{\hbar c} \right) \nu |\vec{X}_{if}|^2 \cdot I_0(\nu), \quad (10)$$

in which $|\vec{X}_{if}|^2 = |x_{if}|^2 + |y_{if}|^2 + |z_{if}|^2$ represents the square-magnitude of the coordinate matrix element associated with the transition i to f , ν the frequency of radiation and $I_0(\nu)$ the intensity of incident radiation. Using (9), the transition matrix element may be written as

$$\begin{aligned} |\vec{X}_{if}|^2 &= |\int \phi^*(\vec{r}; \vec{R}_{\lambda 0}) \vec{X} \phi'(\vec{r}; \vec{R}_{\lambda 0}') d\vec{r}|^2 \cdot \left| \int \prod_q v(x_q) \prod_q v'(x_q') dx_q \right|^2 \\ &= (E^e[\vec{X}|E^e])^2 \cdot \left| \int \prod_q v(x_q) \prod_q v'(x_q') dx_q \right|^2, \end{aligned} \quad (11)$$

where the quantities with and without dashes belong to the final and initial states respectively.

In accordance with the electronic excitation in the final state, the energy surface $E^e(\vec{R}_\lambda)$ and thus the equilibrium positions of lattice points

are generally changed into the new ones, which situation leads to the changes of normal vibration frequencies ω_q and the normal coordinates x_q respectively.

It, therefore, follows,

$$\omega_q'(f'_{\lambda x, \mu y}) = \omega_q(f_{\lambda x, \mu y}) + \Delta\omega_q \quad (12)$$

and

$$x_q'(u_\lambda') = x_q(u_\lambda) + \Delta x_q, \quad (13)^*$$

in which $\vec{u}'_\lambda = \vec{u}_\lambda - \Delta \vec{R}_\lambda$, $\vec{R}_{\lambda 0} + \Delta \vec{R}_\lambda$ denoting the changed position vector of the lattice point λ . When ΔR_λ are sufficiently small compared with $R_{\lambda 0}$, the following relations are easily shown to be valid approximately.

$$\Delta x_q \sim G^{-3/2} \sum_{\lambda} \Delta \vec{R}_\lambda \cdot \vec{e}_q \cdot \cos(\vec{q} \cdot \vec{\lambda}),$$

in which G^3 expresses the total number of atoms involved in the crystal and \vec{e}_q the polarization unit vector of q -th normal vibration.

In the following computation it has been reasonably supposed that $|\Delta\omega_q|/\omega_q$ and Δx_q are of small quantities, with respect to which the expansion of the eigenfunctions of the normal modes of vibrations shall be carried out.

Namely, we have, to a second order of approximation,

$$\begin{aligned} v'(x_q') &= \left(\frac{\sqrt{a'}/\pi}{2^{N_q'} N_q'!} \right)^{1/2} \exp(-a' x_q'^2/2) H_{N_q'}(\sqrt{a'} x_q'), \quad a' = \frac{M\omega_q'}{\hbar} \\ &= [v_{N_q'}(x_q) + \sqrt{2N_q'} \cdot \Delta \xi \cdot v_{N_q'-1}(x_q) + \\ &\quad \sqrt{N_q'}(N_q'-1) \cdot \Delta \xi^2 \cdot v_{N_q'-2}(x_q)] \cdot (1 - \xi \Delta \xi + \frac{1}{2}(\xi^2 - 1) \Delta \xi^2), \end{aligned} \quad (14)$$

in which

$$\left. \begin{aligned} v_{N_q'}(x_q) &= \left(\frac{\sqrt{a}/\pi}{2^{N_q'} N_q'!} \right)^{1/2} \exp(-a x_q^2/2) H_{N_q'}(\sqrt{a} x_q), \quad a = \frac{M\omega_q}{\hbar} \\ \xi &= \sqrt{a} x_q, \\ \Delta \xi &= \sqrt{a} \Delta x_q + \sqrt{a} \left(\frac{\Delta\omega_q}{2\omega_q} - \frac{1}{2} \left(\frac{\Delta\omega_q}{2\omega_q} \right)^2 \right) x_q + \sqrt{a} \Delta x_q \frac{\Delta\omega_q}{2\omega_q} \end{aligned} \right\} \quad (15)$$

and $H_{N_q'}(\sqrt{a} x_q)$ expresses the Hermite polynomial of the N_q' -th order.

* Although Δx_q will be dependent upon x_q from the general point of view, it is tentatively supposed, for simplicity, that Δx_q do not depend upon x_q but only upon q which specifies each normal mode of vibrations.

Thus, for the integral involved in the transition matrix elements, it follows

$$\begin{aligned}
 \int v(x_q) v'(x_q) dx_q = & \left[1 - u(\Delta x_q)^2 (N_q + 1/2) - \frac{1}{32} \left(\frac{\Delta \omega_q}{\omega_q} \right)^2 (3N_q^2 - 10N_q - 4) \right] \times \\
 & \delta_{N_q', N_q} + \sqrt{\frac{a}{2}} \Delta x_q \cdot \sqrt{N_q} \left(-1 + \frac{\Delta \omega_q}{4\omega_q} (N_q - 1) \right) \delta_{N_q', N_q-1} + \sqrt{\frac{a}{2}} \Delta x_q \times \\
 & \sqrt{N_q + 1} \left(1 - \frac{\Delta \omega_q}{4\omega_q} (10N_q - 9) \right) \delta_{N_q', N_q+1} + \frac{1}{4} \sqrt{(N_q + 2)(N_q + 1)} \left(\frac{\Delta \omega_q}{\omega_q} \right. \\
 & \left. + u(\Delta x_q)^2 + \frac{3}{8} \left(\frac{\Delta \omega_q}{\omega_q} \right)^2 (N_q - 2) \right) \delta_{N_q', N_q+2} + \frac{1}{4} \sqrt{N_q(N_q - 1)} \left(-\frac{\Delta \omega_q}{\omega_q} \right. \\
 & \left. + \left(\frac{\Delta \omega_q}{\omega_q} \right)^2 + u(\Delta x_q)^2 \right) \delta_{N_q', N_q-2} + \frac{3}{4} \sqrt{\frac{a}{2}} \Delta x_q \cdot \frac{\Delta \omega_q}{\omega_q} \times \\
 & \sqrt{(N_q + 3)(N_q + 2)(N_q + 1)} \cdot \delta_{N_q', N_q+3} + \frac{1}{4} \sqrt{\frac{a}{2}} \Delta x_q \cdot \frac{\Delta \omega_q}{\omega_q} \times \\
 & \sqrt{(N_q - 2)(N_q - 1)N_q} \cdot \delta_{N_q', N_q-3} + \frac{1}{32} \left(\frac{\Delta \omega_q}{\omega_q} \right)^2 \times \\
 & \sqrt{(N_q + 4)(N_q + 3)(N_q + 2)(N_q + 1)} \cdot \delta_{N_q', N_q+4} + \frac{1}{32} \left(\frac{\Delta \omega_q}{\omega_q} \right)^2 \times \\
 & \sqrt{(N_q - 3)(N_q - 2)(N_q - 1)N_q} \cdot \delta_{N_q', N_q-4} \quad (16)
 \end{aligned}$$

Substituting (16) into (10) and (11), we obtain, for the intensity of the F-absorption band,

$$\begin{aligned}
 S(\nu) = & \frac{4\pi^2}{3} \left(\frac{e^2}{\hbar c} \right) \nu I_0(\nu) \cdot |(E^e | \vec{X} | E^v)|^2 \cdot H_{(N_q' = N_q)} \left[1 - u(\Delta x_q)^2 (2N_q + 1) \right. \\
 & \left. - \frac{1}{16} \left(\frac{\Delta \omega_q}{\omega_q} \right)^2 (3N_q^2 - 10N_q - 4) \right] \cdot H_{(N_q' = N_q+1)} \frac{a}{2} (\Delta x_q)^2 (N_q + 1) \left(1 - \frac{1}{8} \frac{\Delta \omega_q}{\omega_q} \times \right. \\
 & \left. (10N_q - 9) \right) \cdot H_{(N_q' = N_q+2)} \frac{1}{16} \left[\frac{\Delta \omega_q}{\omega_q} + u(\Delta x_q)^2 + \frac{3}{8} \left(\frac{\Delta \omega_q}{\omega_q} \right)^2 (N_q - 2) \right]^2 (N_q + 1) \times \\
 & (N_q + 2) \cdot H_{(N_q' = N_q+3)} \frac{9}{32} u(\Delta x_q)^2 (N_q + 3)(N_q + 2)(N_q + 1) \cdot H_{(N_q' = N_q+4)} \frac{1}{32} \left(\frac{\Delta \omega_q}{\omega_q} \right)^4 \times \\
 & (N_q + 4)(N_q + 3)(N_q + 2)(N_q + 1) \cdot H_{(N_q' = N_q-1)} \frac{a}{2} (\Delta x_q)^2 N_q \left(1 - \frac{\Delta \omega_q}{4\omega_q} (N_q - 1) \right)^2 \times \\
 & H_{(N_q' = N_q-2)} \frac{1}{16} \left(\frac{\Delta \omega_q}{\omega_q} + \left(\frac{\Delta \omega_q}{\omega_q} \right)^2 - u(\Delta x_q)^2 \right)^2 N_q (N_q - 1) \cdot H_{(N_q' = N_q-3)} \frac{1}{32} u(\Delta x_q)^2 \left(\frac{\Delta \omega_q}{\omega_q} \right)^2 \times
 \end{aligned}$$

$$N_q(N_q-1)(N_q-2) \cdot \frac{H}{32^2} \left(\frac{\Delta\omega_q}{\omega_q} \right)^4 N_q(N_q-1)(N_q-2)(N_q-3), \quad (17)$$

($N_q' = N_q - 4$)

and, for the corresponding absorbed frequencies,

$$\begin{aligned} h\nu = (E' - E^e) + \sum_q (N_q + 1/2) \hbar \Delta\omega_q + \sum_{(N_q' = N_q + 1)} \hbar \omega_q' + \sum_{(N_q' = N_q + 2)} 2\hbar \omega_q' + \sum_{(N_q' = N_q + 3)} 3\hbar \omega_q' + \sum_{(N_q' = N_q + 4)} 4\hbar \omega_q' \\ - \sum_{(N_q' = N_q - 1)} \hbar \omega_q' - \sum_{(N_q' = N_q - 2)} 2\hbar \omega_q' - \sum_{(N_q' = N_q - 3)} 3\hbar \omega_q' - \sum_{(N_q' = N_q - 4)} 4\hbar \omega_q', \end{aligned} \quad (18)$$

in which each product factors in (17) are composed of each mode of vibrations concerned with the specified transition processes denoted below the product notations, and the same is valid for the summations of (18)

As seen in (17), the small quantities involving $|\Delta\omega_q|/\omega_q$ or Δx_q appear always in the intensity formula of the light absorption accompanied by the emission or absorption of lattice quanta, on account of which the maximum intensity of F-absorption may be given by the electronic transition process without the change of lattice quanta.

Thus, we have, for the maximum intensity and its corresponding frequency,

$$\begin{aligned} S_m(\nu_m) \propto \nu_m I_0(\nu_m) | \langle E^e | \vec{X} | E^e \rangle \cdot |^2 \cdot \frac{H}{q} [1 - u(\Delta x_q)^2 (2N_q + 1) \\ - \frac{1}{16} \left(\frac{\Delta\omega_q}{\omega_q} \right)^2 (3N_q^2 - 10N_q - 4)], \end{aligned} \quad (19)$$

and

$$h\nu_m = (E' - E^e) + \sum_q (N_q + 1/2) \hbar \Delta\omega_q. \quad (20)$$

In the transition processes mentioned above, only the direct interaction of the incident radiation with the trapped electrons within F-center has been taken into consideration, the interaction of the radiation with the ions situated near the lattice point being neglected in view of a small order of magnitude compared with the former according to the large mass of ions. The strong coupling between the trapped electrons and the lattice vibrations, therefore, may be considered to be responsible for the excitation of the lattice vibrations through the electronic interaction with the incident radiation.

For the band breadth, we have

$$\overline{\Delta\nu^2} = \frac{\int_0^\infty (\nu - \nu_m)^2 S(\nu) d\nu}{\int_0^\infty S(\nu) d\nu} \quad (21)$$

in which the integrations are to be carried out in accordance with the frequency condition (18), although it is too complicated to be proceeded analytically at present.

§ 5. Comparison with Experiments.

Since the energy surface near F-center of the excited states may be considered generally to become flatter in comparison with that of the ground state, the frequency change $\Delta\omega_q$ dependent upon its curvature will be of negative sign ($\Delta\omega_q < 0$) and further, it will actually vanish for the normal modes of vibrations with relatively long wave lengths compared with the lattice constant, since they are less sensitive for the lattice distortion due to the F-center.

Now we shall consider the remarkable influence of temperatures upon the maximum intensity of F-absorption band and its corresponding frequency. As well-known in the theory of the specific heat of crystal,⁽⁸⁾ the average of the number of lattice quanta N_q in thermal equilibrium are given by Planck's formula

$$N_q = (\exp(\hbar\omega_q/kT) - 1)^{-1}, \quad (22)$$

(k =Boltzmann constant, T =absolute temperature)

which increases with temperature of the crystal. Taking into account (19) and (22), the maximum intensity of F-absorption band may be shown to decrease with increasing temperatures, although the simple relation between $S_m(\nu_m)$ and T can not be derived yet on account of lacking of the accurate knowledge of Δx_q and $\Delta\omega_q$ involved. Furthermore, it is to be expected theoretically from (20) that the frequency of maximum intensity displaces toward longer wave length side with temperatures on account of $\Delta\omega_q < 0$. For the two limiting cases of temperatures the simple relations may be derived as follows.

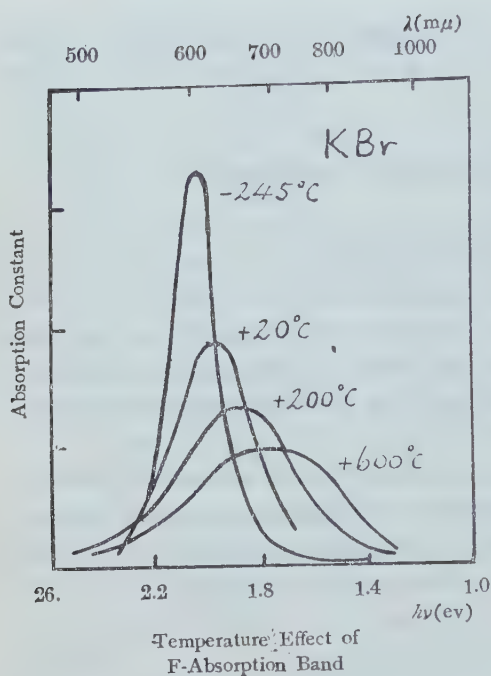
$T \gg \theta$ (Debye temperature),

$$\begin{aligned} \hbar\nu_m &= \alpha - \beta T, \\ (\alpha &= E^e - E^g + \frac{1}{2} \sum_q \hbar \Delta\omega_q, \quad \beta = \sum_q |\Delta\omega_q|/\omega_q) \end{aligned} \quad (23)$$

$T \lesssim \theta$,

$$\hbar\nu_m = \alpha - \hbar \sum_q |\Delta\omega_q| \exp(-\hbar\omega_q/kT). \quad (24)$$

As seen in Fig. 2, the theoretical results mentioned above are actually in good agreements with the experimental observations of Pohl and his



band shows a definite temperature effect, although such identification has been erroneously assumed in the literature⁽¹⁰⁾ without careful consideration.

The temperature-broadening of F-absorption band is easily understood,

coworkers⁽⁹⁾ According to (23) and (24), the frequency shift to the red side behaves linearly with temperatures in the relatively high temperature range while it does rather slowly in the lower temperature range, which theoretical conclusions are clearly reproduced in the experimental results shown in Fig. 2, although the quantitative computations of the coefficients α and β are not yet feasible to be carried through at present. In this connection, it should be pointed out that the photon quanta corresponding to the maximum absorption of band is not to be identified with the electronic energy difference alone when the

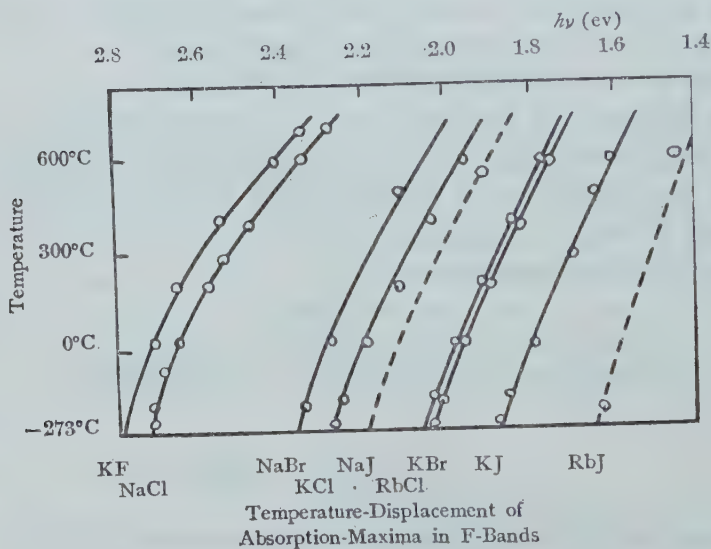


Fig. 2

though qualitatively, by the theoretical formula (17) and (18). Since the number of lattice quanta increases with temperatures according to (22), the intensity of the band corresponding to the transition processes accompanied by the emission or absorption of lattice quanta will become gradually predominant with temperatures as shown by (17), which fact leads to the broadening of band with appreciable intensities in accordance with (18). Further, since the emission probabilities of lattice quanta are larger than the absorption ones (refer to (17)), the asymmetrical intensity distribution of the broadened band is to be expected theoretically from (17) together with the contribution from the higher electronic excitations, which fact will be clearly confirmed by the experimental observations. (Fig. 2)

Finally it will be worth while discussing theoretically whether the F-absorption band approaches a sharp line spectrum or not at $T=0$. According to (17) and (18), the absorption of lattice quanta by the crystal are made possible even at $T=0$ despite of the condition $N_q(T=0)=0$ and then the absorption band may be considered to have the tail part to the violet side of ν_m , having the asymmetrical band form of relatively narrow breadth.

It should be remarked that the effect of thermal expansion of the crystal on the absorption band has not been taken into account in the present computation. In reality we have to consider both effects due to the thermal expansion and the excitation of lattice vibrations in order to obtain a full knowledge of the phenomena concerned, the computation of the former shall be worked out in later occasion.

The author is indebted to Special Research Committee of Theoretical Physics, National Research Council of Education Ministry for financial aid throughout the preparation of the work.

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On the Meson Theory of Nuclear Forces.*

Gentaro ARAKI.

Department of Industrial Chemistry, Kyoto University.

(Received Jan. 23, 1949)

Introduction.

It has generally been believed that the vector, pseudovector, and pseudoscalar meson theories of nuclear forces involve a r^{-3} difficulty. Mixtures of two fields were considered first by Møller and Rosenfeld⁽¹⁾ and later by Schwinger⁽²⁾ in order to eliminate such a term. Further it was noticed by the present author⁽³⁾ that they involve also a divergence difficulty, and another mixture was suggested by him. This mixture could also eliminate a divergent part from the magnetic moment of nucleons.

However, if we examine the procedure of deriving the nuclear force we can find that the above mentioned difficulties are due to the inadequacy of the method of the derivation, and that they have no direct connection with the general defect of the contemporary field theory. For example, the pseudoscalar mesons interact with nucleons by the pseudovector and pseudoscalar couplings. As we shall see later on, the pseudovector coupling is equivalent, except for a factor $2M/\mu$, to the pseudoscalar one in a non-relativistic approximation. If we make use of this relation we can get a non-relativistic nuclear force which is free from the above mentioned difficulties.

It was first suggested by Nelson⁽⁴⁾ that the pseudovector coupling includes a term of a pseudoscalar-coupling type. Later, using a unitary transformation, the latter was derived from the former by Dyson.⁽⁵⁾

One may imagine that this relation can not be applied to the cases of vector and pseudovector meson theories because the couplings are different in these cases. But the term in question comes from the matrix element of the same type in all cases of these three theories. And we can find that the above mentioned result still holds in the vector and pseudovector theories.

* The brief account of the present paper was given in Phys. Rev., 75 (1949), 1101, 1262.

The purpose of the present paper is to show how we can get such a satisfactory two-nucleon potential. It will also be shown that such potentials are the same as the hitherto well-known ones in case of a large distance between two nucleons.

The following calculation will be carried out exclusively in the non-relativistic and static limit. On a theoretical point of view it may be necessary to calculate a non-static and complete expression in order to gain a more cutting insight into the interaction between two nucleons through a meson field. But such an expression, if we could obtain it, can not be used in the relativistic case. In this case it is necessary to consider a two-nucleon system taking into account the meson field around them. The static potential may be sufficient for practical purpose such as a ground state of deuteron. However, if we consider a problem including higher energy of nucleons, for example a collision of a 100 Mev neutron with a proton, the approximation of a static force may become insufficient. In such a case the next approximation, namely the term of the first degree in $1/M$, may be necessary.

§ 1. On the Principle of Calculation.

We shall derive a two-nucleon potential, W , according to the method of transition⁽⁶⁾ because this fits in with our purpose of eliminating the defect in question. In this method the matrix element of W is given by

$$W_{BA} = H_{BA}^{nn} - \sum_I \frac{H_{BI}^{nm} H_{IA}^{nm}}{E_I - E_A} \quad (1.1)$$

where H^{nn} is a direct interaction between nucleons, H^{nm} is an interaction between mesons and nucleons, and A and B are two states of the whole system including no meson. The whole system consists of two nucleons and mesons. Since we want to get W in a non-relativistic and static approximation it is sufficient to consider only non-relativistic states of two nucleons in A and B .

Let \mathbf{p}_1 and \mathbf{p}_2 be the momenta of the two nucleons in the state A and \mathbf{p}_1' and \mathbf{p}_2' be those in B where we adopt the eigenstates of energies and momenta of the two nucleons as states of them in A or B . The momentum of the whole system is conserved in A , I and B . Therefore, if $\hbar\mathbf{k}$ or $-\hbar\mathbf{k}$ is the momentum of a meson in the intermediate state I , we have

$$\mathbf{p}_1' - \mathbf{p}_1 = \hbar \mathbf{k} = -(\mathbf{p}_2' - \mathbf{p}_2) \quad (1.2)$$

The relation between W and W_{RA} is given by the Fourier theorem as follows:

$$W \vartheta_n \exp i(\mathbf{p}_1 \mathbf{x}_1 + \mathbf{p}_2 \mathbf{x}_2) / \hbar = \sum_R \{ \vartheta_n' \exp i(\mathbf{p}_1' \mathbf{x}_1 + \mathbf{p}_2' \mathbf{x}_2) / \hbar \} W_{RA} \quad (1.3)$$

where ϑ_n is a spin function with four components, \mathbf{x}_1 and \mathbf{x}_2 are the position vectors of the two nucleons, and n and n' are the spin states of nucleons in A and B respectively. Since W includes no differential operator we can find W itself, using (1.2), as follows:

$$W = \sum_k W(\mathbf{k}) \exp i \mathbf{k} \mathbf{x} \quad (1.4)$$

where $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ is a relative position vector between the two nucleons and $W(\mathbf{k})$ is defined by

$$W(\mathbf{k}) \vartheta_n = \sum_n' \vartheta_n' W_{RA} \quad (1.5)$$

It should be noted here that, according to the principle of the present method, A and B of (1.1) are non-relativistic states whereas, in the actual calculation, B of (1.3) is extended over all states including a relativistic region. Therefore the non-relativistic form of W_{RA} should be so determined that the contribution from a relativistic region in the sum of (1.3) does not play an important rôle. The customary theories did not satisfy this requirement. It is obvious that the divergent term in the hitherto well-known result is due to this inadequate method. We shall see that the so-called r^{-3} difficulty also comes from the same cause. The following method of calculation will satisfy the above mentioned requirement.

§ 2. Non-relativistic Matrix Elements of $\sigma \text{ grad } U$.

We shall calculate non-relativistic matrix elements of $\sigma \text{ grad } U$ for the later use. Let χ^+ and χ^- be normalized eigenfunctions of ρ_3 belonging to its eigenvalues $+1$ and -1 respectively. The wave function Ψ of a nucleon can be represented by a linear combination of them as follows:

$$\Psi = \chi^+ \phi^+ + \chi^- \phi^- \quad (2.1)$$

ρ_1 , ρ_2 , and ρ_3 transform χ^+ and χ^- as follows:

$$\left. \begin{aligned} \rho_1 \chi^+ &= \chi^- & \rho_2 \chi^+ &= i \chi^- & \rho_3 \chi^+ &= \chi^+ \\ \rho_1 \chi^- &= \chi^+ & \rho_2 \chi^- &= -i \chi^+ & \rho_3 \chi^- &= -\chi^- \end{aligned} \right\} \quad (2.2)$$

If Ψ is a normalized eigenfunction of the energy and momentum of

a free nucleon belonging to an eigenvalue E of the energy, a relation between ψ^+ and ψ^- is given by

$$\psi^- = -i(E + Mc^2)^{-1} \hbar c \sigma \text{grad} \psi^+ \quad (2.3)$$

which can be obtained by substituting (2.1) into Dirac's equation. In this equation M denotes the mass of a nucleon. If we replace ψ^- in (2.1) by (2.3) we have

$$\Psi = \chi^+ \psi^+ - i \chi^- (E + Mc^2)^{-1} \hbar c \sigma \text{grad} \psi^+ \quad (2.4)$$

If Ψ_a and Ψ_b represent two non-relativistic states of the nucleon we can calculate the matrix element of $U(\mathbf{x}) \rho_z$ by making use of (2.4) and (2.2) where U is a point function. Expanding the result in powers of $1/Mc$ we have

$$\begin{aligned} \frac{2Mc}{\hbar} (\Psi_b, U \rho_z \Psi_a) = & (\psi_b^+, (\sigma \text{grad} U) \psi_a^+) \left\{ 1 - \left(\frac{p_b}{2Mc} \right)^2 + \dots \right\} \\ & + (\psi_b^+, U \sigma \text{grad} \psi_a^+) \frac{p_a^2 - p_b^2}{(2Mc)^2} + \dots \end{aligned} \quad (2.5)$$

In quite the same way we have

$$(\Psi_b, (\sigma \text{grad} U) \Psi_a) = (\psi_b^+, (\sigma \text{grad} U) \psi_a^+) \left\{ 1 - \frac{p_a p_b}{(2Mc)^2} + \dots \right\} + \dots \quad (2.6)$$

In these expressions p_a and p_b are eigenvalues of the momentum in Ψ_a and Ψ_b respectively, p_a and p_b are their absolute values, and... means the terms of higher degree in $p_a/(2Mc)$ and $p_b/(2Mc)$. In the non-relativistic approximation it follows from (2.5) and (2.6) that

$$\frac{1}{x} \sigma \text{grad} U = \frac{2M}{\mu} \rho_z U \quad (2.7)$$

where μ is the mass of a meson and x is defined by $\hbar x = \mu c$.

§ 3. Two-Nucleon Potential.

We shall next calculate W_{BA} which satisfies the requirement mentioned at the end of the first section. If a nucleon has a positive energy in A and a negative energy in B then the leading term of $E_B - E_A$ is proportional to Mc^2 . Therefore, in this case, W_{BA} vanishes in the non-relativistic limit. If we substitute the explicit expressions of H^{nn} and H^{nm} into (1.1) we have the following matrix elements⁽³⁾⁽⁷⁾;

(a) the vector theory

$$W_{BA} = F(k) [\{g_1^2 + g_2^2 \sigma_1 \sigma_2 + g_2^2 (\sigma_1 \mathbf{k})(\sigma_2 \mathbf{k})/x^2\} UT]_{ba} \quad (3.1a)$$

(b) the pseudovector theory

$$W_{BA} = F(k) [\{-f_2^2 \sigma_1 \sigma_2 + (g_2^2 - f_2^2)(\sigma_1 \mathbf{k})(\sigma_2 \mathbf{k})/x^2\} UT]_{ba} \quad (3.1b)$$

(c) the pseudoscalar theory

$$W_{BA} = -f_2^2 F(k) \{UT(\sigma_1 \mathbf{k})(\sigma_2 \mathbf{k})/x^2\}_{ba} \quad (3.1c)$$

where

$$F(k) = (4\pi/I)(k^2 + x^2)^{-1} \quad (3.2a)$$

$$T = (\tau_1^{(1)} \tau_1^{(2)} + \tau_2^{(1)} \tau_2^{(2)})/2 \quad (\text{charged theory})$$

$$\text{or } (\tau_1^{(1)} \tau_1^{(2)} + \tau_2^{(1)} \tau_2^{(2)} + \tau_3^{(1)} \tau_3^{(2)})/2 \quad (\text{symmetrical theory}) \quad (3.2b)$$

$$U = \exp \{i\mathbf{k}(\mathbf{x}_1 - \mathbf{x}_2)\} \quad (3.2c)$$

and g_1 , g_2 , and f_2 are respectively the constants of the vector, six-vector, and pseudovector couplings between mesons and nucleons. The indices a and b specify the states of nucleons in A and B respectively. For example G_{ab} stands for $(\psi_b, G\psi_a)$ where ψ_a and ψ_b are the wave functions of the nucleons. In deriving these results for the cases (a) and (b) we have adjusted H^{nm} 's so that the matrix elements of them cancel those of the same type^(c) in the second order terms due to H^{nm} .

Since the absolute values of the matrix elements given by (3.1) never decrease as k increases, they do not satisfy the requirement which was mentioned at the end of the first section. Therefore we should transform them so as to satisfy this requirement. This can be attained by the relation (2.7). Making use of this relation we have

$$UT(\sigma_1 \mathbf{k})(\sigma_2 \mathbf{k})/x^2 = (2M/\mu)^2 UT\rho_1^{(1)}\rho_2^{(2)} \quad (3.3)$$

We replace the terms including $(\sigma_1 \mathbf{k})(\sigma_2 \mathbf{k})$ in (3.1) by the right hand side of this equation, substitute the results into (1.5), and then we have, by (1.4) and (1.5), the non-relativistic two-nucleon potentials as follows:

(a) the vector theory

$$W = \left\{ g_1^2 + g_2^2 \sigma_1 \sigma_2 + g_2^2 \left(\frac{2M}{\mu} \right)^2 \rho_2^{(1)} \rho_2^{(2)} \right\} \frac{e^{-\kappa r}}{r} T \quad (3.4a)$$

(b) the pseudovector theory

$$W = \left\{ -f_2^2 \sigma_1 \sigma_2 + (g_2^2 - f_2^2) \left(\frac{2M}{\mu} \right)^2 \rho_2^{(1)} \rho_2^{(2)} \right\} \frac{e^{-\kappa r}}{r} T \quad (3.4b)$$

(c) the pseudoscalar theory

$$W = -f_2^2 \left(\frac{2M}{\mu} \right)^2 \rho_2^{(1)} \rho_2^{(2)} \frac{e^{-\kappa r}}{r} T \quad (3.4c)$$

These potentials are free from the so-called r^{-3} term and the divergent integral. The defect of the customary theory is due to that the inadequate forms given by (3.1) were adopted as non-relativistic approximations of W_{BA} .

§ 4. Relation of New Formulas with Customary Ones.

Lastly we shall show that the two-nucleon potentials given by (3.4) agree with the hitherto well-known ones⁽⁷⁾ for large distance between two nucleons. We shall consider the pseudoscalar theory for example. The Hamiltonian of a two-nucleon system is given by

$$H = c\rho_1^{(1)}\sigma_1\mathbf{p}_1 + c\rho_1^{(2)}\sigma_2\mathbf{p}_2 + Mc^2(\rho_3^{(1)} + \rho_3^{(2)}) + W \quad (4.1)$$

where $\mathbf{p}_k = -i\hbar\nabla_k$ ($k=1, 2$). Let ψ be an eigenfunction of H belonging to its eigenvalue E . We expand ψ in a linear combination of products of χ^+ and χ^- as follows:

$$\psi = \chi^+(1)\chi^+(2)\phi^{++} + \chi^+(1)\chi^-(2)\phi^{+-} + \chi^-(1)\chi^+(2)\phi^{-+} + \chi^-(1)\chi^-(2)\phi^{--} \quad (4.2)$$

where χ^+ and χ^- are defined by (2.2), and $\chi(1)$ and $\chi(2)$ are the functions of the first and second nucleons respectively.

If we substitute (4.2) into $H\psi = E\psi$ we have a following system of simultaneous equations for ϕ^{++} , ϕ^{+-} , ϕ^{-+} and ϕ^{--} because of the linear independence of χ -functions:

$$(\sigma_1\mathbf{p}_1\phi^{--} + \sigma_2\mathbf{p}_2\phi^{+-})/(Mc) = \varepsilon\phi^{++} - 4\phi\phi^{--} \quad (4.3a)$$

$$(\sigma_1\mathbf{p}_1\phi^{--} + \sigma_2\mathbf{p}_2\phi^{++})/(Mc) = (2 + \varepsilon)\phi^{+-} + 4\phi\phi^{+-} \quad (4.3b)$$

$$(\sigma_1\mathbf{p}_1\phi^{++} + \sigma_2\mathbf{p}_2\phi^{--})/(Mc) = (2 + \varepsilon)\phi^{-+} + 4\phi\phi^{+-} \quad (4.3c)$$

$$(\sigma_1\mathbf{p}_1\phi^{+-} + \sigma_2\mathbf{p}_2\phi^{-+})/(Mc) = (4 + \varepsilon)\phi^{--} - 4\phi\phi^{++} \quad (4.3d)$$

where

$$\phi(r) = a \frac{M}{\mu} \frac{f_2^2}{\hbar c} \frac{e^{-\kappa r}}{\kappa r}, \quad \varepsilon = (E - 2Mc^2)/(Mc^2) \quad (4.4)$$

and a is an eigenvalue of T . ε is very small compared with unity in case of a non-relativistic state. Further if ϕ is also very small compared with unity it follows from (4.3) that ϕ^{+-} , ϕ^{-+} , and ϕ^{--} are very small compared with ϕ^{++} ,

If we eliminate ϕ^{+-} , ϕ^{-+} and ϕ^{--} from (4.2) and $H\psi=E\psi$ by (4.3), the result can be expanded in a power series in $1/c$ provided that

$$\epsilon < 1, \quad |\phi(r)| < 1. \quad (4.5)$$

The zeroth-degree term of ϕ is equal to $\phi_0 = \chi^+(1)\chi^+(2)\phi^{++}$ and the zeroth-degree term of $H\psi=E\psi$ is given by

$$\left\{ -\frac{\hbar^2}{2M}(\mathcal{A}_1 + \mathcal{A}_2) + W' \right\} \phi_0 = (E - 2Mc^2)\phi_0 \quad (4.6)$$

where

$$W' = \frac{f_2^2}{(1+|\phi|)(1+2|\phi|)} \cdot \frac{e^{-\kappa r}}{r} \left\{ \frac{1}{3} \sigma_1 \sigma_2 + \left(\frac{1}{3} + \frac{1}{\kappa r} + \frac{1}{\kappa^2 r^2} \right) A \right\} T \quad (4.7a)$$

$$A = 3(\sigma_1 \kappa)(\sigma_2 \kappa)/r^2 - \sigma_1 \sigma_2 \quad (4.7b)$$

This is a non-relativistic limit of $H\psi=E\psi$. Consequently W' is a non-relativistic two-nucleon potential. It has a validity only for large r such that $|\phi(r)| \ll 1$. Therefore ϕ in W' is unnecessary and it should be omitted. But we reserve it for another reason. If we omit ϕ from W' it is just the same as the hitherto well-known potential.⁽³⁾⁽⁷⁾

When r is so small that $|\phi(r)| \sim 1$ or > 0 , $H\psi=E\psi$ can not be reduced to (4.6). But the potential W' has no term which is too singular because we have reserved ϕ in it. It has only a r^{-1} singularity as W does. If E and ψ are essentially determined by the behaviour of W in the region of larger r and if the behaviour of W' in the region of small r does not play any rôle, we can use (4.6) as an approximation of $H\psi=E\psi$ throughout all range of r . Whether such a device has its validity or not can be decided by solving a practical non-relativistic problem, for example, the ground state of deuteron. The root of $\phi(r)=1$ is given by $\kappa r_0=0.45$ or 0.57 according as $|a|(M/\mu)f_2^2/(\hbar c)=0.7$ or 1 . These values of r_0 are reconcilable with Bethe's result⁽⁸⁾.

The comparison of our new formulas given by (3.4) with the customary result of the vector or pseudovector theory can be done in the same way. We can thus find that both agree with each other for large r . The possibility of getting these results was once imagined by Bethe⁽⁸⁾, but finally discarded by him.

The two-nucleon potential W becomes large as r decreases. This means that our perturbation method loses its validity for small r . Therefore we see that in this case W itself has no definite meaning as well,

The result of the present theory is quite satisfactory from a theoretical point of view, but the vector case still disagrees with the experiment of the electric quadrupole moment of deuteron as the customary one did. The potential given by the pseudovector theory is just in agreement with Bethe's neutral vector theory⁽⁸⁾ so far as the ground state of deuteron is concerned, where $g_2=0$ is assumed. According to this theory the surplus magnetic moment of proton is given by $\mu_p = (e/\kappa)(f_2^2/\hbar c)(1-C)$ where C is a divergent integral⁽⁹⁾. The λ -limiting process can eliminate this divergent part⁽⁹⁾. In this case μ_p is finite and positive and it is equal to $2(M/\mu)f_2^2/(\hbar c)$ nuclear magnetons which is in agreement with experiment⁽¹⁰⁾.

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Note added in proof (I) After the manuscript of the present paper was submitted to the editors, Professor H. Yukawa communicated to me that Dr. L. van Hove calculated the second order matrix element. The non-relativistic term of his result agrees with mine for the case of the pseudoscalar theory whereas for the case of the vector theory it is, according to Yukawa's communication, given by

$$-\left(\frac{g_2}{\kappa}\right)^2 + \frac{g_1^2 + g_2^2}{k^2 + \kappa^2} - \left(\frac{2M}{\mu}\right)^2 \frac{g_2^2}{k^2 + \kappa^2} \rho_1^{(1)} \rho_1^{(2)} \sigma_1 \sigma_2 \quad (\text{n.1})$$

where a common factor is omitted. The first term can be eliminated by its counter term due to the direct interaction. If we supplement the factor we can derive the two-nucleon potential from this matrix element as follows:

$$W = \left\{ g_1^2 + g_2^2 - g_2^2 \left(\frac{2M}{\mu} \right) \rho_1^{(1)} \rho_1^{(2)} \sigma_1 \sigma_2 \right\} \frac{e^{-\kappa r}}{r} \quad (\text{n.2})$$

According to the procedure mentioned in the fourth section of the present paper it can be shown that this expression agrees with the customary one for large r . Therefore the non-relativistic limit of van Hove's result agrees with mine for the case of the vector theory too.

(II) The exact expression corresponding to the relation (2.7) is given by

$$\frac{1}{\kappa} \sigma \text{grad } U = \frac{2M}{\mu} \rho_2 U + \frac{i}{\mu c^2} (H \rho_1 U - \rho_1 U H) \quad (\text{n.3})$$

where H is the Dirac Hamiltonian of a free nucleon.

The Propagation of Order in Crystal Lattices.

Haruo SHIMAZU.

Physical Institute, Nagoya University.

(Received Feb. 1, 1949)

I. In order-disorder problems, only the long and short range orders were considered at the first stage⁽¹⁾. But there is a more general order, for the probabilities of finding various kinds of atoms at every lattice site must be affected by the presence of a certain kind of atom at a certain lattice site. These correlations must necessarily be taken into account for interpreting the diffuse scattering of x-rays by a partially disordered crystal alloy. The long range order is then considered as the limiting case of such intermediate orders. These correlation probabilities were first discussed by Zernike⁽²⁾; he obtained a recurrence relation to relate the probability of finding a certain kind of atom at a given lattice point to the probabilities for various kinds of atoms at the lattice points forming the nearest neighbors of a given one. His method will be touched briefly in the next section.

In the two-dimensional case, a powerful technique has been introduced recently in order-disorder problems, reducing the statistical treatment of binary alloys to an eigenvalue problem of a certain matrix⁽³⁾; and according to this, the propagation of order has been discussed by Ashkin and Lamb⁽⁴⁾, giving the exact solution at low temperatures by series expansions. But this cannot be applied to three-dimensional cases for mathematical difficulties.

Zernike's method, the only one which has hitherto been applicable to three-dimensional cases, uses an idea similar to Bethe's original method, but is more rough an approximation than Bethe's. And as pointed out by Ashkin and Lamb⁽⁴⁾, the approximation becomes successively poorer as the dimension is increased; and in three-dimensional cases, comparing with the exact series expansion for the long and short range orders given by van der Waerden^{*(5)} at low temperatures, there is no difference between

* The writer has had no opportunity to see the original paper, but the results are quoted in (4), and are as follows.

$$S = 1 - 2x^6 + 12x^{10} + 14x^{12} - 90x^{14} + 192x^{16} - 792x^{18} + \dots,$$

$$C = 1 - 2x^6 - 10x^{10} + 14x^{12} - 70x^{14} + 176x^{16} - 626x^{18} + \dots,$$

Zernike's results are

$$S = 1 - 2x^6 + 6x^8 + \dots,$$

$$C_1 = 1 - 2x^6 + 6x^8 + \dots,$$

the long and short range orders given by Zernike up to these terms of his solutions which coincide with the exact ones. It is the object of this paper to extend Bethe's method to a more general problem, the correlation probabilities, giving a better solution than that of Zernike.

II. We consider an alloy containing equal numbers of A and B atoms, and interaction energies only between nearest neighbors. We denote the interaction energy between an A and a B atom by V_{AB} , and between two A atoms and two B atoms by V_{AA} and V_{BB} respectively. Lattice points or atoms at these lattice points are denoted by $[0], [1], [2], \dots$, etc., and the probabilities of finding a right atom at the lattice points $[0], [1], [2], \dots$, etc. by r_0, r_1, r_2, \dots , etc..

The fundamental equation of Zernike is expressed in the case of simple cubic lattice as follows.

$$r_0 = r_1 r_2 r_3 r_4 r_5 r_6 \frac{1}{1+x^6} + (w_1 r_2 r_3 r_4 r_5 r_6 + r_1 w_2 r_3 r_4 r_5 r_6 + \dots) \frac{x}{x+x^6} + \dots + w_1 w_2 w_3 w_4 w_5 w_6 \frac{x^6}{1+x^6}, \quad (1)$$

where

$$w_i = 1 - r_i, \quad x = e^{-V/kT} \left(V = V_{AB} - \frac{1}{2}(V_{AA} + V_{BB}) \right),$$

$[0]$ is the central atom and $[1], [2], \dots, [6]$ form the nearest neighbors of $[0]$. The meaning of (1) can easily be understood without further explanations. The essential approximation involved in this formula is that the probabilities for atoms at surrounding sites are given independently of one another. We shall try to make some improvements in this point by a method similar to that introduced by Bethe of taking into account the mean influence of actions between atoms at surrounding sites through all paths connecting every two atoms of them.

We consider the partition function for a cell containing $1+6$ atoms, one for the central atom and six for nearest neighboring ones. (We consider only a simple cubic lattice in this paper.) If a lattice point is occupied by a right (R) atom, we attach ϵ to the partition function, and if occupied by a wrong (W) atom, we attach $1-\epsilon$. But unlike Bethe's treatment we cannot use the same ϵ for every lattice point, because the probabilities of finding an R or W atom at other lattice points differ with their positions relative to a given atom.

At first, the partition function for the cell having the central atom $[0]$, which we denote by $\{0\}$, is considered. ($[0]$ is not the given atom whose kind is known.)

[1], [2],, [6] relate to the nearest neighboring atoms of [0]. If both [0] and all the nearest neighboring atoms are R , we add to the partition function $\varepsilon(1) \varepsilon(2) \varepsilon(3) \varepsilon(4) \varepsilon(5) \varepsilon(6)$, and if only [1] is W , we add $(1 - \varepsilon(1)) \varepsilon(2) \varepsilon(3) \varepsilon(4) \varepsilon(5) \varepsilon(6)x$, etc..

When we construct the partition function, summing the terms described above over all configurations, we divide it up into two parts, one consisting of all the configurations in which [0] is R , and the other all in which [0] is W , which we denote by $r_0(0)$ and $w_0(0)$ respectively. The suffix indicates the cell {0} of which the partition function is given. Then the partition function becomes

$$Z_0 = r_0(0) + w_0(0), \quad (2)$$

where

$$\left. \begin{aligned} r_0(0) &= \{(1 - \varepsilon_0(1))x + \varepsilon_0(1)\} \{(1 - \varepsilon_0(2))x + \varepsilon_0(2)\} \dots \{(1 - \varepsilon_0(6))x + \varepsilon_0(6)\}, \\ w_0(0) &= \{(1 - \varepsilon_0(1)) + x\varepsilon_0(1)\} \{(1 - \varepsilon_0(2)) + x\varepsilon_0(2)\} \dots \{(1 - \varepsilon_0(6)) + x\varepsilon_0(6)\}. \end{aligned} \right\} \quad (3)$$

The ε 's must also be distinguished by suffixes, indicating cells to which these are related.

We can divide the partition function into two parts in an other manner, i.e. with one consisting of configurations in which an R atom is found at the site [1], and the other of those in which a W atom is there. Then

$$Z_0 = r_0(1) + w_0(1), \quad (4)$$

where

$$\left. \begin{aligned} r_0(1) &= \varepsilon_0(1) \prod_{i=2}^6 \{(1 - \varepsilon_0(i))x + \varepsilon_0(i)\} + x\varepsilon_0(1) \prod_{i=2}^6 \{(1 - \varepsilon_0(i)) + x\varepsilon_0(i)\}, \\ w_0(1) &= (1 - \varepsilon_0(1))x \prod_{i=2}^6 \{(1 - \varepsilon_0(i))x + \varepsilon_0(i)\} + (1 - \varepsilon_0(1)) \prod_{i=2}^6 \{(1 - \varepsilon_0(i)) + x\varepsilon_0(i)\}. \end{aligned} \right\} \quad (5)$$

And likewise

$$Z_0 = r(2) + w_0(2) = \dots = r_0(6) + w_0(6). \quad (6)$$

When we consider the cell {1}, we can obtain as before the series of partition functions concerning this cell.

$$Z_1 = r_1(1) + w_1(1) = r_1(7) = \dots = r_1(12) + w_1(12), \quad (7)$$

where [7], [8],, [12] denote the atoms forming the nearest neighbors of [1], and as one of them is identical with [0], let us consider, [7] = [0].

These quantities must satisfy the condition that the probability for any lattice site must be determined uniquely, i.e.,

$$\frac{r_0(0)}{r_0(0)+w_0(0)} = \frac{r_1(0)}{r_1(0)+w_1(0)} . \quad (8)$$

etc.,

(8) can be written explicitly

$$\begin{aligned} & \left[\prod_{i=1}^6 \{ (1-\varepsilon_0(i))x + \varepsilon_0(i) \} \right] \times \left[\prod_{j=7}^{12} \{ (1-\varepsilon_1(j))x + \varepsilon_1(j) \} + \prod_{j=7}^{12} \{ (1-\varepsilon_1(j)) + x\varepsilon_1(j) \} \right] \\ &= \left[\varepsilon_0 \prod_{j=8}^{12} \{ (1-\varepsilon_1(j))x + \varepsilon_1(j) \} + x\varepsilon_0 \prod_{j=8}^{12} \{ 1-\varepsilon_1(j) \} + x\varepsilon_1(j) \} \right] \\ & \times \left[\prod_{i=1}^6 \{ (1-\varepsilon_0(i))x + \varepsilon_0(i) \} + \prod_{i=1}^6 \{ (1-\varepsilon_0(i)) + x\varepsilon_0(i) \} \right] . \quad (9) \end{aligned}$$

This is the equation relating the probabilities for [1], [2], ..., [6], to those for [0], [8], ..., [12].

III. At first, we shall give the long range order. The correlation probabilities may converge to a finite limit at great distances. Then, when we consider the relation (9) about atoms which are far apart from a given atom, we can put all ε 's equal. Calling this ε , we obtain the next equation,

$$\{ (1-\varepsilon)x + \varepsilon \}^6 = \varepsilon \{ (1-\varepsilon)x + \varepsilon \}^5 + x\varepsilon \{ (1-\varepsilon) + x\varepsilon \}^5 . \quad (10)$$

The long range order is related to ε by

$$\left(\frac{1+S}{1-S} \right)^{5/6} = \frac{\varepsilon}{1-\varepsilon} . \quad (11)$$

The long range order given above is identical with that given by Bethe.

At low temperatures ($x \ll 1$) we can solve Eq. (10) by a power series of x .

$$\varepsilon = 1 - x^5 - 5x^9 + x^{10} + 5x^{11} - 35x^{13} + 10x^{14} - 76x^{15} - 10x^{16} + \dots . \quad (12)$$

From (11) and (12)

$$S = 1 - 2x^6 - 12x^{10} + 14x^{12} - 90x^{14} + 216x^{16} + \dots . \quad (13)$$

IV. Nextly, we shall reduce (9) to a more convenient form by using an approximation. If we put

$$\varepsilon_n(i) = \varepsilon + (1-\varepsilon)p_n(i) , \quad (14)$$

then

$$\begin{aligned} \{ (1-\varepsilon_n(i))x + \varepsilon_n(i) \} &= \{ (1-\varepsilon)x + \varepsilon \} + (1-\varepsilon)(1-x)p_n(i) , \\ \{ (1-\varepsilon_n(i)) + x\varepsilon_n(i) \} &= \{ (1-\varepsilon) + x\varepsilon \} - (1-\varepsilon)(1-x)p_n(i) . \end{aligned} \quad (15)$$

Considering the p 's to be small quantities, we substitute (14) and (15) into (9) and neglect non-linear terms of p . Using (10), we obtain the next equation.

$$\begin{aligned} p_1(0) & \left[\{ (1-\varepsilon)x + \varepsilon \}^5 + \{ (1-\varepsilon) + x\varepsilon \}^5 \} [\{ (1-\varepsilon)x + \varepsilon \}^6 + \{ (1-\varepsilon) + x\varepsilon \}^6] \right. \\ & \quad \left. - (1-x) \{ (1-\varepsilon)x + \varepsilon \}^6 - \{ (1-\varepsilon)x + \varepsilon \}^5 - \{ (1-\varepsilon) + x\varepsilon \}^5 \right] \\ & + (p_1(8) + p_1(9) + p_1(10) + p_1(11) + p_1(12))(1-x) \left[\varepsilon [\{ (1-\varepsilon)x + \varepsilon \}^4 \right. \\ & \quad \left. - x \{ (1-\varepsilon) + x\varepsilon \}^4] [\{ (1-\varepsilon)x + \varepsilon \}^6 + \{ (1-\varepsilon) + x\varepsilon \}^6] - \{ (1-\varepsilon)x + \varepsilon \}^6 \right. \\ & \quad \left. \times [\{ (1-\varepsilon)x + \varepsilon \}^5 - \{ (1-\varepsilon) + x\varepsilon \}^5] \right] \\ & = (p_0(1) + p_0(2) + p_0(3) + p_0(4) + p_0(5) + p_0(6))(1-x) \left[\{ (1-\varepsilon)x + \varepsilon \}^5 \right. \\ & \quad \times [\{ (1-\varepsilon)x + \varepsilon \}^6 + \{ (1-\varepsilon) + x\varepsilon \}^6] - \varepsilon [\{ (1-\varepsilon)x + \varepsilon \}^5 - \{ (1-\varepsilon) + x\varepsilon \}^5] \\ & \quad \left. \times [\{ (1-\varepsilon)x + \varepsilon \}^5 + x \{ (1-\varepsilon) + x\varepsilon \}^5] \right]. \end{aligned} \quad (16)$$

we can rewrite (16) briefly in the next formula.

$$\begin{aligned} p_1(0) & = \{ p_0(1) + p_0(2) + p_0(3) + p_0(4) + p_0(5) + p_0(6) \} \tilde{\varepsilon} \\ & \quad + \{ p_1(8) + p_1(9) + p_1(10) + p_1(11) + p_1(12) \} \eta. \end{aligned} \quad (17)$$

As atom [0] is not only a nearest neighbor of atom [1], but also that of atom [2], [3],....., [6], we have five other equations about atom [0] similar to (17). And similar equations are obtained for all atoms except a given one. As the cells and atoms need only be distinguished by the relative positions to a given atom, we shall now change our notation and use the squares of the distances from a given atom, with the lattice constant as unit, to indicate atoms. Then we have the next equations concerning several atoms around a given atom [0].

$$\begin{cases} p_2(1) = \{ p_1(0) + 4p_1(2) + p_1(4) \} \tilde{\varepsilon} + \{ p_2(1) + 2p_2(3) + 2p_2(5) \} \eta, \\ p_4(1) = \{ p_1(0) + 4p_1(2) + p_1(4) \} \tilde{\varepsilon} + \{ 4p_4(5) + p_4(9) \} \eta, \\ p_1(2) = \{ 2p_2(1) + 2p_2(3) + 2p_2(5) \} \tilde{\varepsilon} + \{ p_1(0) + 3p_1(2) + p_1(4) \} \eta, \\ p_3(2) = \{ 2p_2(1) + 2p_2(3) + 2p_2(5) \} \tilde{\varepsilon} + \{ 2p_3(2) + 3p_3(6) \} \eta, \\ p_5(2) = \{ 2p_2(1) + 2p_2(3) + 2p_2(5) \} \tilde{\varepsilon} + \{ p_5(4) + 2p_5(6) + p_5(8) + p_5(10) \} \eta, \\ p_1(4) = \{ p_4(1) + 4p_4(5) + p_4(9) \} \tilde{\varepsilon} + \{ p_1(0) + 4p_1(2) \} \eta, \\ p_3(4) = \{ p_4(1) + 4p_4(5) + p_4(9) \} \tilde{\varepsilon} + \{ p_3(2) + 2p_3(6) + p_3(8) + p_3(10) \} \eta, \\ p_5(4) = \{ p_4(1) + 4p_4(5) + p_4(9) \} \tilde{\varepsilon} + \{ 4p_5(10) + p_5(16) \} \eta, \end{cases}$$

$$\begin{cases}
 \begin{cases}
 p_2(3) = \{3p_3(2) + 3p_3(6)\} \xi + \{2p_2(1) + p_2(3) + 2p_2(5)\} \eta, \\
 p_6(3) = \{3p_3(2) + 3p_3(6)\} \xi + \{2p_6(5) + 2p_6(9') + p_6(11)\} \eta, \\
 p_2(5) = \{p_5(2) + p_5(4) + 2p_5(6) + p_5(8) + p_5(10)\} \xi + \{2p_2(1) + 2p_2(3) + p_2(5)\} \eta, \\
 p_4(5) = \{p_5(2) + p_5(4) + 2p_5(6) + p_5(8) + p_5(10)\} \xi + \{p_4(1) + 3p_4(5) + p_4(9)\} \eta, \\
 p_6(5) = \{p_5(2) + p_5(4) + 2p_5(6) + p_5(8) + p_5(10)\} \xi + \{p_6(3) + p_6(5) \\
 \quad + 2p_6(9') + p_6(11)\} \eta,
 \end{cases} \\
 \text{and the other two,}
 \end{cases} \quad (18)$$

$$\begin{cases}
 p_4(9) = \{p_9(4) + 4p_9(10) + p_9(16)\} \xi + \{p_4(1) + 4p_4(5)\} \eta, \\
 \text{and the other two,}
 \end{cases}$$

$$\begin{cases}
 \begin{cases}
 p_3(6) = \{p_6(3) + 2p_6(5) + 2p_6(9') + p_6(11)\} \xi + \{3p_3(2) + 2p_3(6)\} \eta, \\
 p_5(6) = \{p_6(3) + 2p_6(5) + 2p_6(9') + p_6(11)\} \xi + \{p_5(2) + p_5(4) + p_5(6) \\
 \quad + p_5(8) + p_5(10)\} \eta, \\
 \text{and the other two,}
 \end{cases}
 \end{cases}$$

where $[9] \neq [9']$, for the two differ from each other by their relative positions to $[0]$, in spite of their distances from it being equal.

When we put $p_1(0)=1$ in (14), $\epsilon_1(0)=1$ and $1-\epsilon_1(0)=0$: it means that the probability for atom $[0]$ to be R is 1. Then we take $p_1(0)=1$ as the boundary condition. If ξ and η are small quantities, we can solve these equations by successive approximations, and obtain the following result, (calculated up to terms of third order in ξ and η).

$$\begin{cases}
 \begin{cases}
 p_2(1) = \xi + 9\xi^3 + 6\xi\eta + 34\xi\eta^2 + \dots, & \begin{cases} p_2(3) = 6\xi^3 + 2\xi\eta + 18\xi\eta^2 + \dots, \\ p_4(1) = \xi + 9\xi^3 + 5\xi\eta + 25\xi\eta^2 + \dots, & p_6(3) = 6\xi^3 + \dots, \end{cases} \\
 p_1(2) = 2\xi^2 + \eta + 27\xi^2\eta + 4\eta^2 + 16\eta^3 + \dots, & \begin{cases} p_2(5) = 3\xi^3 + 2\xi\eta + 18\xi\eta^2 + \dots, \\ p_3(2) = 2\xi^2 + 24\xi^2\eta + \dots, & \begin{cases} p_4(5) = 3\xi^3 + \xi\eta + 9\xi\eta^2 + \dots, \\ p_5(2) = 2\xi^2 + 21\xi^2\eta + \dots, & p_6(5) = 3\xi^3 + \dots, \end{cases} \end{cases}
 \end{cases} \\
 \end{cases} \quad (19)$$

$$\begin{cases}
 \begin{cases}
 p_1(4) = \xi^2 + \eta + 18\xi^2\eta + 4\eta^2 + 16\eta^3 + \dots, & \begin{cases} p_4(9) = \xi^3 + \xi\eta + 9\xi\eta^2 + \dots, \\ p_5(4) = \xi^2 + 12\xi^2\eta + \dots, & \begin{cases} p_3(6) = 6\xi^2\eta + \dots, \\ p_9(4) = \xi^2 + 10\xi^2\eta + \dots, & p_5(6) = 3\xi^2\eta + \dots. \end{cases} \end{cases}
 \end{cases}
 \end{cases}$$

The relative probabilities that $[k]$ is R or W , when a given atom is R , are

$$r_{(R)}(k) = \{(1-\epsilon)x + \epsilon\}^6 + (1-\epsilon)(1-x) \{(1-\epsilon)x + \epsilon\}^5 \sum_{\substack{\text{near.} \\ \text{neigh.}}} p_k(i), \quad (20)$$

$$w_{(R)}(k) = \{(1-\varepsilon) + x\varepsilon\}^6 - (1-\varepsilon)(1-x) \{(1-\varepsilon) + x\varepsilon\}^5 \sum_{\substack{\text{near.} \\ \text{neigh.}}} p_k(i).$$

Then we can obtain these probabilities as functions of ε , which is the quantity relating to the long range order already obtained.

When we put, instead of (14),

$$\varepsilon_n(i) = \varepsilon - \varepsilon p_n(i), \quad (21)$$

and inserting this into (9), we obtain again Eq. (16) by the same procedure described above; $p(0)=1$ means $\varepsilon_1(0)=0$ and $1-\varepsilon_1(0)=1$, i.e. the probability for atom [0 to be W is 1 in this case. Then the relative probabilities that $[k]$ is R or W , when a given atom is W , are

$$r_{(W)}(k) = \{(1-\varepsilon)x + \varepsilon\}^6 - \varepsilon(1-x) \{(1-\varepsilon)x + \varepsilon\}^5 \sum_{\substack{\text{near.} \\ \text{neigh.}}} p_k(i), \quad (22)$$

$$w_{(W)}(k) = \{(1-\varepsilon) + x\varepsilon\}^6 + \varepsilon(1-x) \{(1-\varepsilon) + x\varepsilon\}^5 \sum_{\substack{\text{near.} \\ \text{neigh.}}} p_k(i).$$

The correlation probabilities are defined by

$$C_k = \frac{1+S}{2} \frac{r_{(R)}(k)}{r_{(R)}(k) + w_{(R)}(k)} + \frac{1-S}{2} \frac{w_{(W)}(k)}{r_{(W)}(k) + w_{(W)}(k)}. \quad (23)$$

V. At low temperatures, ξ and η are expressed, from (12) and (16), as power series of x .

$$\xi = x^4 - x^6 - x^8 + 5x^9 + x^{10} + \dots,$$

$$\eta = -(x^8 - 2x^{10} + \dots), \quad (24)$$

Then we can give the correlation probabilities by series expansions as follows.

$$\left. \begin{aligned} S &= 1 - 2x^6 - 12x^{10} + 14x^{12} - 90x^{14} + 216x^{16} + \dots, \\ C_1 &= 1 - 2x^6 - 10x^{10} + 14x^{12} - 70x^{14} + 192x^{16} + \dots, \\ C_2 &= 1 - 2x^6 - 12x^{10} + 16x^{12} - 88x^{14} + 2x^{15} + 236x^{16} + \dots, \\ C_4 &= 1 - 2x^6 - 12x^{10} + 16x^{12} - 89x^{14} + x^{15} + 238x^{16} + \dots, \\ C_3 &= 1 - 2x^6 - 12x^{10} + 16x^{12} - 90x^{14} + 240x^{16} + \dots, \\ C_5 &= 1 - 2x^6 - 12x^{10} + 16x^{12} - 90x^{14} + 240x^{16} + \dots, \end{aligned} \right\} \quad (25)$$

$(C_i, i \geq 6)$ equal \bar{C}_5 to terms of order x^{16} ;

* By Bethe's original method (the first approximation)

\mathcal{S} : same as ours,

$C_1 = 1 - 2x^6 - 10x^{10} + 14x^{12} - 70x^{14} + 42x^{16} + \dots$

and

$$C_{\infty} = \left(\frac{1+S}{2} \right)^2 + \left(\frac{1-S}{2} \right)^2 = \frac{1+S^2}{2}.$$

The long and short range orders coincide up to terms of order x^{14} with the exact solution. The correlation probabilities converge more rapidly in this case than in the two-dimensional case⁽⁴⁾, as to be expected from the general consideration of cooperative phenomena.

In conclusion, the writer wishes to express his cordial thanks to Prof. K. Ariyama and Dr. S. Miyahara for their interest in this work.

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Note on the Formal Solution of the Tomonaga-Schwinger Equation.

Smiō TANI.

Department of Physics, Tokyo University.

(Received Feb. 5, 1949)

We shall present here some remarks on the formal solution of the Tomonaga-Schwinger equation of field theory. However, the physical interpretation of these formal solutions will be left for future works which will come into contact with the fundamental difficulties underlying our present theory.

The fundamental equation of the Tomonaga-Schwinger theory⁽¹⁾⁽²⁾ is of the form

$$\left\{ H(x) - i \frac{\delta}{\delta \sigma(x)} \right\} \Psi[\sigma] = 0, \quad (1)$$

where the interaction density $H(x)$ is subject to the condition

$$[H(x), H(x')] = 0, \quad \text{when } (x_\mu - x'_\mu)^2 > 0. \quad (2)$$

The formal solution of this equation will be given by the following state functional

$$\Psi[\sigma] = S[\sigma, \sigma_0] \Psi[\sigma_0], \quad (3)$$

when we impose an initial condition on the hyper-surface σ_0 and $\Psi[\sigma_0]$ denotes the state functional corresponding to this initial condition. The transformation functional $S[\sigma, \sigma_0]$ is to be determined by the following functional differential equation

$$i \frac{\delta S[\sigma, \sigma_0]}{\delta \sigma(x)} = H(x) S[\sigma, \sigma_0] \quad (4)$$

and it should also have the property

$$S[\sigma_0, \sigma_0] = 1 \quad (5)$$

that is, in form of a functional integral equation, it is given by

$$S[\sigma, \sigma_0] = 1 + \frac{1}{i} \int_{\sigma_0}^{\sigma} H(x') S[\sigma', \sigma_0] dx'. \quad (6)$$

We shall expand the transformation functional $S[\sigma, \sigma_0]$ into a power series of the coupling constant e for the later convenience. (We may assume the interaction density to be of the first order in e without missing the essential feature of our argument.)

$$S[\sigma, \sigma_0] = \sum_{n=0}^{\infty} S^{(n)}[\sigma, \sigma_0]; \quad \text{with} \quad S^{(0)}[\sigma, \sigma_0] = 1. \quad (7)$$

The functional differential equation for the $S^{(n)}[\sigma, \sigma_0]$ is read as

$$i \frac{\partial S^{(n)}[\sigma, \sigma_0]}{\partial \sigma(x)} = H(x) S^{(n-1)}[\sigma, \sigma_0] \quad (8)$$

which has the formal solution

$$S^{(n)}[\sigma, \sigma_0] = \frac{1}{i} \int_{\sigma_0}^{\sigma} H(x') S^{(n-1)}[\sigma', \sigma_0] dx'. \quad (9)$$

We also have the following equations concerning to their Hermitean conjugate $S^{(n)\dagger}[\sigma, \sigma_0]$

$$-i \frac{\partial S^{(n)\dagger}[\sigma, \sigma_0]}{\partial \sigma(x)} = S^{(n-1)\dagger}[\sigma, \sigma_0] H(x), \quad (8')$$

$$S^{(n)\dagger}[\sigma, \sigma_0] = i \int_{\sigma_0}^{\sigma} S^{(n-1)\dagger}[\sigma', \sigma_0] H(x') dx', \quad (9')$$

$$\text{with} \quad S^{(0)\dagger}[\sigma, \sigma_0] = 1. \quad (7')$$

1°) We show explicitly that $S[\sigma, \sigma_0]$ is an unitary operator:

$$\begin{aligned} S^{\dagger}[\sigma, \sigma_0] S[\sigma, \sigma_0] &= 1 + (S^{(1)\dagger} + S^{(1)}) + (S^{(2)\dagger} + S^{(1)\dagger} S^{(1)} + S^{(2)}) \\ &\quad + (S^{(3)\dagger} + S^{(2)\dagger} S^{(1)} + S^{(1)\dagger} S^{(2)} + S^{(3)}) + \dots \\ &= 1. \end{aligned} \quad (10)$$

The proof for the e -part of the above equation is self-evident;

$$S^{(1)\dagger} + S^{(1)} = i \int_{\sigma_0}^{\sigma} H(x') dx' + \left(\frac{1}{i} \right) \int_{\sigma_0}^{\sigma} H(x') dx' = 0.$$

For the e^2 -part we require the following relation

$$\begin{aligned} S^{(2)\dagger} + S^{(1)\dagger} S^{(1)} + S^{(2)} &= (i)^2 \int_{\sigma_0}^{\sigma'} H(x'') dx'' \int_{\sigma_0}^{\sigma} H(x') dx' \\ &\quad + \int_{\sigma_0}^{\sigma} H(x') dx' \int_{\sigma_0}^{\sigma'} H(x'') dx'' + \left(\frac{1}{i} \right)^2 \int_{\sigma_0}^{\sigma} H(x') dx' \int_{\sigma_0}^{\sigma'} H(x'') dx'' = 0 \end{aligned}$$

wherein the point x' lies on the surface σ' . We change the notation in the first integral and have the following three integrals

$$\left[- \int_{\sigma_0}^{\sigma} dx'' \int_{\sigma_0}^{\sigma''} dx' + \int_{\sigma_0}^{\sigma} dx' \int_{\sigma_0}^{\sigma} dx'' - \int_{\sigma_0}^{\sigma} dx' \int_{\sigma_0}^{\sigma'} dx'' \right] H(x') H(x''),$$

which reduce to

$$\left[- \int_{\sigma_0}^{\sigma} dx'' \int_{\sigma_0}^{\sigma''} dx' + \int_{\sigma_0}^{\sigma} dx' \int_{\sigma'}^{\sigma} dx'' \right] H(x') H(x'')$$

In order to see the essential points easier, we shall take the surfaces which are parallel to the spatial coordinate axis. Let $\sigma_0, \sigma', \sigma'', \sigma$ correspond to the surface $x=t, t', t'', t$ respectively. The surface integral (3-dimensional volume integral) of the interaction density over the surface with the time value t is denoted by

$$\bar{H}(t) = \int_{\sigma} H(x) dv; \quad x_0=t \text{ on the surface } \sigma.$$

Then we shall have

$$\left[- \int_{t_0}^t dt'' \int_{t_0}^{t''} dt' + \int_{t_0}^t dt' \int_{t'}^t dt'' \right] \bar{H}(t') \bar{H}(t'')$$

Both integral cover just the same region, though the order of the integration is reversed; so they cancel each other.

The proof for the 3-rd or higher order is a little more complicated, but can be performed similarly by considering the order of the integration in a certain multiple integral.

2°) One can easily verify that there exists the Heisenberg picture of the energy operator in the generalized sense, that is, it is defined in terms of the coordinate of the respective point alone but does not depend on the surface σ : the representative state functional is thereby the same and constant for all world points.

For the proof we have only to show that the transformed operator $iS^+[\sigma, \sigma_0] \cdot \partial/\partial\sigma(x) \cdot S[\sigma, \sigma_0] = S^+[\sigma, \sigma_0] H(x) S[\sigma, \sigma_0]$ is independent on the form of the space like surface σ , or in other words, their functional derivative with respect to another point x which lies upon the surface σ always vanishes.

$$\begin{aligned} & \frac{\partial(S^+[\sigma, \sigma_0] H(x) S[\sigma, \sigma_0])}{\partial\sigma(x')} \\ &= - \frac{\partial S^+[\sigma, \sigma_0]}{\partial\sigma(x')} H(x) S[\sigma, \sigma_0] + S^+[\sigma, \sigma_0] H(x) \frac{\partial S[\sigma, \sigma_0]}{\partial\sigma(x')} \end{aligned}$$

$$= \frac{1}{i} S^+[\sigma, \sigma_0] (H(x') H(x) - H(x) H(x')) S[\sigma, \sigma_0] = 0. \quad (11)$$

We shall employ the following notation hereafter

$$S^+[\sigma, \sigma_0] H(x) S[\sigma, \sigma_0] = \sum_{n=1}^{\infty} E^{(n)}(x); \quad \text{with } E^{(1)}(x) \equiv H(x). \quad (12)$$

(Rigorously speaking, they ought to be written as $E_{[\sigma_0]}^{(n)}(x)$; See Art. 5.)

The statement that the transformed operator of energy is a point function is valid not only for the transformed operator as a whole, but also for the individual part of the power series of the coupling constant.

3°) The condition for an arbitrary operator $O(x)$ to have the Heisenberg picture in the sense of the preceding article is

$$[H(x), O(x')] = 0, \quad \text{when } (x_\mu x_\mu')^2 > 0 \quad (13)$$

The proof is quite similar to that of the preceding article.

4°) The transformation functional $S[\sigma, \sigma_0]$ can be expressed in terms of the 4-dimensional volume integrals of the point function operators of energy mentioned in Art. 2.: the upper and lower limit of these integrals are bounded by the surfaces σ and σ_0 respectively. That is, the $S^{(n)}[\sigma, \sigma_0]$'s are expressed in terms of $\int_{\sigma_0}^{\sigma} E^{(n)}(x') dx'$'s by solving the following equations inversely.

$$\begin{aligned} \frac{1}{i} \int_{\sigma_0}^{\sigma} E^{(n)}(x') dx' &= \sum_{m=1}^n m S^{+(n-m)}[\sigma, \sigma_0] S^{(m)}[\sigma, \sigma_0] \\ &= n S^{(n)}[\sigma, \sigma_0] - \sum_{m=1}^{n-1} m S^{(n-m)}[\sigma, \sigma_0] S^{(m)}[\sigma, \sigma_0] + \sum_{m=1}^{n-2} \sum_{\substack{\text{All permutation} \\ (i) + (ii) = n-m}} m S^{(i)} S^{(ii)} S^{(n-m)} \\ &+ \dots + (-1)^i \sum_{m=1}^{n-2} \sum_{\substack{\text{All perm.} \\ (i) + (ii) + \dots + (\lambda) = n-m}} m S^{(i)} S^{(ii)} \dots S^{(\lambda)} S^{(n-m)} + \dots + (-1)^{n-1} \{ S^{(1)} \}^n. \end{aligned} \quad (14)$$

Proof: we notice first that

$$E^{(n)}(x) = \sum_{m=0}^{n-1} S^{+(m)}[\sigma, \sigma_0] H(x) S^{(n-m-1)}[\sigma, \sigma_0] \quad (12')$$

and that

$$\sum_{m=0}^n S^{+(m)}[\sigma, \sigma_0] S^{(n-m)}[\sigma, \sigma_0] = 0, \quad (10)$$

which relation was used in the third statement of Eq. (14), i.e.

$$\begin{aligned} S^{(n)}[\sigma, \sigma_0] = & -S^{(n)}[\sigma, \sigma_0] + \sum_{m=1}^{n-1} S^{(m)}[\sigma, \sigma_0] S^{(n-m)}[\sigma, \sigma_0] + \dots \\ & + (-1)^n \sum_{\substack{(i)(ii) \dots (\lambda) \\ \text{(All perm.)} \\ (i) + (ii) + \dots + (\lambda) = n}} S^{(i)} S^{(ii)} \dots S^{(\lambda)} + \dots + (-1)^n \{ S^{(0)}[\sigma, \sigma_0] \}^n \end{aligned} \quad (15)$$

By integrating the following equation

$$S^{(n)}[\sigma, \sigma_0] H(x) S^{(m)}[\sigma, \sigma_0] = i \frac{\partial}{\partial \sigma(x)} \left\{ \sum_{l=0}^n S^{(n-l)}[\sigma, \sigma_0] S^{(m+1+l)}[\sigma, \sigma_0] \right\}, \quad (16)$$

we have the relation

$$\frac{1}{i} \int_{\sigma_0}^{\sigma} S^{(n)}[\sigma', \sigma_0] H(x') S^{(m)}[\sigma', \sigma_0] dx' = \sum_{l=0}^n S^{(n-l)}[\sigma, \sigma_0] S^{(m+1+l)}[\sigma, \sigma_0]. \quad (17)$$

Then, from Eqs. (17) and (12') we get the final results.

The results for the first four order in ϵ are tabulated as follows :

$$\begin{aligned} S^{(1)}[\sigma, \sigma_0] &= \frac{1}{i} \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx'; \\ S^{(2)}[\sigma, \sigma_0] &= \frac{1}{2!} \left\{ \frac{1}{i} \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx' \right\}^2 + \frac{1}{2} \frac{1}{i} \int_{\sigma_0}^{\sigma} E^{(2)}(x') dx; \\ S^{(3)}[\sigma, \sigma_0] &= \frac{1}{3!} \left\{ \frac{1}{i} \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx' \right\}^3 + \frac{1}{3!} \left(\frac{1}{i} \right)^2 \int_{\sigma_0}^{\sigma} E^{(2)}(x') dx' \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx' \\ &\quad + \frac{1}{3} \left(\frac{1}{i} \right)^2 \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx' \int_{\sigma_0}^{\sigma} E^{(2)}(x') dx' + \frac{1}{3} \frac{1}{i} \int_{\sigma_0}^{\sigma} E^{(3)}(x') dx'; \\ S^{(4)}[\sigma, \sigma_0] &= \frac{1}{4!} \left\{ \frac{1}{i} \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx' \right\}^4 + \frac{1}{4!} \left(\frac{1}{i} \right)^3 \int_{\sigma_0}^{\sigma} E^{(2)}(x') dx' \left\{ \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx' \right\}^2 \\ &\quad + \frac{1}{4 \cdot 2} \left(\frac{1}{i} \right)^2 \left\{ \int_{\sigma_0}^{\sigma} E^{(2)}(x') dx' \right\}^2 + \frac{1}{4 \cdot 2} \left(\frac{1}{i} \right)^3 \left\{ \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx' \right\}^2 \int_{\sigma_0}^{\sigma} E^{(2)}(x') dx' \\ &\quad + \frac{1}{4 \cdot 3} \left(\frac{1}{i} \right)^2 \int_{\sigma_0}^{\sigma} E^{(3)}(x') dx' \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx' \\ &\quad + \frac{1}{4 \cdot 3} \left(\frac{1}{i} \right)^3 \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx' \int_{\sigma_0}^{\sigma} E^{(2)}(x') dx' \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx' \\ &\quad + \frac{1}{4} \left(\frac{1}{i} \right)^2 \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx' \int_{\sigma_0}^{\sigma} E^{(3)}(x') dx' + \frac{1}{4} \frac{1}{i} \int_{\sigma_0}^{\sigma} E^{(4)}(x') dx'. \end{aligned}$$

5°) Between the $E^{(n)}(x)$'s there exists the following recurrence formula

$$(n-1)E^{(n)}(x) = \sum_{m=1}^{n-1} \left[E^{(n-m)}(x), \frac{1}{i} \int_{\sigma}^{\sigma} E^{(m)}(x') dx' \right], \quad (18)$$

which enables for one to get the higher order terms successively initiating from the $E^{(1)}(x) = H(x)$. This formula is useful in practice, but some points should be noticed as regards to the boundary of the integration region. In

the above equation, the integral $\int_{\sigma}^{\sigma} E^{(m)}(x') dx'$ should have been written as $\int_{\sigma_0}^{\sigma} E^{(m)}(x') dx'$ according to their original meaning, however, the contribution

from the lower surface σ_0 have nothing to do with the character of $E^{(n)}(x)$ as a point function. So it is convenient for one to proceed in the following way: we may neglect the contribution from the lower boundary by employing the Fourier-transform of the integrand and the Gauss' theorem.* This

integral is denoted by $\int_{\sigma}^{\sigma} E^{(m)}(x') dx'$ in the above equation. This procedure corresponds physically to the adiabatic switching on of the interaction: and the principal value should be employed in the integrals which contain processes obeying the energy-momentum conservation law. After we get the general point function of energy, we supplement the contribution which come from the surface σ_0 , if necessary. The supplementary terms are expressed by means of the surface integrals over σ_0 by performing the Fourier-transform of $E^{(m)}(x)$'s.

For the proof of Eq. (18) we should combine the following relations.

$$E^{(n)}(x) = \sum_{n=0}^n S^{(m)}[\sigma,] \cdot i \frac{\partial}{\partial \sigma(x)} \cdot S^{(n-m)}[\sigma,] = i \frac{\partial S^{(n)}[\sigma,]}{\partial \sigma(x)} - \sum_{m=1}^{n-1} S^{(n-m)}[\sigma,] E^{(m)}(x), \quad **$$

or

$$i \frac{\partial S^{(n)}[\sigma,]}{\partial \sigma(x)} = \sum_{m=1}^n S^{(n-m)}[\sigma,] E^{(m)}(x), \quad (19)$$

and

* In Schwinger's notation, $\int_{\sigma}^{\sigma} E^{(m)}(x') dx' = \int_{\sigma}^{\sigma} G_{\mu}^{(m)}(x') d\sigma_{\mu}'$, where $\frac{\partial}{\partial X_{\mu}} G_{\mu}^{(m)}(x) = E^{(m)}(x)$, and $d\sigma_{\mu}'$

is the element at the point x' of the surface σ over which the integration is performed.

** In $S^{(n)}[\sigma,]$, the $\int_{\sigma}^{\sigma} E^{(m)}(x') dx'$'s are exclusively used.

$$-i \frac{\partial S^+[\sigma,]}{\partial \sigma(x)} = \sum_{m=1}^n E^{(m)}(x) S^+[\sigma,], \quad (19')$$

which are revision of (12') by making use of (15); and

$$\frac{1}{i} \int E(x') dx' = - \sum_{m=1}^n m S^{(m)}[\sigma,] S^{(n-m)}[\sigma,] = \sum_{m=1}^n m S^{(n-m)}[\sigma,] S^{(m)}[\sigma,], \quad (14')$$

which can be shown similarly as in Eq. (14) by changing the role of the Hermitian conjugate operators. With the aid of Eq. (10), they yield the results

$$\begin{aligned} E^{(n)}(x) &= n i \frac{\partial S^{(n)}[\sigma,]}{\partial \sigma(x)} + \sum_{m=1}^{n-1} m i \frac{\partial S^{(n-m)}[\sigma,]}{\partial \sigma(x)} S^{(m)}[\sigma,] \\ &\quad + \sum_{m=1}^{n-1} (n-m) i S^{(m)}[\sigma,] \frac{\partial S^{(n-m)}[\sigma,]}{\partial \sigma(x)} \\ &= n E^{(n)}(x) + n \sum_{m=1}^{n-1} S^{(n-m)}[\sigma,] E^{(m)}(x) - \sum_{m=1}^{n-1} \sum_{l=1}^{n-m} m E^{(l)}(x) S^{(n-m-l)}[\sigma,] S^{(m)}[\sigma,] \\ &\quad + \sum_{m=1}^{n-1} \sum_{l=1}^{n-m} (n-m) S^{(m)}[\sigma,] S^{(n-m-l)}[\sigma,] E^{(l)}(x) \\ &= n E^{(n)}(x) - \sum_{l=1}^{n-1} E^{(l)}(x) \frac{1}{i} \int E^{(n-l)}(x') dx' + \sum_{l=1}^{n-1} \frac{1}{i} \int E^{(n-l)}(x') dx' \cdot E^{(l)}(x). \end{aligned}$$

6°) We add that the $E^{(n)}(x)$ of the preceding article is uniquely determined disregarding an arbitrary constant factor, when we impose a condition that they should be a point function and be a linear combination of the products $\int_{\sigma}^{(i)} E(x') dx' \int_{\sigma}^{(ii)} E(x'') dx'' \dots H(x) \dots \int_{\sigma}^{(\lambda)} E(x''') dx''' [(i) + (ii) + \dots + (\lambda) = n-1]$ of the integrals of the point function energy lower than n and $H(x)$. We will not reproduce the proof in detail. But the essential is as follows: when we perform the functional differentiation of this linear combination with respect to the point x' on the surface σ , we have as many condition that the sum of certain coefficients of combination should vanish as there are types of terms such as $\int_{\sigma}^{(i)} E(x') dx' \int_{\sigma}^{(ii)} E(x'') dx'' \dots E^{(x)}(x')$ $\dots H(x) \dots \int_{\sigma}^{(\lambda)} E(x''') dx''' [(i) + (ii) + \dots + (x) + \dots + (\lambda) = n-1]$. In general, the number of the conditions is greater than the number of the unknown coefficients. However, if we eliminate those conditions which are not

linearly independent and take the condition $[H(x), H(x')]=0$ into account, there is left just one less conditions than the number of the unknown coefficients.

7°) For some purposes it is convenient to express the transformation functional in form of an exponential function

$$S[\sigma, \sigma_0] = \exp -i(K^{(1)}[\sigma, \sigma_0] + K^{(2)}[\sigma, \sigma_0] + \dots), \quad (20)$$

where the $K^{(n)}[\sigma, \sigma_0]$ is n -th order in ϵ , and is expressed in terms of $\int_{\sigma_0}^{\sigma} E^{(n)}(x) dx$'s. The results for the first four order are as follows:

$$K^{(1)}[\sigma, \sigma_0] = \int_{\sigma_0}^{\sigma} E^{(1)}(x') dx';$$

$$K^{(2)}[\sigma, \sigma_0] = \frac{1}{2} \int_{\sigma_0}^{\sigma} E^{(2)}(x') dx';$$

$$K^{(3)}[\sigma, \sigma_0] = \frac{1}{3} \int_{\sigma_0}^{\sigma} E^{(3)}(x') dx' + \frac{1}{12} \left[\int_{\sigma_0}^{\sigma} E^{(1)}(x') dx', \int_{\sigma_0}^{\sigma} E^{(2)}(x') dx' \right];$$

$$K^{(4)}[\sigma, \sigma_0] = \frac{1}{4} \int_{\sigma_0}^{\sigma} E^{(4)}(x') dx' + \frac{1}{12} \left[\int_{\sigma_0}^{\sigma} E^{(1)}(x') dx', \int_{\sigma_0}^{\sigma} E^{(3)}(x') dx' \right].$$

The ϵ^n -approximation of the original equation in which the first term of the transformed equation of motion is $E^{(n)}(x)/n$, which is a point function, i.e. in equation

$$\left\{ \frac{1}{i} \frac{\delta}{\delta \sigma(x)} + \frac{E^{(n)}(x)}{n} + (\text{terms of order higher than } n \text{ in } \epsilon) \right\} \Phi[\sigma] = 0. \quad (21)$$

is given by the following canonical transformation of the state functional and the dynamical variables

$$\begin{aligned} \Psi[\sigma] &\longrightarrow \Phi[\sigma]; \quad \Psi[\sigma] = U^{(n)}[\sigma,] \Phi[\sigma] \\ U^{(n)}[\sigma,] &= \exp -i \left\{ \sum_{n=1}^{n-1} K^{(n)}[\sigma,] + \left(K^{(n)}[\sigma,] - \frac{1}{n} \int_{\sigma_0}^{\sigma} E^{(n)}(x') dx' \right) \right\} \\ O(x) &\longrightarrow U^{(n)-1}[\sigma,] O(x) U^{(n)}[\sigma,]. \end{aligned} \quad (22)$$

8°) In concluding this note, we can add that the generalization of the results presented here may be able to include the cases where the

interaction density is no more a point function but contains the normal or other quantity related to the surface and consequently the integrability condition is much more complicated than in Eq. (2). But these results will be discussed in another occasion.

The author wishes to express his cordial thanks to Prof. Tomonaga for the suggestion to this work.

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- (1) S. Tomonaga: *Prog. Theor. Phys.*, **1** (1946) 27, and a series of works following this Part I.
 - (2) J. Schwinger: *Phys. Rev.* **74** (1948) 1439.
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Reactive Corrections for the Elastic Scattering of an Electron.*

Shinji ENDÔ, Tôichirô KINOSHITA, and Zirô KOBA

Institute of Physics, Faculty of Science, Tokyo University.

(Received Apr. 5, 1949)

§ 1. Introduction and Summary.

One of the simplest problems about the reaction of radiation field on the electron is the radiative correction for the elastic scattering cross section of an electron by a fixed electrostatic potential. But attempts to solve this problem have heretofore been beset by divergence difficulties. Recently, however, a certain part of the divergent corrections was found to be just cancelled if the C-meson field is introduced in addition to the radiation field.⁽¹⁾⁽²⁾ At the same time it became clear that the divergences can be attributed to the self-energy and vacuum polarization, so that immediately after that "self-consistent subtraction method" or "amalgamation method" was proposed in order to obtain a finite result.⁽³⁾

Chronologically, the C-meson hypothesis was thus put forward first and then the self-consistent subtraction method was presented as an alternative prescription. But through further considerations we have now the following opinion with regard to the *logical* relation of these two hypotheses: The self-consistent subtraction method analyses in what manner the divergences occur and finds that they are reduced to unobservable renormalization of mass and charge and that one can obtain a finite reactive correction by merely replacing the theoretical infinite mass and charge by the empirical ones. This conclusion has been indeed verified by several experiments, but the fundamental difficulty of divergence remains still unsolved. The C-meson hypothesis, on the other hand, gives a tentative solution as to the origin of one of these divergences. Thus it aims at a higher substantial stage of the theory and seems at the same time inevitably less confirmative before it obtains an experimental support. Another remark is that the

* The main part of this paper was read at the annual meeting of the Physical Society of Japan held on May 22, 1948, at Kyoto University. Also a preliminary report and errata to it have been published in this journal, vol. 3 (1948) 320, and vol. 4 (1949) 100, respectively.

amalgamation procedure of mechanical and reactive mass is necessary also on this hypothesis, but the correction is finite here.

Thus we can now derive the theoretical value to be compared with future experiments. Moreover, it seemed to us interesting to see whether the C-meson hypothesis gives rise to appreciable deviation from the mere subtraction of infinity. So we calculated the actual value of reactive correction in two ways, one using the self-consistent subtraction method for the electron-photon interaction, the other including the C-meson hypothesis.

As formerly pointed out by Pauli and Fierz,⁽⁴⁾ the non-relativistic treatment of the field reaction to a point electron gives rise to a logarithmic divergence for the scattering cross section. Dancoff⁽⁵⁾ showed that this divergence disappears if it is treated relativistically while there arises another divergence of purely relativistic origin whose reasonable management was the very purpose of the above investigations.⁽²⁾⁽³⁾ In the non-relativistic theory of Bloch-Nordsieck⁽⁶⁾ and Pauli-Fierz, it was possible to arrive at the final result in a closed form by a canonical transformation, without expanding into the ascending powers of the electronic charge e , thus avoiding completely the so-called "infrared catastrophe." The circumstance is, however, not so simple in the relativistic theory, and we do not know for the present any other method than that of expansion in e , i.e., the perturbation method, or the equivalent canonical transformation method of Tomonaga-Schwinger theory, etc.⁽⁷⁾ We have resorted here to the usual perturbation method. Of course, the infrared catastrophe appears in such a treatment, and in order to avoid this trouble we have to take care not to miss any possible process in each step of approximation. But since our discussions here are restricted to the e^2 -approximation, processes to be considered are only those that involve 0 or 1 light quantum in the final state. Under these circumstances, the only finite value we can evaluate is that of the cross section for the scattering of an electron into an angle θ both with and without emission of a light quantum.

General evaluation being rather complicated,* we have restricted ourselves to the case of a slow electron and calculated those corrections $\delta\sigma/\sigma_0$ approximately, assuming the velocity p/E of the incident electron small compared with that of light, and expanding them into the ascending powers of p/m , with only the first two terms retained. It must be noticed on the other hand that the magnitude of p/E cannot be taken too small since all our results are based on Born's approximation.

* The general aspects are now being investigated by our group.

In this approximation the self-consistent subtraction method for the electron-photon system yields the following result:

$$\frac{\delta\sigma}{\sigma_0} = \frac{e^2}{\pi} \left(\frac{p}{m}\right)^2 (1 - \cos\theta) \left[-3 + \frac{8}{3} \log \frac{2p}{m} \right] \quad (1)$$

where θ is the scattering angle of electron. We use the natural unit system $\hbar=c=1$ throughout this paper. As is easily seen from (1), this correction is always negative for the values of p considered here. One can also verify that the non-relativistic formula derived by Pauli and Fierz gives the same value as (1), provided that the diverging integral is cut off at about m , what is to be expected.

As to the reactive correction when the electron is surrounded by the radiation field and the C-meson field, it is found that its magnitude is almost the same with that obtained by the self-consistent subtraction method for the electron-photon system. Therefore, it is concluded that the C-meson field has hardly any detectable effect* on the correction for scattering cross section. By the way, we had better consider in this case the world scalar potential besides the electrostatic one, if the problem is to be treated consistently. But in the present approximation, we find that the above conclusion is scarcely affected by this conclusion.

§ 2. Calculations.

As the starting point let us investigate what result is obtained for the radiative correction to the cross-section of elastic scattering by means of the ordinary formalism of quantum electrodynamics. Under the circumstances explained in § 1, the objects of our calculation are confined to (I) the radiationless correction in the e^2 -approximation to the zeroth order elastic scattering, and (II) the correction due to the process of emitting a single light quantum.

The procedure to calculate the correction (I) is given in detail elsewhere⁽²⁾, so we give here only its outline.

i) To find the wave function Ψ , which represents the stationary state of a free electron interacting with its own radiation field. When expanded in powers of e , Ψ may be written in the form $\Psi = \Psi^{(0)} + e\Psi^{(1)} + e^2\Psi^{(2)} + \dots$, where the first term represents a free electron state accompanied by no

* This is a conclusion contrary to the one given in the first letter. With regard to the circumstance of this revision, see the discussions in § 2.

②

light quantum and the following terms, in general, states with light quanta and electron-positron pairs present.

ii) To calculate the probability of transition from a state $\Psi(\mathbf{p})$ to $\Psi(\mathbf{q})$ induced by the electrostatic potential V , where \mathbf{p} and \mathbf{q} are the initial and final momenta of an electron respectively and $\mathbf{p}^2 = \mathbf{q}^2$ in the present case. The matrix element of V with regard to states $\Psi^{(0)}(\mathbf{p})$ and $\Psi^{(0)}(\mathbf{q})$ leads to the zeroth order cross section σ_0 for the elastic scattering. We then pick up the terms in $\Psi(\mathbf{p})$ which combine with those of $\Psi(\mathbf{q})$ through the scattering, even or odd, to give corrections $\delta\sigma$ of order ϵ^2 to σ_0 . By an odd scattering is meant a pair creation or a pair annihilation by V .

The correction term for (I) can be classified, according to Dancoff,⁽⁵⁾ into the following three groups.

(A) Terms involving no pairs in initial and final states. This group represents a relativistic modification of the terms occurring in the non-relativistic treatment. This correction is given by (I.3.7)* Terms of the relative correction in the order p^2 are collected into

$$\delta(A) = \frac{e^2}{\pi} \left(\frac{p}{m} \right)^2 (1 - \cos \theta) \left[\frac{1}{12} + \frac{\pi}{6} + \frac{4}{3} \lim_{k \rightarrow 0} \log k \right]. \quad (2)$$

(B) Terms involving pairs in initial and final wave functions which are combined through an even scattering of electron or positron by the scattering potential. See (I.3.8) and (I.3.9). The p^2 -term is

$$\delta(B) = \frac{e^2}{\pi} \left(\frac{p}{m} \right)^2 (1 - \cos \theta) \left[-\frac{1}{4} - \frac{\pi}{6} + \frac{4}{3} \log 2 \right] \quad (3)$$

(C) Terms involving pair creation and annihilation by the scattering potential, which can be subdivided into three parts:

i) Processes which arise, e.g., according to the following schema

$$p \xrightarrow{\text{Hint.}} p, q, (-q-k)^+, \tilde{k} \xrightarrow{\text{Hint.}} p+k, q, (-q-k)^+ \xrightarrow{V} q.$$

One reads this as follows; in the wave function $\Psi(\mathbf{p})$ we consider the term which contains a pair $\mathbf{p}+\mathbf{k}, (-\mathbf{q}-\mathbf{k})^+$ created through the two steps

a) pair creation of $q, (-q-k)^+$ with the emission of \tilde{k} and b) absorption of \tilde{k} by p ; this combines, by means of the term in the scattering potential

* In the paper I, the correction is calculated for the general case where the scattering potential consists of electrostatic potential V and world scalar potential W . As we consider in this paragraph electrostatic potential only, we cite formulae of the paper I with the specialization $W=0$.

annihilating $\mathbf{p} + \mathbf{k}$, $(-\mathbf{q} - \mathbf{k})^+$, with $\mathcal{V}^0(\mathbf{q})$ in $\mathcal{V}(\mathbf{q})$. (I.3.10~14) belong to this type. The \mathcal{P}^2 -term is

$$\delta(C_i) = \frac{7e^2}{18\pi} \left(\frac{\mathcal{P}}{m}\right)^2 (1 - \cos\theta). \quad (4)$$

ii) There are several processes such as

$$\mathbf{p} \xrightarrow{\text{Hint.}} \mathbf{p}, \mathbf{q}, (-\mathbf{q} - \mathbf{k})^+, \tilde{\mathbf{k}} \xrightarrow{\text{Hint.}} \mathbf{p}, \mathbf{q}, (-\mathbf{q})^+ \xrightarrow{\mathbf{V}} \mathbf{q}.$$

This is a process in which the effect of the interaction of electron with radiation field repeated twice results in the mere creation of a pair \mathbf{q} , $(-\mathbf{q})^+$, the initial electron \mathbf{p} being unaltered at all. This process may well be called of a "self-energy type." From (I.3.15~17) we get

$$\delta(C_{ii}) = \frac{e^2}{\pi} \left(\frac{\mathcal{P}}{m}\right)^2 (1 - \cos\theta) \left[\frac{3}{2} \lim_{K \rightarrow \infty} \log 2K - \frac{17}{12} \right]. \quad (5)$$

iii) Processes in which an arbitrary virtual electron-positron pair plays an essential role. For example

$$\mathbf{p} \xrightarrow{\text{Hint.}} \mathbf{q}, \widetilde{\mathbf{p} - \mathbf{q}} \xrightarrow{\text{Hint.}} \mathbf{q}, \mathbf{r}, (-\mathbf{r} + \mathbf{p} - \mathbf{q})^+ \xrightarrow{\mathbf{V}} \mathbf{q}.$$

These consist of (I.3.19~22). The first two terms of expansion in the power of \mathcal{P} are :

$$\delta(C_{iii}) = - \left[\frac{4e^2}{3\pi} \left(-\frac{5}{6} + \lim_{K \rightarrow \infty} \log 2K \right) \right] + \frac{4e^2}{15\pi} \left(\frac{\mathcal{P}}{m}\right)^2 (1 - \cos\theta). \quad (6)$$

This represents an infinite effect of vacuum polarization, the first term diverging logarithmically. This divergence is to be subtracted in a positron-theoretical treatment. But in the present stage of positron theory, there is no definite rule how to subtract the diverging effect of vacuum polarization.* The simplest manner is of course to drop off this term entirely regarding it as an unobservable quantity. Another conceivable way is indicated by the self-consistent subtraction method applied to the vacuum polarization phenomena⁽³⁾; according to which only the term in the square bracket on the right hand side of (6) is to be subtracted. The difference between these two methods, i.e., the term with \mathcal{P}^2 in $\delta(C_{iii})$, is, however, rather small. Therefore, we shall prefer the former standpoint for simplicity and always drop off $\delta(C_{iii})$ in the following discussions.

* Recently Schwinger has treated this problem in detail. (Phys. Rev. 75 (1949), 651).

The correction (I) is obtained adding the terms (2), (3), (4), and (5) together:

$$\left(\frac{\delta\sigma}{\sigma_0}\right)_I = \frac{c^2}{\pi} \left(\frac{p}{m}\right)^2 (1 - \cos\theta) \left[-\frac{43}{36} + \frac{4}{3} \lim_{k \rightarrow 0} \log 2k + \frac{3}{2} \lim_{K \rightarrow \infty} \log 2K \right]. \quad (7)$$

As is seen at once, there appear two logarithmic divergences, one representing the so-called "infrared catastrophe," the other being of the ultraviolet origin. The former can be eliminated if we take another e^2 -correction corresponding to an emitting process, i.e., correction (II) into account. As we assume $p/E \ll 1$ throughout this paper, we may be justified to calculate this correction non-relativistically. We also assume that the electrostatic potential is a Coulombian one. Making use of the formula first derived by Mott⁽⁶⁾

$$d\sigma = \frac{32}{3\pi} e^2 v^2 \left(\frac{Ze^2}{2mv^2} \right)^2 \frac{dk}{k} \frac{\sin\theta d\theta d\varphi}{v/v' + v'/v - 2\cos\theta} \quad (8)$$

which gives a cross-section for the scattering of an electron with velocity v into the solid angle $\sin\theta d\theta d\varphi$ with final velocity v' emitting a light quantum of energy between k and $k+dk$ induced by a nuclear Coulomb field with charge Ze , the correction (II) can be written in the following way:

$$\begin{aligned} \left(\frac{\delta\sigma}{\sigma_0}\right)_{II} = & \frac{c^2}{\pi} \left(\frac{p}{m}\right)^2 (1 - \cos\theta) \left[-\frac{4}{3} \lim_{k \rightarrow 0} \log k + \frac{8}{3} \log \frac{p}{m} + \frac{4}{3} \log 2 \right] \\ & + \frac{4c^2}{3\pi} \left(\frac{p}{m}\right)^2 (1 - \cos\theta) \left[\frac{2\cos\theta}{1 + \cos\theta} \log \sin \frac{\theta}{2} - (\pi - \theta) \tan \frac{\theta}{2} \right]. \quad (9) \end{aligned}$$

Now, as the second term of (9) is almost equal to

$$-2e^2 p^2 / \pi m^2 (1 - \cos\theta) \quad (10)$$

for all values of θ between 0 and π , it is possible to write

$$\left(\frac{\delta\sigma}{\sigma_0}\right)_{II} = \frac{c^2}{\pi} \left(\frac{p}{m}\right)^2 (1 - \cos\theta) \left[-2 + \frac{4}{3} \log 2 + \frac{8}{3} \log \frac{p}{m} - \frac{4}{3} \lim_{k \rightarrow 0} \log k \right] \quad (9')$$

with a good approximation, error comitted by the substitution of the second term of (9) by (10) amounting to $0.15 e^2 p^2 / \pi m^2$ under the worst conditions.

The radiative correction to be expected from the usual theory of interaction between electron and radiation field is obtained adding the corrections (I) and (II).

$$\left(\frac{\partial\sigma}{\sigma_0}\right)_I + \left(\frac{\partial\sigma}{\sigma_0}\right)_{II} = \frac{e^2}{\pi} \left(\frac{p}{m}\right)^2 (1 - \cos\theta) \left[-\frac{115}{36} + \frac{8}{3} \log \frac{2p}{m} + \frac{3}{2} \lim_{K \rightarrow \infty} \log 2K \right]. \quad (11)$$

This involves yet a logarithmic divergence, which, different from other divergences appeared above, has been found impossible to remove in the frame of ordinary quantum field theory and has been regarded as an expression of its characteristic difficulties. Recently, however, it has been noticed that this divergence is of the same type as that of the electromagnetic self-energy of an electron in the e^2 -approximation, which has offered a new indication how to deal with this difficulty. That is the reason why both the self-consistent subtraction method and the C-meson hypothesis have been successful in getting rid of this divergence.

In the self-consistent subtraction method for the electron-photon system, the Hamilton function describing it:

$$H = H_{\text{electron}} + H_{\text{radiation}} + H_{\text{interaction}} \quad (12)$$

is rewritten in the form

$$H = (H_{\text{elec.}} + \delta m \int \psi^* \beta \psi d\mathbf{r}) + H_{\text{rad.}} + (H_{\text{int.}} - \delta m \int \psi^* \beta \psi d\mathbf{r}), \quad (13)$$

where δm is the electromagnetic mass of an electron at rest⁽⁹⁾

$$\delta m = \left[\frac{3}{2} \lim_{K \rightarrow \infty} \log 2K - \frac{1}{4} \right], \quad (14)$$

and $m + \delta m$ is replaced by the empirical value of the electron mass. Applying the new interaction Hamiltonian $H_{\text{int.}} - \delta m \int \psi^* \beta \psi d\mathbf{r}$ to the present problem, the counter-self-energy term $-\delta m \int \psi^* \beta \psi d\mathbf{r}$ gives rise to the additional correction

$$\delta(m) = -\frac{e^2}{\pi} \left(\frac{p}{m}\right)^2 (1 - \cos\theta) \left[\frac{3}{2} \lim_{K \rightarrow \infty} \log 2K - \frac{1}{4} \right], \quad (15)$$

which involves a logarithmic divergence of the same magnitude and opposite sign compared with the one appeared in (11). Adding (15) to (11), the divergence just cancel each other, and one can arrive at a convergent result

$$\frac{\partial\sigma}{\sigma_0} = \frac{e^2}{\pi} \left(\frac{p}{m}\right)^2 (1 - \cos\theta) \left[-3 + \frac{8}{3} \log \frac{2p}{m} \right]. \quad (1)$$

As a closer examination shows, this correction consists of the following two contributions

$$\left. \begin{aligned} \frac{\delta\sigma}{\sigma_0} &= \frac{e^2}{\pi} \left(\frac{p}{m} \right)^2 (1 - \cos\theta) \left[-2 + \frac{8}{3} \log 2 \right] \dots\dots\dots (a) \\ &+ \frac{e^2}{\pi} \left(\frac{p}{m} \right)^2 (1 - \cos\theta) \left[-1 + \frac{8}{3} \log \frac{p}{m} \right] \dots\dots\dots (b) \end{aligned} \right\} \quad (1')$$

In this expression, the first term (a) represents the sum of two corrections, one due to the actual emission of light quantum of energy less than $p^2/2m$, the other due to the corresponding radiationless process. Though light quanta of energy larger than $p^2/2m$ cannot be emitted actually on account of the energy conservation, the corresponding radiationless process survives all over the energy range between $p^2/2m$ and ∞ and gives rise to the correction (b). As is easily seen from (1'), (a) \ll (b) holds as far as the momentum p remains small.

Correction due to the reaction of C-meson field on the electron can be calculated almost in the same way: The results of calculations are (1.3.24~36). The magnitude of correction of order p^2 is

$$-\frac{f^2}{2\pi} \left(\frac{p}{m} \right)^2 (1 - \cos\theta) \left[-\frac{7}{8} + \frac{3}{4} \log 2 - \frac{3}{2} \log \frac{\mu}{m} + \frac{3}{2} \lim_{K \rightarrow \infty} \log 2K \right], \quad (16)$$

where μ is the C-meson mass ($\mu \sim 100m$). Applying the condition $f^2 = 2e^2$, we obtain the finite reactive correction when the C-meson field is considered besides the radiation field,

$$\frac{\delta\sigma}{\sigma_0} = \frac{e^2}{\pi} \left(\frac{p}{m} \right)^2 (1 - \cos\theta) \left[-\frac{7}{3} - \frac{3}{4} \log 2 + \frac{8}{3} \log \frac{2p}{m} + \frac{3}{2} \log \frac{\mu}{m} \right] \quad (17)$$

This is a result obtained by applying directly the C-meson hypothesis, and a one reported in the first letter. However, when the C-meson field exists besides the radiation field, the electron undergoes a finite non-vanishing mass change through interaction with these fields, and, therefore, we must take into account of its effect on the reactive correction for electron scattering in order to compare the result with experiments. This was the very point overlooked in the first letter.

The relative correction due to the change of mass is

$$\frac{\delta\sigma}{\sigma_0} = -\frac{1}{\sigma_0} \frac{d\sigma_0}{dm} \delta m, \quad (18)$$

where δm is the sum of rest mass modifications of an electron by the two fields;

$$\delta m = \frac{e^2}{\pi} m \left(\frac{5}{8} - \frac{3}{4} \log 2 + \frac{3}{2} \log \frac{\mu}{m} \right) \quad (19)$$

Adding (18) to (17), we can get a result which is almost the same with (1), i.e., it is very close to the one obtained considering radiation field alone and subtracting the infinite correction due to mass change.

The calculation in which only the first non-vanishing term is retained might appear unreliable in the limit of long wave length, since higher order terms in ϵ then become important. But in fact we can ascertain that the real emission process of arbitrary order and the corresponding radiationless process cancel each other and that the result in ϵ^2 -approximation is really very close to the one which should be obtained by an exact calculation.

§ 3. Discussions.

Our main interests in this investigation are to see whether the actual magnitude of relative correction is experimentally detectable or not, and to know whether the reactive correction for electron scattering can serve as a test for the existence of the C-meson field or not. As to the magnitude of relative correction $\delta\sigma/\sigma_0$, it does not exceed 10^{-3} for the momentum range of electron considered here, as is easily computed from (1). Since this is extremely small, it seems impossible to detect the calculated effects by the existing experimental apparatus. Thus one is justified to calculate the elastic scattering cross section of an electron entirely neglecting the reaction of radiation field (and of C-meson field). The C-meson hypothesis gives rise to a correction which is nearly equal to that obtained by the self-consistent subtraction method for the electron-photon system, as is shown in § 2. Therefore, we arrive at the conclusion that *C-meson field shows hardly any detectable effect with respect to the correction for elastic scattering cross section of an electron by a fixed center of force, at least in the case in which the electron has velocity small compared to that of light.*

The above result indicates that the total amount of C-mesonic correction, i.e., the sum of the reactive correction relating only to the C-meson field and the correction due to the mass change by surrounding C-meson field, is very small compared with the correction obtained by the self-consistent subtraction method for the electron-photon system. The reason will be clarified in the following.

As is seen from integrals (3.24)~(3.26) of paper I, the integrands of corrections A and B due to the C -meson field have denominators of almost equal magnitude because of $\sqrt{m^2 + p^2} \ll \sqrt{\mu^2 + k^2}$ for every momentum k of virtual photon, and thorough calculations of their numerators show that they give contributions essentially of the same magnitude and of opposite signs. Therefore the corrections A and B almost cancel each other with only terms of order $10^{-4} f^2/\pi \cdot (p/m)^2$ remaining. The correction C_{ii} is expected to be exactly cancelled by the correction due to the mass change from the covariant theory of Tomonaga and Schwinger. In fact, in the perturbation calculation they do not cancel themselves exactly and there remains a finite term of order $10^{-2} f^2/\pi \cdot (p/m)^2$, which is practically compensated by the correction C_i . Thus the net correction due to the C -meson field only does not exceed $10^{-3} f^2/\pi (p/m)^2$, which is below the error committed in the approximate computation of correction obtained by self-consistent subtraction method. The main cause of extreme smallness of this correction is attributed to the large mass ratio $\mu/m \sim 100$, i.e., the smallness of the C -mesonic force range in comparison with the Compton wave-length of an electron.

The correction due to the effect of vacuum polarization, which was divergent in the radiative correction, vanishes exactly in the case of C -meson field. It is not yet obvious whether this is an event of general character due to the property of the C -meson field or merely an accident.

According to the analysis of Pauli and Fierz, in which the scattering of a charged particle of finite size by a fixed center of force is treated non-relativistically, the relative correction for the elastic scattering cross section becomes

$$\frac{\delta\sigma}{\sigma_0} = (p^2/2m\omega_1)^0 - 1, \quad (20)$$

where

$$C = \frac{4}{3} \frac{e^2}{\pi} \left(\frac{p}{m} \right)^2 (1 - \cos \theta), \quad (21)$$

and ω_1 is the cut-off frequency. The correction in the e^2 -approximation can be obtained expanding the above expression in power of C ,

$$\left(\frac{\delta\sigma}{\sigma_0} \right)_{e^2} = C \log \frac{p^2}{2m\omega_1} = \frac{4e^2}{3\pi} \left(\frac{p}{m} \right)^2 (1 - \cos \theta) \log \frac{p^2}{2m\omega_1}. \quad (22)$$

This obviously diverges logarithmically when the cut-off frequency ω_1 tends

to infinity, i.e., when the size of the particle reduces to a point. We have seen, however, that the relativistic treatment of this process with suitable subtraction prescription does converge. Thus the contributions of the photons in relativistic energy region could safely be neglected. Once this is admitted we might expect the same result again from the non-relativistic formula (22) if we only give an appropriate value, e.g. m , to the cut-off frequency ω_1 .

This is the very procedure that was employed by Bethe when he calculated the level shift of the hydrogen atom preliminarily in a non-relativistic approximation.⁽¹⁰⁾ In the present case, too, Bethe's method proves correct. In fact one finds, by equating (19) and (1), $\omega_1 = 1.19 m$, which seems a quite reasonable value.

In the above calculation we have assumed that the scattering potential is the fourth component of a four-vector. Now the fundamental idea of the cohesive force theory is that the electromagnetic field is always accompanied by the C-meson field, so that it will be more consistent to introduce a scattering potential of the world scalar type besides the fourth component one. In the same way as the latter can, in usual, be considered a fixed Coulombian potential, a static solution of the Maxwell's equation, the former may be regarded as a static solution of the wave equation of the C-meson field, $e^{-\mu r}/r$ with a very short force range since $\mu \sim 100 m$. In the paper I, the whole problem was treated according to this general view point. But in our present approximation in which $p/E \ll 1$, this circumstance does not affect the above result at all, correction to be added to (1) on this account amounting at most to 0.01 %. This is a result to be expected, since such a slow electron is usually deflected by the long range Coulombian force before it can reach the immediate neighborhood of the scattering center where the short range force is effective. On the other hand, if we treat the vacuum polarization effect of the C-meson field in this general case

$$\frac{f^2 [2mEV + (E^2 - (\mathbf{p}\mathbf{q}) + m^2)W] W}{\pi E^2 (\mu^2 + (\mathbf{p} - \mathbf{q})^2) S} \left[4m^2 \int_0^\infty \frac{l' dl'}{(1+l'^2)^{3/2}} - (\mathbf{p} - \mathbf{q})^2 \int_0^\infty \frac{l^2 dl}{(1+l^2)^{3/2}} \right],$$

where

$$E = (m^2 + p^2)^{1/2}, \quad S = [(E^2 + (\mathbf{p}\mathbf{q}) + m^2)l^2 + 2mEVW + (E^2 - (\mathbf{p}\mathbf{q}) + m^2)W^2] / 2E^2$$

and V and W are respectively the matrix element of the fourth component and world scalar potential, according to the second alternative method of

subtraction mentioned in §2, C_{III} , only the logarithmically diverging part of the first term in the square bracket is subtracted, one quadratic and one logarithmic divergence still remaining. Thus, when the scattering potential transforms as a world scalar, the two manners of subtracting the effect of vacuum polarization give quite different results. This is the case also when one is concerned with space-component of electromagnetic potential, for instance, in the case of magnetic scattering. This difficulty could not be overcome in the frame of the usual perturbation method and we shall not enter into it here. The readers are referred to the recent work of Schwinger* and others.

Concluding remarks. We have shown that the reactive correction for electron scattering cannot serve as means to find out the effect of C-meson field on account of the large mass ratio. However, when $\mu/m \sim 1$, this gives rise to a correction comparable to the radiative correction. This is expected in the case of reactive correction for elastic scattering of μ -meson by a nucleus if it is really a Fermi particle with spin $1/2$ as is now becoming plausible. In this case the above calculations stand as they are, and yet the mass of C-meson (100 electron mass) is by no means largely different from that of μ -meson (217 electron mass). Therefore, the C-meson theory gives considerably different result from the mere subtraction, and it might be permitted for us to dream of the possibility of finding some evidence about the existence of C-meson field in future experiments of meson scattering.

Acknowledgement. The authors express their sincere gratitude to Prof. S. Tomonaga for having suggested them this problem and for having helped and encouraged them throughout the work. They are also very much obliged to Mr. Y. Fujimoto, Mr. H. Fukuda, Mr. Y. Miyamoto, and Mr. G. Takeda for their discussions and advices.

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* See the footnote on page 222.

Letters to the Editor.

On the Nucleon Component in Cosmic Rays.

Y. Fujimoto and Y. Yamaguchi
Physical Institute, Tokyo University.

Mar. 28, 1949

Stars in emulsion and bursts in thin-walled ionization chambers show the same altitude dependence, and both may be considered due to the nuclear events caused by moderate energy (from several hundred Mev) nucleons⁽¹⁾. Here we call these nucleons "B-nucleons." As is shown by Wilson chamber pictures, most of B-nucleons are neutrons. We have estimated the intensity of B-neutrons assuming the following simple model. i) High energy (more than several hundred Mev) nucleons—we call them "A nucleons"⁽²⁾—produce B-neutrons in penetrating showers. ii) Angular divergence of a penetrating shower may be neglected. iii) Absorption coefficients of A-nucleons and B-neutrons are the same and is equal to $1/l \sim 1/125$ g cm⁻² in air. iv) The cross section for producing a penetrating shower is the geometrical cross section of an air nucleus.

Taking the A-nucleon intensity at x g cm⁻² as

$$A(x) = a \exp(-x/l)$$

and then the B-neutron intensity is given by

$$B(x) = 2am(x/l) \exp(-x/l)$$

where m means an average number of B-neutrons emitted in one penetrating shower. The factor 2 is due to the fact that an A-nucleon makes two penetrating showers on the average in its mean path l .

To compare with experimental data of stars and bursts, we must calculate the integral intensity as follows.

$$\int B(x/\cos \theta) d\Omega = 4\pi am(x/l) [-E_i(-x/l)] \quad (1)$$

The experimental data⁽³⁾ and curve (1) are shown in Fig. 1, and we obtain con-

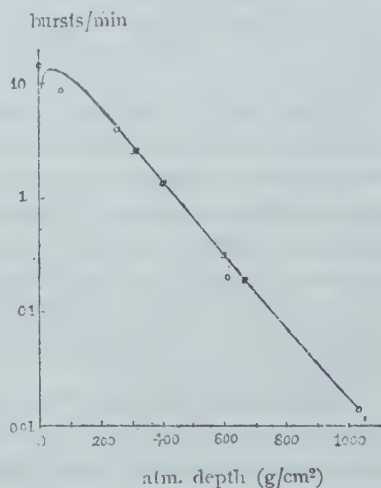


Fig. 1

siderable agreement. For $x \geq 200$ g cm⁻² (1) is $\sim \text{const.} \exp(-x/132)$.

Comparing the absolute value of $A(x)$ with that of $B(x)$, both being obtained experimentally, the undetermined factor m is ~ 0.8 . If we assume that nearly the same number of B-protons are also produced in a penetrating shower, then the mean number of emitted B-nucleons is ~ 1.6 in one penetrating shower.

Now we can estimate the intensity of B-protons. If we take the mean energy of B-protons as ~ 200 Mev, their mean range is ~ 20 g cm⁻², which is shorter than l . Treating B-protons as in equilibrium with A-nucleons, we see the inten-

sity of B-protons are roughly one third of that of A-nucleons. So only 6% of B-nucleons are consisted of protons. This agrees approximately with Rossi's estimation⁽⁴⁾.

More quantitative studies are now in progress. We wish to thank Mr. Hayakawa for his kind interest taken in this work.

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- (2) We do not distinguish protons and neutrons in A-nucleons, since the ionization loss is not important for such high energies.
- (3) B. Rossi: *Rev. Mod. Phys.* **20** (1948) 537.

Note on the Deuteron Problem of the Mixed Meson Theory of Pseudovector and Pseudoscalar Fields.

S. Hiroishi and H. Tanaka.

Institute of Physics, Kyoto University.

April 7, 1949

Recently, Araki⁽¹⁾ proposed the mixed meson theory of pseudovector and pseudoscalar field, which gives the finite magnetic moment of the nucleon and seems to agree with the other facts, i.e. the sign of quadrupole moment of the deuteron, nuclear forces, β -decay, etc.

We calculated exactly the deuteron problem on this line with the value of the coupling constant $\frac{f^2}{\hbar c} \frac{M}{\mu} = 0.5$, where M , μ , are the mass of the nucleon and pseudoscalar meson, respectively, and the ratio of the pseudovector and pseudo-

scalar meson mass is assumed to be 4.72, which comes from the above value of the coupling constant and the experimental value of the magnetic moment of the proton.

But the obtained results are, unfortunately, negative. First, the scattering cross section of slow neutrons by proton is too small (order of 10^{-27} cm²) to account for the experimental value (order of 10^{-24} cm²).

Secondly, the binding energy of the deuteron is not positive.

To find the reason of this discrepancy, let us compare the Araki's potential with the Schwinger's⁽²⁾ and Bethe's⁽³⁾ one.

Araki's potential:

$$V_A = \frac{(\tau^{(1)}\tau^{(2)})}{2} \{(\sigma^{(1)}\sigma^{(2)})J_A + S_{12}K_A\}$$

Schwinger's Potential:

$$V_S = (\tau^{(1)}\tau^{(2)}) \{(\sigma^{(1)}\sigma^{(2)})J_S + S_{12}K_S\}$$

Bethe's potential:

$$V_B = \{(\sigma^{(1)}\sigma^{(2)})J_B - S_{12}K_B\}$$

where

$$S_{12} = \frac{3(\sigma^{(1)}r)(\sigma^{(2)}r)}{r^2} - (\sigma^{(1)}\sigma^{(2)})$$

and

$$J_A = \frac{a_A}{3} \left(\frac{\exp(-x)}{x} - \delta_A^2 \frac{\exp(-\delta_A x)}{x} \right)$$

$$J_S = \frac{a_S}{3} \left(\frac{\exp(-x)}{x} + 2\delta_S^2 \frac{\exp(-\delta_S x)}{x} \right)$$

$$J_B = -\frac{2a_B}{3} \frac{\exp(-x)}{x}$$

$$K_A = a_A \left\{ \frac{\exp(-x)}{x} \left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{3} \right) \right.$$

$$\left. - \delta_A^2 \frac{\exp(-\delta_A x)}{x} \left(\frac{1}{\delta_A^2 x^2} + \frac{1}{\delta_A x} + \frac{1}{3} \right) \right\}$$

$$K_S = a_S \left\{ \frac{\exp(-x)}{x} \left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{3} \right) \right.$$

$$\left. - \delta_S^2 \frac{\exp(-\delta_S x)}{x} \left(\frac{1}{\delta_S^2 x^2} + \frac{1}{\delta_S x} + \frac{1}{3} \right) \right\}$$

$$K_R = a_R \left\{ \frac{\exp(-x)}{x} \left(\frac{1}{x^2} + \frac{1}{x} + \frac{1}{3} \right) \right\}$$

a is $\frac{M}{\mu} \frac{f^2}{\hbar c}$ in each cases: $a_A \approx a_S \approx 0.5$, $a_R \approx 0.8$ and $\delta_A = 4.72$, $\delta_S = 1.6$. In V_A , J_A is negative and $|J_A| > |K_A|$ near the origin. Because of this repulsive character of Araki's potential for the small distance, the scattering cross section becomes very small and the deuteron is not bound. In V_S , J_S is positive and much more effective than K_S , so if the binding energy is coincide with the experimental value, K_S becomes not so effective that the quadrupole moment can not be explained.⁽¹⁾

In contrast to the above two cases, K_R is more effective than J_R in V_R . This great effectiveness of the tensor forces, in Bethe's case, gives the quadrupole moment enough to explain the experimental results.

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Phenomenological Treatment on the Production of Cosmic-Ray Mesons.

S. Hayakawa and J. Nishimura.
Central Meteorological Observatory and
Scientific Research Institute.

Apr. 13, 1949

The analysis of cosmic-ray intensity in the atmosphere gives ample informations about the production of mesons for which any satisfactory theory is yet unknown⁽¹⁾.

So, we attempt to get forth phenomenologically to determine the mode of meson production, which is represented by a set of parameters, on the basis of experimental facts.

For this purpose it is of importance to choose the primary spectrum more carefully. Referring the rocket measurements at $40^\circ N$ ⁽²⁾ and the energy flow obtained from the balloon experiments⁽³⁾, we assume the differential energy spectrum of primaries as

$$p(\epsilon) d\epsilon = [(17.6/\epsilon^3) - (59/\epsilon^{3.8})] d\epsilon, \quad (1)$$

where the energy of primary ϵ is measured in Bev. At usual the index of ϵ will be denoted by $\gamma+1$.

Secondly, we consider the decrement of primaries through the atmosphere. Denoting the probability that the primary proton with energy ϵ loses the energy between ϵ' and $\epsilon'+d\epsilon'$ by a collision with an air nucleus by $\Phi(\epsilon'/\epsilon) d\epsilon'/\epsilon$. If we take the collision mean free path, 65 g cm^{-2} , as the unit of length, the diffusion equation for primaries is given by

$$\frac{\partial p(\epsilon, x)}{\partial x} = -p(\epsilon, x) \int_0^1 \Phi(v) dv + \int_0^1 p\left(\frac{\epsilon}{1-v}, x\right) \Phi(v) \frac{dv}{1-v}, \quad (2)$$

neglecting the ionization loss of the protons. This equation is solved by means of Mellin transformation and the result is

$$p(\epsilon, x) = (1/2\pi i) \int \{1/(\gamma-s)\} (\epsilon_s^{s-\gamma}/\epsilon^{s+1}) \times \exp\{-A(s)x\} ds, \quad (3)$$

$$A(s) = \int_0^1 [1 - (1-v)^s] \Phi(v) dv = [1 - (1-v_0)^s]; \quad (4)$$

the last expression being obtained by assuming $\Phi(v) = \delta(v-v_0)$. ϵ_0 means the magnetic cut off energy. For $\epsilon > \epsilon_0$, (3) is represented by the residue at $s=\gamma$, so

we have

$$p(\epsilon, x) = \text{const. } \epsilon^{-\gamma-1} \exp\{-A(\gamma)x\}. \quad (5)$$

(5) shows that $A(\gamma)$ is to be identified with the inverse of the absorption coefficient for primaries (measured in our unit). Thus $A(\gamma) = 65/125$, which results in $v_0 = 0.31$ for $\gamma = 2$.

Thirdly, the mode of meson production is treated on following assumptions. (a) A primary proton with energy ϵ produces pi-mesons with energy between E and $E+dE$ by the probability

$$\Psi(\epsilon, E)dE = C\epsilon^\alpha E^{\beta-1}dE, \quad E \leq v_0\epsilon^\eta, \quad (6)$$

(b) The spectrum of the produced pi-mesons is the same as that of primaries. (c) The number of mesons produced at a collision is Be^δ . (d) The energy lost by a primary at a collision, $v_0\epsilon$, is transported into charged mesons by a factor H . (e) The ratio of the intensities of primaries and mu-mesons at sea level is 70 in the high energy region. From these assumptions we can determine the value of various parameters introduced above:

$$\delta = 1/3, \alpha = 1/3, \beta = -1/3, \gamma = 1, \\ H = 2/3, C = 1/3, B = 1. \quad (7)$$

Fourthly, we calculate the intensity of mu-mesons assuming that the pi-mesons disintegrate immediately into mu-mesons. The energy spectrum of the hard component at sea level is calculated in the approximation as was used by Heitler and Walsh⁽⁴⁾. The result obtained gives the higher intensity than the experimental one in the low energy region, while in the high energy region a fairly good agreement is found. This seems to indicate that Heitler's approximation is too rough and may be the reason why Janossy got the too large index of the power in the spectrum. We can verify that this fault

can be mended by the improvement of the calculation. The still remaining discrepancy may be removed by taking into consideration the ionization loss protons in low energy region.

Consideration must be extended to include also the soft component at the upper atmosphere, but this problem is left to the future occasion, and here we remark only the fact that the bulk of the soft component may be the disintegration products of neutral mesons which are believed to decay into photons with considerable short life.

Detailed account will be published elsewhere.

- (1) The rather satisfactory treatments are H. Tamaki: *Rinken Tho* **21** (1942), 891; G. F. Chew: *Phys. Rev.* **73** (1948), 1128.
- (2) A. V. Gengnes, J. F. Jenkins and J. A. Van Allen: *Phys. Rev.* **75** (1949), 57.
- (3) R. S. Millikan, E. V. Neher and W. H. Pickering: *Phys. Rev.* **63** (1943), 234.
- (4) W. Heitler and P. Walsh: *Rev. Mod. Phys.* **17** (1945), 252.
- (5) L. Janossy: *Nature* **158** (1946), 450.

Cosmic-Ray Bursts under Thick Shields.

Y. Fujimoto, S. Hayakawa,
and Y. Yamaguchi

*Physics Department, Tokyo University
and Research Institute, Central
Meteorological Observatory*

Apr. 13, 1949

The size-frequency relation of cosmic-ray bursts has been analyzed in detail by Christy and Kusaka⁽¹⁾, and it was concluded that the bursts were caused by mesons not of spin 1. This remarkable result seems, however, to meet with a

difficulty, if one will account for the variation of the frequency with altitude; the frequency of bursts at 3000 m amounts about 4~6 times larger than that at sea level^(2,3). The intensity of energetic mu-mesons, which are supposed to be the main agents of bursts at sea level, can not increase so strongly with altitude, because at such high energy the ionization loss and the disintegration of the mu-mesons are negligible. This suggests that some other components must produce most part of bursts at higher altitude.

As for above unexplained altitude dependence, Bethe⁽⁴⁾ suggested that the high-density cores of showers would be the main agents of them. But later experiment⁽⁵⁾ showed that the coincidences of bursts with air showers amount at most 5%.

Here we remark the possible contribution of nucleons and pi-mesons. From the analysis of cosmic-ray intensities at the upper atmosphere⁽⁵⁾ and underground⁽⁶⁾, we can estimate their relative intensities and energy distributions at 3000 m as follows.

$$\left. \begin{aligned} \text{mu-mesons: } & B/(B+E) \cdot dE/E^{\tau+1}, \\ \text{pi-mesons: } & 0.013 \times E/(B+E) \\ & \times dE/E^{\tau+1}, \\ \text{nucleons: } & 0.069 \times dE/E^{\tau+1}, \end{aligned} \right\} (1)$$

where B means the decay constant of pi-meson, 3.4×10^{11} ev. On the other hand the cross-section for the production of electronic component with energy between ϵ and $\epsilon+d\epsilon$ in lead are as follows.

(a) bremsstrahlung of mu-meson (spin 1/2):

$$3.3 \times 10^{-28} \ln(E/\mu c^2) d\epsilon/\epsilon \text{ cm}^2, \quad (2)$$

(b) bremsstrahlung of pi-meson (spin 1):

$$1.8 \times 10^{-27} d\epsilon \text{ cm}^2, \quad (3)$$

(c) charge acceleration of nucleon⁽⁷⁾:

$$k \times 6.5 \times 10^{-26} \ln(E/\text{Me}^2) d\epsilon/\epsilon \text{ cm}^2, \quad (4)$$

Other processes are known to be so small that we need not consider them here.

From the above considerations we get the following conclusions: (1) Even if we assume the spin of pi-meson to be 1, the contribution of pi-mesons is much smaller than that of nucleons. The previous note of one of us (S. H.) is erroneous because there the angular distribution of mu-mesons was not properly considered. (2) At 3000 m, bursts caused by the charge acceleration of nucleons are 7 times or more frequent than that caused by the radiation of mu-mesons, if k is assumed to be nearly equal to 1. The former process gives a main contribution to the frequency of large bursts at higher altitude since the intensity of mu-mesons varies little with altitude.

We have further evidences which support the above interpretation by referring a recent experiment⁽⁸⁾. Firstly, the absorption coefficient of burst-producing rays, $1/434 \text{ g cm}^{-2}$, is in accordance with our view, because the absorption coefficient of nucleons in lead may be about $1/300$, corresponding to a half of the geometrical cross-section as was confirmed by the experiments about stars⁽⁹⁾. Secondly, we can explain the observed sharp angular distribution, since the nucleon component falls on more vertically than mu-mesons.

Detailed account will be soon published elsewhere.

(1) R. F. Christy and S. Kusaka: *Phys. Rev.* **59** (1941), 414.

(2) R. E. Lapp: *Phys. Rev.* **69** (1946), 321.

(3) E. F. Fahy and M. Schein: *Phys. Rev.* **75** (1949), 207.

(4) H. A. Bethe: *Phys. Rev.* **72** (1947), 172.

- (5) S. Hayakawa and J. Nishimura: to be published in this issue.
 (6) S. Hayakawa and S. Tomonaga: to be published.
 (7) S. Hayakawa: to be published.
 (8) S. Hayakawa: Prog. Theor. Phys. **3** (1948), 458.
 (9) c.f. G. Bernardini, G. Cortini and A. Manfredini: Phys. Rev. **74** (1948), 845.

On the γ -Decay of Neutral Meson.

H. Fukuda and Y. Miyamoto.

Physics Institutes, Tokyo University.

April 28, 1949

As was first suggested by Sakata and Tanikawa⁽¹⁾ the neutral meson can disintegrate into several number of photons through creation and subsequent reabsorption of virtual proton-antiproton pairs. Tanikawa⁽²⁾ and Finkelstein⁽³⁾ treated this problem by perturbation theory, and obtained logarithmically diverging results, but we found that the results converge in contradiction with these authors. This is, as we think, due to some overlook in their calculations. On closer investigation, however, it is found that the life time of the neutral meson is indeterminate, even if it converges, and assumes various values, depending on the way of evaluation. Such a circumstance is often encountered in the treatment of such quantities as photon self-energy and is connected with the difficulty which we meet with when we will draw some conclusion from the improper expression $\infty - \infty$. In order to treat our problem correctly, it is necessary to use the covariant formulation of the theory of wave fields. Thus it is worth while fully to investigate our problem in the light of Tomonaga-Schwinger theory.

In Tomonaga-Schwinger theory, the matrix elements for the two quantum

disintegration of the neutral meson are given as follows:
 for scalar meson

$$fU(\varphi^+\varphi)_2 \quad (1)$$

and for pseudoscalar meson with pseudoscalar meson with pseudoscalar and pseudovector couplings

$$ifU(\varphi^+\gamma_5\varphi)_2 \quad (2)$$

and

$$\frac{if}{\mu} \sum_{(l,ijk)} (\varphi^+\gamma_{ijk}\varphi)_2 \partial_l U \quad (3)$$

where U is the potential for meson field, φ the spinor for protonic field, f the coupling constant, and μ the reciprocal Compton wave length of the meson. $\sum_{(l,ijk)}$ means the sum over $(l,ijk) = (1,234), (2,134), (3,124)$ and $(4,123)$. $(\varphi^+L\varphi)_2$ is the vacuum expectation value with respect to the protonic field, and explicitly it is

$$\left(\frac{e}{hc}\right)^2 \int dx' \int dx'' \langle [\varphi''(\gamma A'')\varphi', [\varphi^+(\gamma A')\varphi', \varphi^+L\varphi]] \rangle_0 \quad (4)$$

where A_i the potential for electromagnetic field. $(\varphi^+L\varphi)_2$ has the following two important properties: (I) $(\varphi^+L\varphi)_2$ is gauge-invariant, that is, invariant under transformation $A_\mu \rightarrow A_\mu + \partial_\mu A$ ($\square A = 0$) and (II) $\sum_{(l,ijk)} \partial_l (\varphi^+\gamma_{ijk}\varphi)_2 = -2\kappa(\varphi^+\gamma_5\varphi)_2$, κ being the reciprocal Compton wavelength of the proton.

$(\varphi^+L\varphi)_2$ can be found by using the same method as that used by Schwinger⁽⁴⁾ in his calculation of the intrinsic moment of the electron. They are

$$(\varphi^+\varphi)_2 = \kappa \left(\frac{e}{hc}\right)^2 \frac{1}{8\pi^2} (A^2 + G_1(\square) \sum_{ij} F_{ij}^2) \quad (5)$$

$$(\varphi^+\gamma_5\varphi)_2 = \frac{1}{\kappa} \left(\frac{e}{hc}\right)^2 \frac{1}{8\pi^2} (1 + G_2(\square))$$

meson	coupling	first term retained	first term discarded
scalar	scalar	1.1×10^{-18} (1.7×10^{-18})	7×10^{-14} (2.1×10^{-14})
pseudoscalar	pseudoscalar	1×10^{-16} (0.6×10^{-16})	1×10^{-11} (1.3×10^{-11})
pseudoscalar	pseudovector	5.4×10^{-16} (5.6×10^{-16})	1×10^{-10} (1.3×10^{-11})
pseudovector	pseudovector	∞	∞

$f_s^2/\hbar c = f_{\pi^0}^2/\hbar c = 0.05$, $f_{\pi^+}^2 = (2\kappa/\mu)^2 0.05$, $\mu = 200 m_\pi (3.0 m_\pi)$

$$\times \frac{\square}{4}) (F_{12} F_{34} + F_{21} F_{31} + F_{14} F_{23}) \quad (6)$$

and

$$(\varphi^+ \gamma_{ijk} \varphi)_2 = \left(\frac{e}{\hbar c} \right)^2 \frac{1}{8\pi^2} \left(-\frac{2}{3} \sum_{\text{cyclic } ijk} A_i F_{jk} \right. \\ \left. + G_2(\square) \sum_{s=1}^3 F_{s2} \partial_s F_{jk} \right) \quad (7)$$

where G_1 and G_2 are function of \square , which, expanded in powers of \square/x^2 , are

$$G_1(\square) = \frac{-1}{3\pi^2} + \dots, \quad G_2(\square) = \frac{1}{3\pi^2} + \dots \quad (8)$$

(5) and (7) are not gauge covariant in contradiction with (I), since A_i appears explicitly. This circumstance is very similar to the discussions by Schwinger⁽⁴⁾ and Wentzel⁽⁵⁾ on the photon self energy. Further, between the first terms in (6) and (7), the identity (II) does not hold, whereas it does so between the terms with G_2 . Evidently these inconsistent results arise from the mathematical difficulty of obtaining the definite expression using the singular delta function of Jordan and Pauli. At present we know no appropriate prescription which makes one free from the ambiguity of this kind. If such prescription were found, and it were proved that (I) and (II) hold simultaneously, the first terms in (5), (6) and (7) would vanish. At present state of the theory of wave field, in which the photon self energy problem is still unsettled, we cannot give the unique answer for the γ -decay of the neutral meson. Thus we shall give below two values of the life time for the neutral meson, the one ob-

tained when we retain the first term in (5), (6) and (7) and the other obtained by discarding these terms.

The evaluation of three quanta disintegration is now being made. Detailed account will be published in the Progress of Theoretical Physics.

We wish to express our cordial thanks to Prof. Tomonaga for the encouragement in this work.

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- (2) Y. Tanikawa, Proc. Phys. Math. So. **24** (1940) 610.
- (3) R. J. Finkelstein, Phys. Rev. **72** (1947), 414.
- (4) J. Schwinger: his unpublished manuscript and Phys. Rev. **75** (1949), 651.
- (5) G. Wentzel, Phys. Rev. **74** (1948), 1070.

Note on the Interaction Representation in case of Meson Field interacting with Electromagnetic Field.

S. Kanesawa.

Institute of Physics,
Tokyo Bunrika Daigaku

Apr. 29, 1949

In the ordinary meson theory, the tensor equation of meson is used, but in this note the following Duffin-Kemmer equation⁽¹⁾ is taken instead of the tensor

one in order to give a unified formulation of the interaction representation in case of mesonic field interacting with electromagnetic field:

$$\left(\beta_\lambda \frac{\partial}{\partial x_\lambda} + \kappa\right)u = 0, \quad (\lambda=1, 2, 3, 4) \quad (1)$$

where κ is connected to the rest mass m of the meson by the relation $\kappa = \frac{mc}{\hbar}$, and β 's satisfy the commutation relation

$$\beta_\lambda \beta_\mu \beta_\nu + \beta_\nu \beta_\mu \beta_\lambda = \beta_\lambda \delta_{\mu\nu} + \beta_\nu \delta_{\mu\lambda}. \quad (2)$$

As is well known, there exist two inequivalent irreducible representations of β -algebra⁽²⁾ except a trivial one, the one is given by five-row and five-column matrix, the other ten-row and ten-column one. The former representation leads to the scalar (pseudoscalar) meson, the latter the vector (pseudovector) meson. This is the very fact which enables the formalism described in this note to be possible.

Now the interaction representation of the system considered here is given by

$$\left\{L_F[C] - \frac{\hbar}{i} \frac{\delta}{\delta C_P}\right\} \Psi[C] = 0 \quad (I)$$

with

$$\begin{aligned} L_F[C] = & \frac{1}{4\pi} \left[\left(\frac{e}{\hbar c} \right) A_\lambda u^\dagger \beta_\lambda u \right. \\ & + \frac{i}{\kappa} \left(\frac{e}{\hbar c} \right)^2 A_\lambda A_\mu u^\dagger \{ \beta_\lambda \beta_\mu \\ & \left. + \beta_\lambda (\beta_\nu N_\nu)^2 \beta_\mu \} u \right], \end{aligned} \quad (I')$$

where u^\dagger is defined by the equation

$$\begin{cases} u^\dagger = iu^* \eta_4 \\ \eta_4^2 = 2\beta_4^2 - 1. \end{cases} \quad (3)$$

The field variables contained in (I') are the solutions of the free field equations and satisfy the following four dimensional commutation relations;

$$\begin{cases} [A_\lambda(x), A_\mu(x')] = -4\pi i \hbar c \delta_{\lambda\mu} D_0(x-x') \\ [u^l(x), u_m^\dagger(x')] = -4\pi \hbar c \left\{ \beta_\lambda \frac{\partial}{\partial x_\lambda} \right. \\ \quad \left. - \frac{1}{\kappa} \beta_\lambda \beta_\mu \frac{\partial^2}{\partial x_\lambda \partial x_\mu} \right\}_{lm} D_\kappa(x-x') \\ [u_l(x), u_m(x')] = [u_l^\dagger(x), u_m^\dagger(x')] = 0, \end{cases} \quad (II)$$

where the suffices l and m of the mesonic field variables u and u^\dagger run respectively from one to five and one to ten according to the scalar (pseudoscalar) or the vector (pseudovector) mesons. The auxiliary condition in this case has the form

$$\begin{cases} \Xi_l[C] \Psi[C] = 0 \\ \Xi_l[C] = \frac{\partial A_\lambda}{\partial x_\lambda} + \left(\frac{e}{\hbar c} \right) \int_0 u^\dagger(P') \beta_\lambda u \\ \quad \times (P') N_\lambda(P') D_\kappa(P'-x) dF_{P'} \end{cases} \quad (III)$$

That the equation (I) satisfies the intergrability condition and the auxiliary condition (III) is compatible, is examined directly on account of the relation (2) and (II). Moreover the formulation of Kanesawa and Tomonaga⁽³⁾ which treated the interaction representation in case of the scalar and vector meson fields coexisting with the electromagnetic field separately by means of the tensorial type equations of mesons, is derived from the formalism stated above by the following substitution:

$$\begin{aligned} (u_l) = & \left\{ \frac{1}{\sqrt{\kappa}} \left(\frac{\partial \phi}{\partial x_0}, -\frac{\partial \phi}{\partial x_1}, -\frac{\partial \phi}{\partial x_2}, \right. \right. \\ & \left. \left. -\frac{\partial \phi}{\partial x_3} \right) \sqrt{\kappa} \phi \right\} \text{(for scalar meson)} \end{aligned} \quad (4)$$

$$\begin{aligned} (\bar{u}^l) = & \left\{ -\frac{i}{\kappa} (\chi_{14}, \chi_{24}, \chi_{34}), \right. \\ & \left. -\frac{1}{\sqrt{\kappa}} (\chi_{23}, \chi_{31}, \chi_{12}), \sqrt{\kappa} (\phi_1, \phi_2, \phi_3, \phi_4) \right\} \\ & \text{(for vector meson)} \end{aligned} \quad (4')$$

Detailed accounts of this and related problems will appear in a later issue of this journal.

- (1) R. J. Duffin: Phys. Rev. **54** (1938) 1114.
N. Kemmer: Proc. Roy. Soc. **173** (1939) 91.
(2) N. Kemmer: *ibid.*
(3) S. Kanesawa and Tomonaga: Prog. Theor. Phys. **3** (1948) 1, 101.

On the Meson Pair Theory of Nuclear Forces in the Interaction Representation.

S. Kanesawa.

*Institute of Physics,
Tokyo Bunrika Daigaku*

Apr. 29 1949

In a foregoing letter⁽¹⁾ in this issue of the Progress of Theoretical Physics the author described the interaction representation in case of mesonic field interacting with electromagnetic field by assuming the Duffin-Kemmer equation of meson. Also in the present letter the first order wave equation is used as the free mesonic field equation, and investigation of the interaction between neutron and proton will be made by means of the symmetrical meson pair theory in the interaction representation.

One of the possible meso-nucleonic interaction Lagrange densities which are Lorentz-invariant is given by

$$L = ig(Wu^\dagger v + W^\dagger v^\dagger u) \quad (1)$$

with

$$\begin{cases} W = -i\phi^\dagger \tau_{np} \psi \\ W^\dagger = -i\psi^\dagger \tau_{pn} \phi \end{cases} \quad (2)$$

where v means the wave function of neutral meson and v is connected to the transposed \tilde{v} of v by the relation

$$v^\dagger = \tilde{v} \gamma_4 \quad (3)$$

Other invariant interaction Lagrangian beside (1) might not be considered here, because they give the infinitely large interactions between nucleons.

Now it is clear that the system given by (1) does not allow the ordinary canonical description on account of the fact that the equation (1) becomes to contain the bilinear terms of the derivatives of mesonic field variables as will be shown by the following equation (4) if it be written in the usual tensor notation by making use of the relations (4) and (4') in A:

$$L = \begin{cases} -g \left\{ \sqrt{x x_0} \phi^* \dot{\phi} - \frac{1}{\sqrt{x x_0}} \frac{\partial \phi^*}{\partial x_\lambda} \frac{\partial \dot{\phi}}{\partial x_\lambda} \right\} \\ \quad \times W + \text{adjoint (for scalar} \\ \quad \text{or pseudoscalar meson)} \\ -g \left\{ \sqrt{x x_0} \phi_\lambda^* \dot{\phi}_\lambda - \frac{1}{\sqrt{x x_0}} \frac{1}{2} \chi_{\lambda\mu}^* \dot{\chi}_{\lambda\mu} \right\} \\ \quad \times W + \text{adjoint, (for vector} \\ \quad \text{or pseudovector meson)} \end{cases} \quad (4)$$

where $\dot{\phi}$ and $\dot{\phi}_\lambda$ mean respectively the field variables of neutral scalar (pseudoscalar) and vector (pseudovector) mesons, and x_0 is defined by $x_0 = \frac{m_0 c}{\hbar}$, m_0 being the rest mass of the neutral meson. According to the prescription given by the author the system corresponding to (I) is described by the equation

$$\begin{aligned} & \left\{ L[C] - \frac{\hbar \delta}{i \delta C_p} \right\} \Psi[C] = 0 \\ L_\eta[C] = & ig \left[(Wu^\dagger v + W^\dagger v^\dagger u) - \right. \\ & - 4\pi g N_\lambda N_\mu \left\{ \frac{1}{2x_0} (W^2 u^\dagger \beta_\lambda \beta_\mu u^* \right. \\ & + W^{\dagger 2} (u^*)^\dagger \beta_\lambda \beta_\mu u) + W^\dagger W \left(\frac{1}{x_0} u^\dagger \beta_\lambda \beta_\mu u \right. \\ & \left. \left. + \frac{1}{x} v^\dagger \beta_\lambda \beta_\mu v \right) \right\} + \dots \left. \right] \quad (1) \end{aligned}$$

in which all of the field variables are the solutions of the free field equations and satisfy the four dimensional commutation relations and u^* is defined by $(u^*)_i = u_i^*$.

In order to calculate the neutron-proton interaction, one introduces the unitary transformation by which the first order term in g is eliminated from the equation (I), and performs the rearrangement of the g^2 -terms in the transformed equation so as to make them "well order," then an appropriate part of Møller terms gives the neutron-proton interaction. It is easily seen that both of the second term in (4) or (4') give diverging interaction between nucleons even if the distance r or neutron and proton be finite or infinitely large. Therefore one should take only the first term in (4) or (4') as the meson-nucleonic interaction Lagrange density. Finally one gets the same finite neutron-proton interactions as for the all four types of mesonic fields except the constant factors 1 (for scalar or pseudoscalar meson) and 3 (for vector or pseudovector meson) although they have $1/r^3$ singularity at the origin. Moreover one knows, as is expected, that they tend to zero rapidly when the distance r of neutron and proton becomes infinitely large.

Detailed accounts will be given in a later issue of this journal.

- (1) S. Kanesawa: Prog. Theor. Phys. (1948).
One cites this paper as A.
- (2) S. Kanesawa and Z. Koba: Prog. Theor. Phys. in press.

On the Multiple Production of Mesons.

K. Sawada and S. Takagi
Dept. of Physics, Kyoto University.
May 4, 1949

It has been well established that the π -mesons interact strongly with nucleons and the nuclear interactions are mainly responsible for the production of them. The multiplicity of the production of mesons was investigated by Lewis, Oppenheimer and Wouthuysen⁽¹⁾, but since their treatment was semi-classical we think that it does not describe satisfactorily these phenomena. We shall consider the production of mesons using the method of mixed representation developed by Tomonaga and Schwinger.

Probabilities of occurring various processes which arise from interaction $H(x)$ can be inferred from the matrix elements of the generalized transformation function $U(\sigma, \sigma_0)$, which can be written in the form of series expansion with the unit system $\hbar=c=1$.

$$U(\sigma, \sigma_0) = 1 - i \int_{\sigma_0}^{\sigma} H(x) d^4x - \int_{\sigma_0}^{\sigma} d^4x_1 \times \int_{\sigma_0}^{\sigma_1(x_1)} d^4x_2 H(x_1) H(x_2) + \dots$$

Now we consider the symmetric pseudo-scalar meson field interacting with nucleons. Then $H(x)$ can be written as

$$H(x) = i f_1 \psi^\dagger(x) \gamma^5 C_i(x) \tau_i \psi(x) + i \frac{f_2}{\mu} \psi^\dagger(x) \gamma^5 \gamma^\mu \frac{\partial}{\partial x_\mu} C_i(x) \tau_i \psi(x)$$

where ψ , C_i ($i=1, 2, 3$) are the field variables of the nucleon and the mesons respectively, τ the isotopic spin, μ the mass of meson. Performing a canonical transformation

$$\Psi(\sigma) = \exp\left(i \frac{f_2}{\mu} \int_{\sigma} \psi^\dagger \gamma^5 \gamma_\mu C_i \tau_i \psi d\sigma_\mu\right) \Psi'(\sigma)$$

$H(x)$ is transformed to

$$H'(x) = i f_1 \psi^\dagger \exp\left(2i \frac{f_2}{\mu} \gamma^5 C_i \tau_i\right) \gamma^5 C_j \tau_j \psi + M \psi^\dagger \left(\exp\left(2i \frac{f_2}{\mu} \gamma^5 C_i \tau_i\right) - 1 \right) \psi$$

where M is the mass of the nucleon. Though U is transformed to U' , it is seen that $U'(\infty, -\infty) = U(\infty, -\infty)$ provided σ, σ_0 are shifted to ∞ and $-\infty$ respectively. First we shall treat the multiple production of mesons by the collision of a meson and the nucleon at rest. For instance, in the case where a neutral meson with energy E_0 collides with a nucleon at rest, the probability of emission of $2n-1$ neutral mesons is proportional to

$$\left(2nf_1 + 2\frac{M}{\mu}f_2\right)^2 f_2^{1n-2} \cdot 2^{2n-1} \cdot \frac{\pi^{2n-2}}{(2n-2)! (2n-3)!} \left(\frac{E_0}{\mu}\right)^{4n-7} \frac{1}{\mu}$$

using Born approximation and the non-relativistic approximation for the nucleon. Making $f_1=0$ for simplicity, the number n with maximum probability is ~ 5 for $E_0 \sim 10 \mu$, so that 9 mesons are emitted most probably.

If the external potential $V(x)$ exists, we can take in the place of above $H(x)$, $H_{\text{ext}}(x) = H(x) + H^{(e)}(x)$, $H^{(e)}(x) = \psi^\dagger(x) \gamma^4 \times V_e(x) \psi(x)$.

After similar calculations, the cross section of emission of n positively charged mesons and n negatively charged mesons by a nucleon with energy E_0 and momentum P_0 in this external potential field is found to be

$$\sigma(n, n, \varepsilon) = \frac{|\overline{V}(if)|^2 (f_2^2)^{2n} (2\pi)^{5-4n}}{(2n!)^2 (2n-1)! (2n-2)!} \left(\frac{\varepsilon}{\mu}\right)^{4n-4} \times \frac{E_0 P(E_0 - \varepsilon)}{P_0 \mu^4}$$

for $\varepsilon \gg \mu$, where ε is the energy loss of the nucleon in this process, P the momentum of the nucleon after the collision, $\overline{V}(if)$ the average value of the matrix element of V corresponding to the transi-

tion of the nucleon from the initial to the final state. The number n with maximum probability is proportional to $\varepsilon^{1/4}$. This result is in contrast with that obtained by Lewis et al. For $\varepsilon \sim 10 \mu$, $n=1 \sim 2$ and for $\varepsilon \sim 100 \mu$ $n=2 \sim 3$.

Detailed account of this work and the investigations of the phenomena concerning the production of mesons will soon appear in the later issue of this journal.

(1) H. W. Lewis, J. R. Oppenheimer and S. A. Wouthuysen, Phys. Rev. 73 (1948) 127.

Note on the Relativistic Formulation of Belinfante's Transformation.

Y. Katayama and S. Takagi
Dept. of Physics, Kyoto University.

April 10, 1949

Recently, Belinfante proposed in his paper⁽¹⁾ that it is very convenient to replace the electromagnetic field by the neutral vector meson field with the negligible small mass, because in the latter we are free from the gauge invariance and the auxiliary condition. The essential point of his theory is to apply the (non-relativistic) Fermi-Belinfante transformation⁽²⁾, which eliminates the longitudinal field in the electromagnetic field, to the Proca's field.

Here we show that this procedure can be formulated quite relativistically and that in this relativistic theory the electromagnetic field can be replaced with Proca's field.

At first, we separate the generalized longitudinal field using the invariant delta function and the induced generalized longitudinal delta function. That is, accord-

ing to the Tomonaga-Schwinger theory,

$$\left. \begin{aligned} U_{\mu}^L(x) &= \int_0 [U_V(x') (N_{\sigma} \nabla_{\sigma}') \Delta_{\mu\nu}^L(x' - x) \\ &\quad - \Delta_{\mu\nu}^L(x' - x) (N_{\sigma} \nabla_{\sigma}') \\ &\quad \times U_V(x')] dF_x' \\ U_{\mu}^T(x) &= \int_0 [U_V(x') (N_{\sigma} \nabla_{\sigma}') \Delta_{\mu\nu}^T(x' - x) \\ &\quad - \Delta_{\mu\nu}^T(x' - x) (N_{\sigma} \nabla_{\sigma}') \\ &\quad \times U_V(x')] dF_x' \end{aligned} \right\} \quad (1)$$

where

$$\Delta_{\mu\nu}^L(x) = [\nabla_{\mu} \nabla_{\nu} + (N_{\mu} \nabla_{\nu} + N_{\nu} \nabla_{\mu}) \times (N_{\sigma} \nabla_{\sigma}) - x^2 N_{\mu} N_{\nu}] \vartheta(x) \quad (2)$$

$$\Delta(x) = [(N_{\sigma} \nabla_{\sigma})^2 + x^2] \vartheta(x) \quad (3)$$

$$\Delta_{\mu\nu}^T(x) = \Delta_{\mu\nu}^L(x) + \Delta_{\mu\nu}^T(x) \quad (4)$$

From these definitions, we conclude, using the properties of $\Delta(x)$, $\Delta_{\mu\nu}^L(x)$, that

$$N_{\mu} U_{\mu}^L(x) = N_{\mu} U_{\mu}(x), \quad N_{\mu} U_{\mu}^T(x) = 0$$

$$\nabla_{\mu} U_{\mu}^L(x) = \nabla_{\mu} U_{\mu}(x), \quad \nabla_{\mu} U_{\mu}^T(x) = 0$$

therefore it is clear that our definitions are quite adequate. And from these, the commutation relations of separated fields become, using the original one,

$$\left. \begin{aligned} [U_{\mu}^L(x), U_{\nu}^L(x')] &= i\hbar c (\Delta_{\mu\nu}^L(x - x') \\ &\quad - \frac{1}{x^2} \nabla_{\mu} \nabla_{\nu} \Delta(x - x')) \\ [U_{\mu}^L(x), U_{\nu}^T(x')] &= 0 \\ [U_{\mu}^T(x), U_{\nu}^T(x')] &= i\hbar c \Delta_{\mu\nu}^T(x - x') \end{aligned} \right\} \quad (5)$$

which are similar to the case of Maxwell field in Schwinger's paper⁽³⁾.

Next, we take the generalized Schrödinger equation (or the interaction representation) of the Proca field developed by Miyamoto⁽⁴⁾, but the original U -field scheme, not the Stückelberg's form⁽⁵⁾:

$$i\hbar c \frac{\delta \Psi}{\delta C_x} = \left[-\frac{1}{c} j_{\lambda} U_{\lambda}^T - \frac{1}{c} j_{\lambda} U_{\lambda}^L + \frac{1}{2x^2} \frac{1}{c} N_{\lambda} j_{\lambda} \cdot \frac{1}{c} N_{\sigma} j_{\sigma} \right] \Psi \quad (6)$$

Here we must take the canonical transformation of state vector,

$$\Psi = e^{-iS} \Psi_1 \quad (7)$$

where

$$\begin{aligned} S = & \frac{1}{\hbar c} \int_0 \frac{1}{c} N_{\rho} j_{\rho}(x') \frac{1}{\nabla_{\mu}'} [U_{\mu}^L(x') \\ & - N_{\mu} \int_{0'} [U_{\mu}^L(x'') (N_{\sigma} \nabla_{\sigma}'') (\nabla_{\sigma}'') \\ & \times (N_{\lambda} \nabla_{\lambda}'') - x'^2 N_{\sigma}) \vartheta(x'' - x') \\ & - (N_{\sigma} \nabla_{\sigma}'') U_{\sigma}^T(x'') \cdot (\nabla_{\sigma}'') \\ & \times (N_{\mu} \nabla_{\mu}'') - x'^2 N_{\sigma}) \vartheta(x'' - x')] \\ & \times dF_x''] dF_x' \end{aligned} \quad (8)$$

After some calculations, we find

$$\begin{aligned} i\hbar c \frac{\delta \Psi_1}{\delta C_x} = & \left[-\frac{1}{c} j_{\lambda}(x) U_{\lambda}^T(x) - x^2 \right. \\ & \times \int_0 [U_{\sigma}^L(x') (N_{\sigma} \nabla_{\sigma}') \vartheta(x' - x) - \vartheta(x' - x) \\ & \times (N_{\sigma} \nabla_{\sigma}') U_{\sigma}^L(x')] N_{\sigma} \frac{1}{c} N_{\rho} j_{\rho}(x) dF_x' \\ & - \int_0 \left(\frac{1}{2} \nabla_{\lambda}' + N_{\lambda} N_{\sigma} \nabla_{\sigma}' \right) \vartheta(x' - x) \\ & \times \frac{1}{c} N_{\rho} j_{\rho}(x') \cdot \frac{1}{c} j_{\lambda}(x) dF_x' \left. \right] \Psi_1 \quad (9) \end{aligned}$$

The first term in the right-hand side is the usual interaction of transversal field, the second term the interaction of longitudinal field and the third term the Coulomb interaction. Therefore, these are the invariant representation of Belinfante's results and if we tend x to zero these reduce to the Maxwellian case.

Of course, we can also formulate the theory in which the Yukawa interaction appears in place of the Coulomb poten-

tial, but in this case we can also conclude that his proposal is still correct.

Applications of this theory to other problems, especially to the problems of nuclear force are now being investigated.

We should like to express our gratitude to Prof. M. Kobayasi for his kind interest in this work.

- (1) F. J. Belinfante*, Phys. Rev. **75** (1949) 337.
 (2) F. J. Belinfante, Physica **12** (1946) 1.

- (3) J. Schwinger, Phys. Rev. **74** (1948) 1439.
 (4) Y. Miyamoto, Prog. Theor. Phys. **3** (1948) 134.
 (5) Here we retain the δ function type interaction of the third term, which plays the very important role in this discussion. If this term is omitted, there appears the opposite term again.

* We wish to express our cordial thanks to Prof. F. J. Belinfante for his kindness to give the chance to see his paper before publication. (cf. Prog. Theor. Phys. **4** (1949) 165).

Errata (Vol. IV, No. 1)

P. 25 Theory of the Interaction of Elementary Particles IV₁.

The last term of the second line in Eq. (16) should be replaced by:

$$\left\{ 1 + \left(\frac{M^2}{4} - \mu^2 \right)^{-1/2} \times \frac{M}{2} \right\}.$$

The condition $\mu \geq M$ in Eq. (16) should be replaced by: $2\mu \geq M$.

The first term in Eq. (18), $\frac{-g^2}{2\pi M} p^3$, should be replaced by $-\frac{g^2}{2\pi M} p^2$.

The Eq. (20) should be replaced by:

$$\begin{aligned} W_{C.V} = & -\frac{1}{\pi M x^2} \left(\frac{g^2}{4} + \frac{5f^2}{16} \right) p(p^2 + x^2)^{3/2} - \frac{1}{\pi M} \left\{ \frac{7}{8} g^2 - \frac{f^2}{32} \left(13 - \frac{M^2}{x^2} \right) \right\} \\ & \times p(p^2 + x^2)^{1/2} + \frac{1}{\pi M} \left\{ \frac{9}{8} x^2 g^2 - \frac{f^2 x^2}{32} \left(19 - \frac{M^2}{x^2} + \frac{M^4}{2x^4} \right) \right\} \\ & \times \log \frac{2p}{x} + \frac{x^2 f^2}{\pi M} \left\{ \left(1 + \frac{M^4}{8x^4} \right) \sqrt{x^2 - \frac{M^2}{4}} \right\} \varphi(xM) \end{aligned}$$

P. 39 Effect of the C-Meson Field on the Anomalous

Magnetic Moment of the Electron. (Letter)

for Z. Koba, Y. Nambu and S. Tani,

read Z. Koba, Y. Nambu and S. Tani.

Theory of the F-centers of Coloured Alkali Halides. Part II. Electronic Structure of F-centers—General Theory†.

Toshinosuke MUTO.

Institute of Science and Technology, Tokyo University.

(Received Jan. 15, 1949)

1. Introduction.

As stressed in Introduction of Part I, it is the important problem in the theory of semiconductor to work out theoretically the so-called quasi-atomic states lying between the energy bands of the body crystal, which states may be considered to behave partly in an atomic way and partly in a crystal lattice one. For the paramagnetic crystal⁽¹⁾ of the complex salts of rare earth elements the electronic system within the incompleting shell of the mentioned atoms has been dealt with approximately in an atomic way in which the system can be reduced to an isolated one under the influence of the electrostatic field of crystal symmetry due to the surrounding ions, taking into account of the screening effect of the outer electrons belonging to the rare earth atoms. Such a method of approximation can not be applied generally to the quasi-atomic states in semiconductors owing to the strong coupling with the ions in crystal. As shown in the comparison with the experimental observations, Tibbs' theory⁽²⁾ for the F-centers seems to be unsatisfactory mainly due to the disregard of the atomic structure in the neighbourhood of negative ion vacancy. Actually his wave functions of trapped electron in F-center are shown to extend over the region involving several atomic ions, which fact leads us to suggest the group binding of electron due to neighbouring ions in the crystal. In order to improve Tibbs' theory from the viewpoint discussed above, we shall here propose a new method of approximation to the quasi-atomic states in the crystal in a close connection with the electronic structure of F-center in coloured alkali halide crystals.

II. A General Method of Approximation to treat the Quasi-Atomic States in the Crystal.

For convenience, we shall confine ourselves to the F-center problem but the method of approximation here proposed will easily be extended to the more complicated one with a trivial modification. As shown in Part I, the wave equation of a trapped electron within F-center may be written as

† Read in the Semiconductor-Symposium of the Physical Society of Japan held at University of Tokyo, on October 15, 1946.

$$-\frac{\hbar^2}{2m} \Delta \psi + \sum_{\lambda}' V_{\lambda}(\vec{R}_{\lambda} - \vec{r}) \psi = E(\vec{R}_{\lambda}) \psi, \quad (1)$$

in which $E(\vec{R}_{\lambda}) = E^c(\vec{R}_{\lambda}) - \sum_{\lambda\mu}' V_{\lambda\mu}(\vec{R}_{\lambda} - \vec{R}_{\mu})$ and the other quantities have the same meanings as in Part I.

Now the wave function of conduction electrons in the crystal is well known to be expressed by $\exp(i\vec{k}\cdot\vec{r})u_{\vec{k}}(\vec{r})$ of Bloch type, the former factor of which represents the free motion throughout the crystal while the latter the binding around each atomic ion within the crystal. On account of the mentioned situation, the electron moving through a perfect lattice behaves partly as a free electron even under the influence of each ion situated at a lattice point. Along a similar consideration to the above case, the wave function $\psi(\vec{r})$ in (1) for the electron trapped in the quasi-atomic state will be allowed to be expressed by the product of two kinds of wave functions $\psi_1(\vec{r})$ and $\psi_2(\vec{r})$, one of which is responsible for the influence of the crystal as a whole and the other for the influence of the local binding in the neighbourhood of negative ion vacancy.

To proceed actually the method of perturbation, $\psi_1(\vec{r})$ and $\psi_2(\vec{r})$ are supposed to satisfy the following wave equations

$$-\frac{\hbar^2}{2m} \Delta^{(1)} \psi_1(\vec{r}) + V_1(\vec{r}) \psi_1(\vec{r}) = E_1 \psi_1(\vec{r}), \quad (2)$$

$$-\frac{\hbar^2}{2m} \Delta^{(2)} \psi_2(\vec{r}) + V_2(\vec{r}) \psi_2(\vec{r}) = E_2 \psi_2(\vec{r}), \quad (3)$$

$$V_1(\vec{r}) + V_2(\vec{r}) = \sum_{\lambda}' V_{\lambda}(\vec{R}_{\lambda} - \vec{r}), \quad (4)$$

where $\Delta^{(1)}$ and $\Delta^{(2)}$ are Laplacian operators operating on $\psi_1(\vec{r})$ and $\psi_2(\vec{r})$ respectively.

Taking into account the adopted form of wave function $\psi(\vec{r}) = \psi_1(\vec{r}) \cdot \psi_2(\vec{r})$ discussed above, Hamiltonian operator H corresponding to the wave equation (1) may be transformed into

$$H = -\frac{\hbar^2}{2m} \Delta^{(1)} - \frac{\hbar^2}{2m} \Delta^{(2)} + V_1(\vec{r}) + V_2(\vec{r}) - \frac{\hbar^2}{m} \nabla^{(1)} \cdot \nabla^{(2)}, \quad (5)$$

where $\nabla^{(1)} \cdot \nabla^{(2)}$ expresses the scalar product of two gradient operators, and further, we shall take, for the unperturbed Hamiltonian H_0 and perturbation δH ,

$$H_0 = -\frac{\hbar^2}{2m} \Delta^{(1)} - \frac{\hbar^2}{2m} \Delta^{(2)} + V_1(\vec{r}) + V_2(\vec{r}) \quad (5a)$$

and

$$\delta H = -\frac{\hbar^2}{m} \nabla^{(1)} \cdot \nabla^{(2)} \quad (5b)$$

respectively.

As easily seen, the rigorous solutions of $H_0\psi=E\psi$ are composed of those of (2) and (3). Namely, when the eigenfunctions and the corresponding energies of (2) and (3) are written as

$$E_1^{(l)}, \psi_1^{(l)}(\vec{r}); E_2^{(k)}, \psi_2^{(k)}(\vec{r})$$

(l, k =quantum numbers)

respectively, and, moreover, $\psi_0(\vec{r})=\psi_1^{(0)}(\vec{r})\psi_2^{(0)}(\vec{r})$ and $E_0=E_1^{(0)}+E_2^{(0)}$ represent the eigenfunction and the corresponding eigenvalue which approach most nearly the real solution of (1) under consideration, the ordinary perturbation theory gives us

$$\psi(\vec{r})=\psi_1^{(0)}(\vec{r})\psi_2^{(0)}(\vec{r})+\psi'(\vec{r})+\psi''(\vec{r})+\dots \quad (6)$$

and

$$E=(E_1^{(0)}+E_2^{(0)})+E'+E''+\dots \quad (7)$$

respectively, in which

$$\psi'(\vec{r})=\sum_{k,l} a_{k,l} \psi_1^{(l)}(\vec{r}) \psi_2^{(k)}(\vec{r}), \quad (8)$$

$$\psi''(\vec{r})=\sum_{k,l} b_{k,l} \psi_1^{(l)}(\vec{r}) \psi_2^{(k)}(\vec{r}), \quad (9)$$

$$a_{k,l} = \frac{\int \psi_1^{(l)*} \psi_2^{(k)*} \left[-\frac{\hbar^2}{m} \nabla^{(1)} \psi_1^{(0)} \cdot \nabla^{(2)} \psi_2^{(0)} \right] d\tau}{E_1^{(l)} + E_2^{(k)} - E_1^{(0)} - E_2^{(0)}}, \quad (k \neq 0, l \neq 0), \quad a_{00}=0 \quad (10)$$

$$E' = -\frac{\hbar^2}{m} \int \psi_1^{(0)*} \psi_2^{(0)*} \nabla^{(1)} \psi_1^{(0)} \cdot \nabla^{(2)} \psi_2^{(0)} d\tau \quad (11)$$

$$\left. \begin{aligned} b_{k,l} &= \frac{E' a_{k,l} + \sum_{k',l'} a_{k',l'} (l, k | \partial H | l', k')}{E_1^{(l)} + E_2^{(k)} - E_1^{(0)} - E_2^{(0)}}, \\ E'' &= \sum_{k,l} \frac{(0, 0 | \partial H | l, k)(l, k | \partial H | 0, 0)}{E_1^{(l)} + E_2^{(k)} - E_1^{(0)} - E_2^{(0)}}, \end{aligned} \right\} \quad (12)$$

$$(l, k | \partial H | l', k') = -\frac{\hbar^2}{m} \int \psi_1^{(l)*} \psi_2^{(k)*} \nabla^{(1)} \psi_1^{(l')} \cdot \nabla^{(2)} \psi_2^{(k')} d\tau. \quad (12')$$

As seen is the above formula, the convergency of our perturbation method depends essentially upon the smallness of the matrix elements of (5b), which states of affairs shall be numerically justified for the case of F-centers in NaCl as shown in Part III. Moreover, it should be stressed here that the perturbing term in our case represents a part of the kinetic energy operator in the Hamiltonian in contradiction to the case of the ordinary perturbation theory in which a part of the potential energy operator is taken as a perturbing term in the successive approximation. The merit of our perturbation method in the actual problem, therefore, will be seen to depend sensibly upon the suitable choice of $V_1(\vec{r})$ and $V_2(\vec{r})$ in (4).

III. Application to the F-centers.

As shown in the following, we have now two kinds of choices of $V_1(\vec{r})$ and $V_2(\vec{r})$ in (4) along a similar way to the theory of molecular structure, one of which is usually called the atomic orbital method and the other the molecular orbital one.

a). Atomic orbital method.

The following decomposition of the potential field will be adopted as one of the above mentioned choices.

$$\sum_{\lambda}' V_{\lambda}(\vec{r}-\vec{R}_{\lambda}) = \sum_{\lambda} V_{\lambda}(\vec{r}-\vec{R}_{\lambda}) - V_{-}(r), \quad (13)$$

in which the coordinate origin lies on the lattice point of negative ion vacancy. the first term on the right hand side of (13) expresses the potential energy of the perfect lattice and the second the electrostatic potential energy of a missing negative ion.

It, therefore, follows

$$V_1(\vec{r}) = \sum_{\lambda} V_{\lambda}(\vec{r}-\vec{R}_{\lambda}) \quad (13)'$$

and

$$V_2(\vec{r}) = -V_{-}(r). \quad (13)''$$

Thus, we have, for the solution of (2).

$$\phi_1(\vec{r}) = \exp(i\vec{k} \cdot \vec{r}) u_k^+(r), \quad (14)'$$

$$E_1 = E_{\bullet}(k), \quad (14)''$$

in which \vec{k} expresses the wave number vector, n the energy band number and $u_k^+(r)$ the periodic function with the crystal period. The actual evaluation of (14)' and (14)'' are to be referred to the ordinary text on the theory of solids⁽³⁾.

The electrostatic potential energy $V_{-}(r)$ of a negative ion (i.e. Cl^-) has been already worked out numerically by the use of Hartree's self-consistent field method and $V_2(\vec{r})$ is obtained through changing its sign, which general behaviour is represented schematically in Fig. 1. Using the Hartree data, we have to solve numerically the wave equation of (3), thus $\phi_2(\vec{r})$ and E_2 being obtained quantitatively. The next processes of perturbation method are to be proceeded in accordance with the suc-

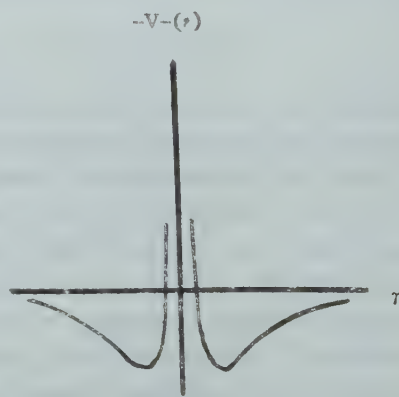


Fig. 1.

cessive approximation method in the preceeding Chapter. As for the convergence of our perturbation method it may be considered to be safely confirmed since our numerical calculation of F-center in NaCl shows $E'/(E_1^{(0)} + E_2^{(0)}) \sim 30\%$, which detailed result shall be described in Part III.

b). Molecular orbital method.

In this case the potential energy is decomposed as follows :

$$V_1(\vec{r}) = \sum_{\lambda}'' V_{\lambda}(\vec{r} - \vec{R}_{\lambda}) \quad (15)$$

and

$$V_2(\vec{r}) = \sum_{i=1}^6 V(\vec{r} - \vec{R}_i) \quad (15)'$$

The summation in (15) extends over all the lattice points except for the nearest neighbours surrounding a vacant lattice point, while that in (15)' is responsible for the contributions from the nearest neighbours which are Na^+ for the case of NaCl, $V(\vec{r} - \vec{R}_i)$ being the potential energy between the trapped electron and Na^+ situated at \vec{R}_i .

Along a similar consideration to that of computing the electrostatic potential at a given point within the dielectric medium, the atomic structure of the medium in the neighbourhood of vacant lattice point is taken account accurately, while its exterior region is approximately replaced by the continuum with excess charge $(-5e)$ distribution.

Therefore, we have

$$V_1(\vec{r}) \simeq V(\vec{r}), \quad (16)$$

in which the excess charge will be supposed provisionally to be distributed uniformly over the spherical surface of a suitable radius R or to be exponentially decreased outwards from the mentioned surface in order to make the actual computation most feasible. By the use of such approximate potential (16) $\psi_1(\vec{r})$ and E_1 in (2) are easily obtained quantitatively.

The solutions $\psi_2(\vec{r})$ and E_2 in (3) are shown to be obtained similarly to the case of hydrogen molecule ion as follows. When the surrounding Na^+ are numbered as in Fig. 2, the eigenfunction of an electron trapped within $V_2(\vec{r})$ is

$$\psi_2(\vec{r}) = \sum_{i=1}^6 a_i \phi(\vec{r} - \vec{R}_i), \quad (17)$$

in which $\phi(\vec{r})$ represents the eigenfunction of a valency electron of a single sodium atom. According to the ordinary perturbation theory, the co-

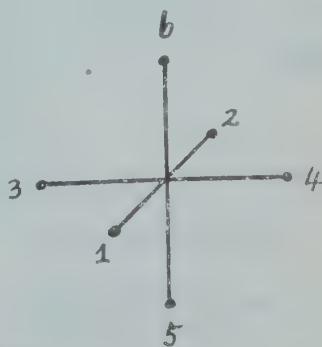


Fig. 2.

efficient a_i are determined by the following linear equation.

$$\sum_{i=1}^n a_i [(h | H | i) - E_2 \cdot \Delta(h, i)] = 0, \quad (18)$$

in which

$$\left. \begin{aligned} H_0 &= (1 | H | 1) = \dots = (6 | H | 6) \\ &= \sum_{j=1}^5 \int \phi^* (\vec{r} - \vec{R}_1) V(\vec{r} - \vec{R}_j) \phi(\vec{r} - \vec{R}_1) d\tau + E_0 \cdot \Delta_0, \\ H_1 &= (1 | H | 3) = \dots = (5 | H | 4) \\ &= \sum_{j=1}^5 \int \phi^* (\vec{r} - \vec{R}_1) V(\vec{r} - \vec{R}_j) \phi(\vec{r} - \vec{R}_3) d\tau + E_0 \cdot \Delta_1 \\ H_2 &= (1 | H | 2) = \dots = (5 | H | 6) \\ &= \sum_{j=1}^5 \int \phi^* (\vec{r} - \vec{R}_1) V(\vec{r} - \vec{R}_j) \phi(\vec{r} - \vec{R}_2) d\tau + E_0 \cdot \Delta_2 \end{aligned} \right\} \quad (19)$$

$$\left. \begin{aligned} \Delta_0 &= \Delta(i, i) = \int \phi^* (\vec{r}) \phi(\vec{r}) d\tau, \\ \Delta_1 &= \Delta(1, 3) = \dots = \Delta(5, 4) = \int \phi^* (\vec{r}) \phi(\vec{r} - \vec{R}_3) d\tau, \\ \Delta_2 &= \Delta(1, 2) = \dots = \Delta(5, 6) = \int \phi^* (\vec{r}) \phi(\vec{r} - \vec{R}_2) d\tau, \end{aligned} \right\} \quad (20)$$

$$-\frac{\hbar^2}{2m} \Delta \phi + V(r) \phi = E_0 \phi, \quad (21)$$

(wave equation of an isolated atom)

The secular equation of (18) is written as

$$\begin{vmatrix} b_0 & b_2 & b_1 & b_1 & b_1 & b_1 \\ b_2 & b_0 & b_1 & b_1 & b_1 & b_1 \\ b_1 & b_1 & b_0 & b_2 & b_1 & b_1 \\ b_1 & b_1 & b_2 & b_0 & b_1 & b_1 \\ b_1 & b_1 & b_1 & b_1 & b_0 & b_2 \\ b_1 & b_1 & b_1 & b_1 & b_2 & b_0 \end{vmatrix} = (b_0 - b_2)^2 (b_0 + b_2 - 2b_1)^2 (b_0 + b_2 + 4b_1) = 0, \quad (22)$$

where

$$\left. \begin{aligned} b_0 &= H_0 - E_2 \cdot \Delta_0, \\ b_1 &= H_1 - E_2 \cdot \Delta_1, \\ b_2 &= H_2 - E_2 \cdot \Delta_2. \end{aligned} \right\} \quad (23)$$

The solutions of (22) are given by

$$E_2^{(1)} = \frac{H_0 + H_2 + 4H_1}{\Delta_0 + \Delta_2 + 4\Delta_1}, \quad \psi_2^{(1)} = [6(\Delta_0 + \Delta_2 + 4\Delta_1)]^{-1/2} (\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6), \quad (24)$$

$$E_2^{(2)} = \frac{H_0 + H_2 - 2H_1}{\Delta_0 + \Delta_2 - 2\Delta_1}, \quad \left. \begin{aligned} \psi_2^{(2)} = & \left\{ \begin{aligned} & [12(\Delta_0 + \Delta_2 - 2\Delta_1)]^{-1/2} (\phi_1 + \phi_1 + \phi_2 + \phi_2 - \phi_3 - \phi_4 - \phi_5 - \phi_6), \\ & [4(\Delta_0 + \Delta_2 - 2\Delta_1)]^{-1/2} (\phi_1 - \phi_1 + \phi_2 - \phi_2 + \phi_3 + \phi_4 - \phi_5 - \phi_6), \end{aligned} \right\} \quad (25) \end{aligned}$$

$$E_2^{(3)} = \frac{H_0 - H_2}{\Delta_0 - \Delta_2}, \quad \psi_2^{(3)} = \left\{ \begin{aligned} & [2(\Delta_0 - \Delta_2)^{-1/2}] (\phi_1 - \phi_2) \\ & [\quad \quad \quad] (\phi_3 - \phi_4) \\ & [\quad \quad \quad] (\phi_5 - \phi_6) \end{aligned} \right\}, \quad (26)$$

where $\phi_i = \phi(\vec{r} - \vec{R}_i)$.

As seen, $E_2^{(1)}$ represents a single term, $E_2^{(2)}$ a doublet one and $E_2^{(3)}$ a triplet one, the corresponding eigenfunctions being described in (24), (25) and (26).

Adding the ion-ion interaction energy $\sum_{\lambda\mu}' V_{\lambda\mu}(\vec{R}_\lambda - \vec{R}_\mu)$ to the above mentioned energies, we have finally the electronic energy $E^*(\vec{R}_\lambda)$; which minimum value with respect to \vec{R}_λ is found to give a stable configuration of the crystal lattice at absolute zero temperature. Being $E'/(E_1^{(0)} + E_2^{(0)}) < 1$ as shown numerically in part III, the convergence criterion has been actually confirmed in the molecular orbital case.

IV. Ritz's Method of Variation.

In order to find a better solution of the problem under consideration, it will be considered to adopt Ritz's method of variation in which the trial form of eigenfunction will be reasonably suggested from the actual result of the atomic or molecular orbital method mentioned above. For example, we shall be able to take the following form of eigenfunction for the ground state of an electron in the potential field $V_2(\vec{r})$,

$$\psi_2(\vec{r}) = \sum_{i=1}^6 \phi_i(Z') + \sigma \sum_{i=1}^6 \varphi_i(Z''), \quad (27)$$

where Z' and Z'' represent the effective nuclear charge respectively, $\varphi_i(Z'')$ the p -eigenfunction of a single Na Atom denoted by i , being responsible for the effect of polarization of each atom as in the case of hydrogen molecule ion, and $|\sigma|$ measures the amount of the mentioned electric polarization.

Our Hamiltonian is written as

$$H = -\frac{\hbar^2}{2m} \Delta + \sum_{i=1}^6 V(\vec{r} - \vec{R}_i) + \sum' V_{\lambda\mu}(\vec{R}_\lambda - \vec{R}_\mu); \quad (28)$$

and its energy is given by

$$E = \int \psi_2^*(\vec{r}) H \psi_2(\vec{r}) d\tau / \int \psi_2^*(\vec{r}) \psi_2(\vec{r}) d\tau. \quad (29)$$

According to Ritz's method of variation, the parameters Z' , Z'' , σ and the energy E are uniquely determined by the minimum condition

$$\frac{\partial E}{\partial Z'} = \frac{\partial E}{\partial Z''} = \frac{\partial E}{\partial \sigma} = 0, \quad (30)$$

and the stable configuration of crystal lattice is obtained in a usual way as mentioned above.

V. Conclusion.

As stressed in the introduction, a suitable method of approximation based on the physical consideration of the phenomena is required essentially in order to work theoretically the quasi-atomic states in the crystal, and a possible one has been developed here with a special application to F-center in alkali halide crystals, although the mentioned method may be considered to be applicable to the more general type of quasi-atomic states in semiconductors or to the luminescent-centers in some crystalphosphors. The appropriateness of our method of treatment, however, should be criticized quantitatively in the detailed comparison of the numerical works of our method with the available experimental observations, which shall be described in Part III.

The author is indebted to Special Research Committee of Theoretical Physics, National Research Council of Education Ministry for the financial aid throughout the preparation of the present work.

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- (2) S. R. Tilbs: *Trans. Faraday Soc.* **35** (1939) 1491.
- (3) Seitz: *Modern Theory of Solids*, 1940.

Note added in proof: After our sending of the manuscript to the (editor) of *Progress of Theoretical Physics*, the interesting paper by Markham and Seitz (*Phys. Rev.* **74**, (1948,) 1014) has come into our notice, in which Landau trapping has been worked out theoretically by the use of the perturbation method for the quasi-atomic states within the crystal proposed here. In this connection we should like to add some remark that the system specified by (15) in the text may be reduced approximately, as an alternative to that given by (16), to the distribution of point charge and electric dipole situated at each lattice point respectively, the computation of the energy of such system being carried through by the conventional method of treatment.

Erratum to Part I (4 (1949), 181)

Fig. 1 (p. 182) is upside down.

Progress of Theoretical Physics, Vol. IV, No. 3, July~Sept., 1949.

Theory of the Interaction of Elementary Particles. V.

The Problem of Vacuum Polarization (3)

Hiroomi UMEZAWA and Eiji YAMADA

Institute of Theoretical Physics, Nagoya University.

(Received Feb. 1, 1949)

§ 1. Introduction.

We have hitherto* investigated, analysed and studied the physical causes of the divergences in problems related to vacuum polarization among the divergence difficulties in quantum electrodynamics, and attempted to solve them in correlation with the mutual interaction between elementary particles, taking up in particular the self-energies of the photon and the C-meson, and the Compton effect in the fourth order approximation of perturbation theory.

The divergence difficulties appearing up to now were those of the type of the self-energies of elementary particles and those that required charge renormalization, of which the latter were characterized in that they could be regarded as charge deviations due to charged particle-pairs created from vacuum in the intermediate states. We intend, in the following, to investigate whether new divergences of different natures are possible in cases other than these, and if so examine their characters, and also see if the method of charge-renormalization can always be rationally applied. Thus, we examine firstly the divergence appearing in the fourth-order perturbation calculation of the scattering of a photon by charged scalar (or pseudo-scalar) mesons, and next compare the charge-renormalizations due to vacuum polarization in the cases of Compton scattering and that of the elastic scattering of a charged particle with an electrostatic field as medium, and finally examine, as an example of a field other than an electromagnetic, the meson f -charge-renormalization of the C-meson field.

§ 2. Compton Scattering due to charged Scalar or Pseudo-scalar Mesons.

We denote the momenta of the meson and photon by P and k respectively, and distinguish between initial and final states by the suffices 0 and f respectively. Writing w for the transition probability, it is, in the fourth order approximation of perturbation theory,

$$w = 2\pi |e^2 H_2 + e^4 H_4|^2 \rho_f \quad (1)$$

* H. Umezawa, J. Yukawa and E. Yamada; Prog. Theor. Phys., 4 (1949), 25, 113. These will hereafter be referred to as papers (I) and (II) respectively.

where ρ_f is the number of final states in the energy range dE_f , while $e^2 H_2$ and $e^4 H_4$ are the contributions of the second and fourth approximations respectively. In the following we treat only the diverging terms in $e^4 H_4$. As the method of calculation is very similar to that for the Compton scattering due to an electron performed in paper (II), we shall give hereunder only those aspects of the calculation differing notably from the electron case. The interaction between a charged meson and a photon is

$$H' = -\frac{ie}{4\pi} \mathbf{A} (U^* \text{grad } U - \text{grad } U^* \cdot U) + \frac{e^2}{4\pi} \mathbf{A}^2 U^* U = e H_1 + e^2 H_2 \quad (2)^*$$

and the appearance of a term proportional to e^2 is a point very different from the Compton effect due to an electron. Furthermore, the contributions from $e H_1$ mostly become zero in the system where the charged particle is at rest in the initial state because of the presence of $\text{grad } U$ and \tilde{U} in $e H_1$, so the calculations must be performed in a system for which $\mathbf{p} \neq 0$. The self-energy $W_{\text{s.f.}}$, due to an electro-magnetic field, of a charged scalar or pseudo-scalar meson in motion with a momentum \mathbf{P}_0 is

$$W_{\text{s.f.}} = \left[-\frac{2e^2}{3\pi\epsilon_{p_0}} \mathbf{P}_0^2 \int \frac{dk}{k} \right] + \left[-\frac{1}{\pi\epsilon_{p_0}} \int k dk \right] + \left[\frac{e^2}{2\pi\epsilon_{p_0}} \int k dk + \frac{3e^2\mu^2}{4\pi\epsilon_{p_0}} \int \frac{dk}{k} \right] + f. t. = \frac{3e^2}{2\pi\epsilon_{p_0}} \int k dk + \frac{3e^2}{4\pi\epsilon_{p_0}} \int \frac{dk}{k} + f. t. \quad (3)$$

coinciding exactly with that of a meson at rest⁽¹⁾.

Here the terms in the first, second and third brackets are contributions from $e H_1$, $e^2 H_2$ and the Coulomb interaction respectively. The self-energy of the above particle due to the C-meson field is also of the same form as that for the particle at rest, so that it is possible, in the case of motion too, to prevent the divergence of the electromagnetic self-energy by means of considering the C-meson. The calculations hereunder are all performed in a system for which $\mathbf{P}_0 \neq 0$.

(a. 1) Cases in which the divergences are due to a photon created in the intermediate states or a Coulomb field.

Exactly as in the electron case, the divergence due to (i), (ii) and (iii) in paper (II) corresponds to the self-energy of the charged scalar (or pseudo-scalar) meson, and, as may be seen from (2), is equivalent to the divergence resulting from the presence of the term

$$H^{(1)} = \left[\frac{3e^2}{4\pi^2} \int k dk + \frac{3e^2\mu^2}{8\pi^2} \int \frac{dk}{k} \right] U^* U \quad (4)$$

in the Hamiltonian.**

* We use the natural units throughout our work, i.e., $\hbar=c=1$. μ and ϵ are mass and energy of the meson.

** Though we mentioned, in our lecture at the Elementary Particle Symposium in October 1948, that a new type of divergence besides (4) is incurred, we cancel our statement for it was erroneous, as pointed out by Mr. Koba and Mr. Takeda.

(a. 2) Cases in which the divergences are due to a charged scalar (or pseudo-scalar) meson-pair created in the intermediate states.

From processes (i) and (ii) in (a. 2) of paper (II) we obtain

$$H^{(2)} = -\frac{e^2}{6\pi^2} \int p dp \cdot A^2 \quad (5)$$

and from those processes among (i), (ii), (iii) and (iv) of the same paper due to eH_1 ,

$$H^{(3)} = -\frac{e^3}{12\pi} \int \frac{dp}{p} H_1 = -\frac{e^3}{12\pi} \int \frac{dp}{p} \left[-\frac{i}{4\pi} A (U^* \text{grad } U - \text{grad } U^* \cdot U) \right] \quad (6)$$

while those among (i), (ii) and (iv) due to $e^2 H_2$, which fall into the following two types corresponding to (iv)

$$\begin{aligned} & \{ (\tilde{l}_0, P_0) \rightarrow (\tilde{l}_0, \tilde{l}_f, P_0, P-l_f, -P) \dots \rightarrow (\tilde{l}_f, \tilde{l}_f, P_f, P-l_f, -P) \rightarrow (\tilde{l}_f, P_f) \\ & \{ (\tilde{l}_0, P_0) \rightarrow (\tilde{l}_0, \tilde{l}_0, P_0, P-l_0, -P) \dots \rightarrow (\tilde{l}_0, \tilde{l}_f, P_f, P-l_0, -P) \rightarrow (\tilde{l}_f, P_f) \end{aligned}$$

give

$$H^{(1)} = -\frac{e^4}{6\pi} \int \frac{dp}{p} H_2 = -\frac{e^4}{6\pi} \int \frac{dp}{p} \left[\frac{i}{4\pi} A^2 U^* U \right] \quad (7)$$

All taken into account, the resulting divergence is

$$\begin{aligned} H^{(1)} + H^{(2)} + H^{(3)} + H^{(4)} = & \left[\frac{3e^2}{4\pi^2} \int k dk + \frac{3e^2 l^2}{8\pi^2} \int \frac{dk}{k} \right] U^* U - \frac{e^2}{6\pi^2} \int p dp A^2 \\ & - \frac{e^3}{12\pi} \int \frac{dp}{p} H_1 - \frac{e^4}{6\pi} \int \frac{dp}{p} H_2. \end{aligned} \quad (8)$$

(b) These are processes of the same nature as those in (b) of Paper (II), and give rise to no divergence.

When the C-meson is considered, the following divergence, which plays the rôle of cancelling $H^{(1)}$ out, is caused:

$$H^{(1)'} = \left[-\frac{g^2}{4\pi^2} \int l dl - \frac{\mu^2}{8\pi^2} \left(f^2 - \frac{M^2}{\mu^2} g^2 \right) \int \frac{dl}{l} \right] U^* U, \quad (9)$$

$$g^2 = 3e^2, \quad f^2 = 3e^2 \left\{ 1 + \left(\frac{M^2}{\mu^2} \right) \right\}. \quad (10)^*$$

On the other hand, if charged Fermi particles too are considered to be creatable in the intermediate states, they give rise the following divergence:

$$-2H^{(2)} + 4H^{(3)} + 4H^{(4)}. \quad (11)$$

Hence, if we assume n such particles existing in nature for every $2n$ charged scalar (or pseudo-scalar) mesons, the divergence (9) is cancelled out just as in the case of Compton scattering due to an electron, and the finally remaining divergence is

$$H'' = -\frac{ne^3}{2\pi} \int \frac{dp}{p} H_1 - \frac{ne^4}{\pi} \int \frac{dp}{p} H_2. \quad (12)$$

* M is the mass of C-meson.

The cause for this divergence lies in the charged particle pair created in the intermediate states.

§ 3. The Nature of the Divergences and Charge-Renormalization.

In view of the fact that positron theory was originally developed in the form of a many body problem introducing an infinite number of negative-energy electrons, with the aim of dissolving the difficulty of the negative-energy solutions of the Dirac equation, without regarding the infinite number of freedoms thus brought about as something suggestive of the nature of the field resulting from the quantization of an electron, it was inevitable that an examination of difficulties was not made in the structure of an adequately developed quantum field-theory but only in the Hartree approximation through the density matrix of Dirac and Heisenberg with the result that the efforts to dissolve the difficulties took the course of solely seeking to exclude the singularities mathematically. As a result, the method of subtraction was found, as pointed out by Serber⁽³⁾, to contain inconsistencies within its own framework. Further, the difficulties appearing in positron theory were considered to be exclusive to the electron. Even the noteworthy work by Weisskopf⁽⁴⁾, which made the physical significance of the subtraction method clear did not take up the stand-point of quantum field-theory. However, the second quantization of the electron field subsequently introduced by Iwanenko and Sokolow has shown, it seems to us, that the difficulties of positron theory must be regarded as difficulties common to all charged particle fields, and not confined to the electron field alone. We therefore took the standpoint of the quantum field-theory thus developed, and performed our calculations with the aim of analysing the physical nature of the divergences in quantum electrodynamics in the light of the correlation among various quantized charged particle-fields and also of investigating the correlation between the divergences appearing respectively in self-energy and collision processes, at the same time endeavouring to discover the direction in which the solutions to these difficulties would be found.

According to our calculations up to now, the natures of the divergences fall into two types, *viz*: (a) Those due to the self-energy of elementary particles, and (b) those due to vacuum polarization and which have undergone charge renormalization.

(a) is further divided into (ai) the self-energy of charged particles, and (a ii) those of fields produced by charged particles—such as photons and C-meson. And against the former, the mixing of a C-meson, while against the diverging self-energy of the photon, the mixing of charged particles, has provided a tentative remedy, so that, though further examination is pending, the method of mixed fields proves an efficient measure.

It is convenient that the method of mixed fields is effective concerning self-energy, for, historically speaking, it was necessary even in classical electromagnetic

theory to introduce a field of cohesive force in order to account for the stability of charged particles, and also, it has been shown by Pais⁽²⁾ that the same measure holds for the self-energy of nucleons due to nuclear-force mesons. Of course (a) and (b) are not to be completely separated from one another, nor are separate methods stageantly employed in the respective cases, but in making systematic analyses, we may tentatively employ different methods, hence the classification. As to the self-energy of the C-meson itself, we consider it, as stated in detail in paper (I), a problem to be taken up again in future in correlation with nuclear force, since there are various possibilities for preventing its divergence, such as assuming mutual interactions among several types of cohesive force mesons instead of assuming the C-meson to be unique in kind. (c) are the divergences appearing when two or more particles are present and are due to electron-pairs created in the intermediate states. The method of charge renormalization has been applied in these cases too, but we must examine to what extent this is rationally done, so we take up only the divergences due to vacuum polarization in Compton scattering and the elastic scattering of charged particles by an electrostatic field, making comparisons for various charged particles.

Furthermore, as we will use them in our following calculations, we write out the Coulomb interactions $V_{s,s}$ and $V_{s,e}$ respectively between scalar (or pseudo-scalar) mesons themselves and between these and electrons as follows. Denoting by ρ_e and ρ_s the charge densities of the electron and charged scalar (or pseudo-scalar) mesons respectively,

$$\rho_e = e\psi^*\psi, \quad \rho_s = -ie(U^*U - U^*U) \quad (13)$$

$$U = i \sum_p \sqrt{\frac{2\pi}{E_p}} (c_p - d_p^*) e^{i(p \cdot r)}, \quad U^* = \frac{1}{4\pi} \sum_p \sqrt{2\pi E_p} (c_p^* + d_p) e^{-i(p \cdot r)}. \quad (14)$$

Hence

$$\begin{aligned} V_s = \frac{e}{2} \left[(c_{p'}^* c_p - d_{p'}^* d_{p'}) \left(\sqrt{\frac{\epsilon_{p'}}{\epsilon_p}} + \sqrt{\frac{\epsilon_p}{\epsilon_{p'}}} \right) - (c_{p'}^* d_{p'}^* - d_{p'} c_{p'}) \right. \\ \left. \times \left(\sqrt{\frac{\epsilon_{p'}}{\epsilon_p}} - \sqrt{\frac{\epsilon_p}{\epsilon_{p'}}} \right) \right] e^{i(P-F) \cdot r} \quad (15) \end{aligned}$$

$$\begin{aligned} V_{s.e.} = \frac{\rho_e \rho_s}{r_{12}} = \frac{2\pi e^2}{|\mathbf{P}_0 - \mathbf{P}_1|^2} (u_{K_0}^* u_{K_1}) \left[(c_{P_1}^* c_{P_0} - d_{P_0}^* d_{P_1}) \left(\sqrt{\frac{\epsilon_{P_1}}{\epsilon_{P_0}}} + \sqrt{\frac{\epsilon_{P_0}}{\epsilon_{P_1}}} \right) \right. \\ \left. - (c_{P_1}^* d_{P_0}^* - d_{P_0} c_{P_1}) \left(\sqrt{\frac{\epsilon_{P_1}}{\epsilon_{P_0}}} - \sqrt{\frac{\epsilon_{P_0}}{\epsilon_{P_1}}} \right) \right] \delta(\mathbf{P}_0 + \mathbf{K}_0 - \mathbf{P}_1 - \mathbf{K}_1) \quad (16) \end{aligned}$$

$$\begin{aligned} V_{s.s} = \frac{1}{2} \frac{\rho^2}{r_{12}} = \sum_{P_1, P_{1+l}} \frac{2\pi e^2}{4l^2} \left[\left(\sqrt{\frac{\epsilon_{P_1}}{\epsilon_{P_1+l}}} + \sqrt{\frac{\epsilon_{P_1+l}}{\epsilon_{P_1}}} \right) \left(\sqrt{\frac{\epsilon_{P_{1+l}}}{\epsilon_{P_{1+l}+l}}} + \sqrt{\frac{\epsilon_{P_{1+l}+l}}{\epsilon_{P_{1+l}}}} \right) \right. \\ \left. (c_{P_1}^* c_{P_{1+l}} - d_{P_{1+l}}^* d_{P_1}^*) (c_{P_{1+l}} c_{P_{1+l}+l} - d_{P_{1+l}+l}^* d_{P_{1+l}}^*) + \left(\sqrt{\frac{\epsilon_{P_1}}{\epsilon_{P_{1+l}}}} - \sqrt{\frac{\epsilon_{P_{1+l}}}{\epsilon_{P_1}}} \right) \right. \end{aligned}$$

$$\begin{aligned}
& \left(\sqrt{\frac{\epsilon_{P''}}{\epsilon_{P''+l}}} - \sqrt{\frac{\epsilon_{P''+l}}{\epsilon_{P''}}} \right) (c_{P'}^* d_{P'+l}^* - d_{P'} c_{P'+l}) (c_{P''} d_{P''+l} - d_{P''}^* c_{P''+l}^*) \\
& - \left(\sqrt{\frac{\epsilon_{P'}}{\epsilon_{P'+l}}} + \sqrt{\frac{\epsilon_{P'+l}}{\epsilon_{P'}}} \right) \left(\sqrt{\frac{\epsilon_{P''}}{\epsilon_{P''+l}}} - \sqrt{\frac{\epsilon_{P''+l}}{\epsilon_{P''}}} \right) (c_{P'}^* c_{P'+l} - d_{P'+l}^* d_{P'}) \\
& (c_{P'} d_{P''+l} - d_{P''}^* c_{P''+l}^*) - \left(\sqrt{\frac{\epsilon_{P'}}{\epsilon_{P'+l}}} - \sqrt{\frac{\epsilon_{P'+l}}{\epsilon_{P'}}} \right) \left(\sqrt{\frac{\epsilon_{P''}}{\epsilon_{P''+l}}} + \sqrt{\frac{\epsilon_{P''+l}}{\epsilon_{P''}}} \right) \\
& (c_{P'}^* d_{P'+l}^* - d_{P'} c_{P'+l}) (c_{P''} c_{P''+l}^* - d_{P''+l} d_{P''}^*) \quad (17)^*
\end{aligned}$$

Nextly, the electrostatic field W produced by the center of the elastic scattering is taken to be of the following form:

$$W = ew\rho \quad (17)$$

so that, for the electron and charged scalar (or pseudoscalar) meson, it becomes respectively

$$W_e = ew\rho_e = e^2 \psi^* w \psi \quad (18.e)$$

and

$$W_s = ew\rho = -ie^2 w (U^+ U^* - U^{**} U) \quad (18.s)$$

Here w is a quantity related to the source of the electrostatic field and does not involve the wave function of the incident particle.

As for the elastic scattering of an electron by an electrostatic field, Itô and Koba⁽⁵⁾ have calculated for the case in which the electron alone is considered to exist, but the various processes appearing in their work can be directly applied to other charged particles, so we borrow those parts of their results related to vacuum polarization. These comprise the following four processes.

$$\begin{aligned}
& [a_r^0] \quad q_0 \rightarrow q_0, \quad q, \quad -q + q_0 - q_f, \quad q_f - q_0 \equiv \rangle \quad q_f - q_0, \quad q_0 \rightarrow q_f \\
& [b_r^0] \quad q_0 \rightarrow q_f, \quad q_0 - q_f \rightarrow q_f, \quad q, \quad -q + q_0 - q_f \equiv \rangle \quad q_f \\
& [c_r^0] \quad q_0 \rightarrow q_0, \quad q, \quad -q + q_0 - q_f, \quad q_f - q_0 \rightarrow q_f, \quad q, \quad -q + q_0 - q_f \equiv \rangle \quad q_f \\
& [g_r^0] \quad q_0 \equiv \rangle \quad q_f, \quad q, \quad -q + q_0 - q_f \equiv \rangle \quad q_f,
\end{aligned}$$

Particles of momenta involving q are pairs of particles created in the intermediate states, and there are two cases, one of electrons (K) and the other of charged scalar (or pseudoscalar) mesons (P), each of these again falling into two cases depending on whether the incident charged particle (q_0) is an electron (K_0) or a charged scalar (or pseudoscalar) meson (P_0). Doubled and tripled arrows denote the mediation of a Coulomb field and that of an electrostatic field respectively. For an elastic scattering of a charged particle A occurring through the medium of a pair of charged particles B , we write " $A \rightarrow B$ scattering."

* Hereunder P and K shall denote the momenta of the charged meson and electron respectively.

[1] For an *electron* \rightarrow *electron scattering*, as has been shown by Itô and Koba⁽⁶⁾, the processes $[a_r^0]$, $[b_r^0]$ and $[c_r^0]$ do not diverge, while $[g_r^0]$ falls into two types which give the following divergence

$$H^{(1)} = -\frac{2e^2}{3\pi} \int \frac{dk}{k} W_e \quad (19)$$

[2] For an *electron* \rightarrow *charged scalar (or pseudo-scalar) meson scattering*, since $[a_r^0]$, $[b_r^0]$ and $[c_r^0]$ are processes caused by the incident electron and a charged meson-pair created by W_s through the medium of a photon ($\mathbf{K}_0 - \mathbf{K}_f$ or $\mathbf{K}_f - \mathbf{K}_0$), they are easily seen to present no divergence if W_s is taken to be of the form (18s), and the interaction (1) between photon and charged meson is considered. $[g_r^0]$ becomes, on using (16) and (18s) for $V_{s,e}$ and W_s respectively (writing E and ϵ for the energies of the electron and charged meson respectively),

$$[g_r^0] = \sum_p \frac{(\mathbf{K}_0 | V_{s,e} | \mathbf{K}_f, \mathbf{P}, -\mathbf{P} + \mathbf{K}_0 - \mathbf{K}_f)(\mathbf{K}_f, \mathbf{P}, -\mathbf{P} + \mathbf{K}_0 - \mathbf{K}_f | W_s | \mathbf{K}_f)}{E_{\mathbf{K}_0} - E_{\mathbf{K}_f} - \epsilon_{\mathbf{P}} - \epsilon_{-\mathbf{P} + \mathbf{K}_0 - \mathbf{K}_f}} \\ \sim -\frac{e^2}{12\pi} \int \frac{dp}{p} (\mathbf{K}_0 | e^2 \bar{\psi} \psi | \mathbf{K}_f) = -\frac{e^2}{12\pi} \int \frac{dp}{p} (\mathbf{K}_0 | e^2 W_e | \mathbf{K}_f)$$

of which there are two types of processes, and the final diverging term obtained is

$$H^{(2)} = -\frac{e^2}{6\pi} \int \frac{dp}{p} W_e \quad (20)$$

Therefore, when n and m types respectively of Fermi and scalar (or pseudoscalar) particles are considered in the elastic scattering of an electron the divergence due to vacuum polarization is

$$nH^{(1)} + mH^{(2)} = \left(-\frac{2ne^2}{3\pi} - \frac{me^2}{6\pi} \right) \int \frac{dp}{p} W_e \quad (21)$$

[3] *Charged scalar (or pseudoscalar) meson* \rightarrow *charged scalar (or pseudoscalar) meson scattering*.

$[a_r^0]$, $[b_r^0]$ and $[c_r^0]$ are exactly the same as in [2] except that $(\mathbf{K}_0 | a_{e, \mathbf{K}_f - \mathbf{K}_0} \times \mathbf{A}_1 | \mathbf{K}_f)$ has to be replaced by $(\mathbf{P}_0 | eH_1 | \mathbf{P}_f)$ (with eH_1 , given by (1)), so that they do not diverge in this case either. As for $[g_r^0]$, if we use (17) and (18s) for $V_{s,s}$ and W_s respectively, the only difference from the calculation for $[g_r^0]$ in [2] is that the E 's in the denominators are replaced by ϵ , and $V_{s,e}$ by $V_{s,s}$, which is equivalent to replacing $(u_{k_0}, u_{k_f}^*)$ by $\frac{1}{2} \left(\sqrt{\frac{\epsilon_{P_0}}{\epsilon_{P_f}}} + \sqrt{\frac{\epsilon_{P_f}}{\epsilon_{P_0}}} \right) c_{P_0} c_{P_f}^*$ which in turn corresponds, as may be seen by comparing (18c) and (18s), to taking W_s instead of W_e , and the divergence becomes, instead of (20)

$$H^{(3)} = -\frac{e^2}{6\pi} \int \frac{dp}{p} W_s \quad (22)$$

[4] *charged scalar (or pseudoscalar) meson* \rightarrow *electron scattering* replacing the Coulomb interaction V between the electrons in the calculations for [1] by $V_{\pi, \phi}$, $[a_r^0]$, $[b_r^0]$ and $[c_r^0]$ are seen not to diverge, while $[c_r^0]$, on replacing W_e by W_s , gives the following divergence.

$$H^{(4)} = -\frac{2e^2}{3\pi} \int \frac{dk}{k} W_s. \quad (23)$$

If we consider n and m types respectively of Fermi and scalar (or pseudoscalar) particles, the total divergence becomes

$$nH^{(3)} + mH^{(4)} = \left(-\frac{2ne^2}{3\pi} \int \frac{dk}{k} - \frac{me^2}{6\pi} \int \frac{dp}{p} \right) W_s. \quad (24)$$

Thus, for divergences of the type (b), the results may be tabulated as follows:

$v.p.$		F		S		$nF+mS$	
$i.p.$		(i)	(ii)	(i)	(ii)	(i)	(ii)
F	d	$2g_1W_e$	g_1eH_e	$2g_2W_e$	g_2eH_e	$2g_3W_e$	g_3eH_e
	$c.r.$	$(1-g_1)^2W_e$	$(1-g_1)eH_e$			$(1-g_3)^2W_e$	$(1-g_3)eH_e$
S	d	$2g_1W_s$	$g_1eH_1 + 2g_1^2e^2H_2$	$2g_2W_s$	$g_2eH_1 + 2g_2^2e^2H_2$	$2g_3W_s$	$g_3eH_1 + 2g_3^2e^2H_2$
	$c.r.$			$(1-g_2)^2W_s$	$(1-g_2)eH_1 + (1-g_2)^2e^2H_2$	$(1-g_3)^2W_s$	$(1-g_3)^2e^2H_1 + (1-g_3)^2e^2H_2$

where

$$g_1 = -\frac{e^2}{3\pi} \int \frac{dk}{k}, \quad g_2 = -\frac{e^2}{12\pi} \int \frac{dp}{p}, \quad g_3 = \left(-\frac{ne^2}{3\pi} - \frac{me^2}{12\pi} \right) \int \frac{dp}{p}$$

and $eH_e = e\psi\alpha A\psi$, while eH , e^2H_2 are as given in (1), (i) and (ii) are the cases of elastic scattering and that of a photon by an electron. Further, F and S denote Fermi and scalar particles respectively, $v.p.$ and $i.p.$ particles created in the intermediate state and the incident particle, while $d.$ and $c.r.$ respectively denote a divergence of type (b) and the charge renormalization required in cancelling it out.

In the charge renormalization for the coefficient of W , account is taken of the fact that, as seen from (18c) and (18s), W is proportional to e^2 , and the renormalization is applied not only to ρ but W itself as well. This is based on the idea that W is, as pointed out by Prof. Sakata, produced by charged particles other than those in immediate consideration, and hence the calculation for elastic scattering should, it seems, more exactly speaking, be calculated as collision processes among elementary particles. If W is thus assumed to be proportional to e^2 , the charge renormalization for both Compton and elastic scattering can as seen from the above table, be taken exactly equal. Furthermore, although in a single-field theory the charge renormalization when Fermi particles alone are considered differs by a factor $1/2$ from that for charged scalar (or pseudoscalar) mesons alone,

it is physically natural to consider all the charged particles existing in nature, and if these are assumed to comprise n and m types respectively of Fermi and scalar (or pseudoscalar) particles, the charge renormalizations for both electron and charged scalar (or pseudoscalar) meson become equally $-\left(\frac{ne^2}{3\pi} + \frac{mc^2}{12\pi}\right)$ so that charge renormalization may be performed independent either of the type of charged particle or whether the scattering is elastic or Comptonian. This is also to be expected from the physical significance of charge renormalization which assumes the charge deviation due to vacuum polarization to be already included in the observed charge when a measurement is made.

Thus, it is seen that the method of charge renormalization can be rationally applied to divergences of type (b) due to vacuum polarization when and only when all the charged particles existing in nature are taken into account; but of course this only a tentative method in setting the difficulties in order, and a more substantialistic method of solution is to be desired.

§ 4. The Mesonic-Charge Renormalization of the C-Meson.

As shown in the foregoing, the method of charge-renormalization can, to a certain extent, be rationally applied to the case of a charged particle interacting with an electromagnetic field. In order to see if it is effective also for fields other than the electromagnetic, we calculate for the case in which a C-meson k_f is emitted instead of the photon l_f in the final state in the Compton scattering due to an electron, comparing the f -charge renormalization thereby involved with that in the case of elastic scattering of an electron by a meson field. As our object is only to examine the nature of the divergence and the validity or not of f -charge renormalization, we consider, for the sake of simplicity, electrons alone as the charged particles created in the intermediate states. The calculations being exactly similar to those for the ordinary Compton scattering in paper (II), we will dispense with detailed explanations. The divergence arising from the creation, in the intermediate state, of a photon l^- is

$$\frac{3e^2}{2\pi} \int \frac{dl}{l} \psi^* \beta_{\mu} \psi + \frac{3e^2}{2\pi} \int \frac{dl}{l} f \psi^* \beta_{\phi} \psi \quad (25)$$

The first of these terms is related to the mass of the electron.

The divergence arising from the creation, in the intermediate state, of a C-meson k^- is

$$-\frac{3f^2}{4\pi} \int \frac{dk}{k} \psi^* \beta_{\mu} \psi - \frac{3f^2}{4\pi} \int \frac{dk}{k} \psi^* \beta_{\phi} \psi \quad (26)$$

Thus, the divergence having no connection with vacuum polarization can be made to converge in this case too, as seen by adding (25) and (26) together, under the condition that $f^2 = 2e^2$. Among the divergences arising from the electron l^-

created in the intermediate state, that due to the vacuum polarization caused by the C-meson k_f is, if we take the self-energy, due to the electron, of the C-meson (whose energy is ϵ_i) to be $f^2 \frac{\delta m}{\epsilon_i}$ *

$$H = \frac{f^2 \delta m}{2(2\pi)} \phi^2 - \frac{f^2}{2\pi} \int \frac{dp}{p} \phi^* \beta \phi \phi. \quad (27)$$

Of these, the first term is of the type of self-energy of the c-meson while the second is to be subjected to the f -charge renormalization as follows:

$$f \rightarrow f \left(1 - \frac{f^2}{2\pi} \int \frac{dp}{p} \right) \quad (28)$$

The divergence arising from the vacuum polarization due to the electron is precisely that which appeared in the ordinary Compton scattering (paper (II)), and has already been thoroughly investigated.

Further, the second term of (27) takes a form similar to that appearing in the calculation of the charge-renormalization type divergence in Compton scattering (cf. Eq. (26) in paper (II)). The first term in (27), i.e. the divergence of the type of self-energy of the C-meson being a problem as such, has already been discussed; so in the following we examine if the f -charge renormalization required in the case of the elastic scattering of an electron by a static C-meson field W turns out to be as stipulated in (28). W can be written as follows:

$$W = f \psi \phi^* \beta \phi \quad (29)$$

The diverging terms in the elastic scattering of an electron by this field has been computed by Itô and Koba⁽⁴⁾ and that among these which are due to vacuum polarization is shown to be quadratic in P and hence by comparison with (28), the divergence which is to be subjected to charge-renormalization does not seem to coincide in order. However, on making detailed calculations, the divergence in elastic scattering becomes of the following form:

$$\left(f^2 \frac{\delta m}{\epsilon_i} - \frac{f^2}{\pi} \int \frac{dp}{p} \right) W \quad (30)$$

Therefore, the above disagreement must be considered as being due to the fact that the first of these terms which really originates from the first term of (27) has been confused with a charge-renormalization type of divergence. This becomes obvious when bear in mind that, in C-meson theory, the charged particle emits and or absorbs the C-meson and these two can interact with each other, so that the field (29) must be considered as being really due to the mutual exchange of C-mesons between the source of the field and the incident electron. Then the self-energy of the C-meson created in the intermediate state must give rise to the

* For the expression of δm , see § 3 of paper (I).

divergence given in the first term of (27). Therefore, in order to show that the first term of (30) should really not be subjected to f -charge renormalization, we must consider the elastic scattering not as being due to a field such as (29), but as resulting from the exchange of C-mesons taking place between the charged particle (acting as source of the field) and the incident electron, and perform calculations accordingly. In the following, we take this line of thought and compute the divergence due to vacuum polarization. We denote the interaction between the C-meson and the elementary particle acting as centre of the scattering by $f'H_a$, while the momentum of the latter and that of the electron is written as \mathbf{K} and \mathbf{P} respectively.

The total number of processes due to vacuum polarization is 24, among which the representative ones are the four given below. (The sign \sim after a letter denote that that quantity relates to the C-meson)

$$(i) \quad \frac{P_0}{\mathbf{K}} \rightarrow (P_0 - P_f, \frac{P_f}{\mathbf{K}}) \rightarrow (\frac{P_0 - P_f}{P_f, P - P_0 + P_f, -P}, \frac{P_0 - P_f}{\mathbf{K}}) \rightarrow (P_0 - P_f, \frac{P_f}{\mathbf{K}}) \rightarrow (P_f, K_f)$$

$$(ii) \quad \frac{P_0}{\mathbf{K}} \rightarrow (P_0 - P_f, P_f) \rightarrow (P_f, P + P_0 - P_f, -P) \rightarrow (P_0 - P_f, P_f) \rightarrow (P_f, K_f)$$

$$(iii) \quad \frac{P_0}{\mathbf{K}} \rightarrow (P_0 - P_f, P_f) \rightarrow (P_0 - P_f, P_0 - P_f, P_f, P + P_0 - P_f, -P) \rightarrow (\frac{P_0 - P_f}{P + P_0 - P_f, -P}, \frac{P_f}{K_f}) \rightarrow (P_f, K_f)$$

$$(iv) \quad \frac{P_0}{\mathbf{K}} \rightarrow (P_0 - P_f, P_0 - P_f, P_f, P - P_0 + P_f, -P) \rightarrow (P_0 - P_f, P_0 - P_f, P_f, P - P_0 + P_f, -P) \rightarrow (P_0 - P_f, P_f) \rightarrow (P_f, K_f)$$

It is easily seen that (i) and (ii) involve divergences of the type of self-energy of the C-meson, and in fact they yield divergences due to $\frac{f^2 \delta m}{2(2\pi)} \phi^2$. Adding a part of the remaining contributions from these processes to those from (iii) and (iv) the resulting divergence is

$$\frac{\delta m}{2(2\pi)} \phi^2 - \frac{f^3}{2\pi} \int \frac{d\rho}{\rho} \phi^* \beta \phi \phi - \frac{f f^2}{2\pi} \int \frac{d\rho}{\rho} H_a \quad (31)$$

The divergences due to the remaining twenty processes all coincide with one or another of those due to the above four, so that the total divergence due to vacuum polarization is sufficiently expressed by (31). Thus it is seen that the f -charge

renormalization required here is precisely that stipulated in (28). Incidentally, this charge-renormalization type of divergence takes a form similar to that in the case of ordinary Compton scattering (cf. eq. (26) of paper (II)).

Thus, we have shown that, provided we make calculations in accordance with the strict idea of interpreting the elastic scattering of an electron by a static C-meson field as really being due to the exchange of C-mesons between elementary particles, the same f -charge renormalization as that for the case in which an electron absorbs a photon and emits a C-meson is valid here, too, in preventing divergent results, and hence that the renormalization of f -charge is an effective measure in setting divergence difficulties in order.

In conclusion the authors wish to express their sincere gratitude to Prof. Sakata and Dr. Tanikawa for their guidance throughout this work.

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Eqs. (24') (28') (P. 119) should be replaced by

$$H_2' = -\frac{e^2}{6\pi^2} \int \dot{\rho} \, d\rho \, A^2 \quad \text{and}$$

$$H_3' = -\frac{e^3}{12\pi} \int \frac{d\rho}{\rho} (\psi^* a A \psi), \text{ respectively.}$$

The Phase Transition and the Piezoelectric Effect of KH_2PO_4 .*

Shigeo YOMOSA and Takeo NAGAMIYA.

Department of Physics, Osaka University.

(Received Feb. 21, 1949)

1. Introduction.

Anomalous behaviors of the dielectric constants⁽¹⁾ and the phase transition⁽²⁾ at 122°K of KH_2PO_4 were treated by the method of statistical mechanics by Slater⁽³⁾ and others.⁽⁴⁾ In this crystal, PO_4 groups are linked by hydrogen bonds which are directed nearly parallel to the axis a or b . In Slater's model it is assumed that two hydrogen atoms (or protons) out of the four hydrogen bonds linking each PO_4 group to its neighbors lie nearer to it, and the resulting $(\text{H}_2\text{PO}_4)^-$ group has a dipole moment directed to one of the crystalline axes. It is further assumed that the energy of the crystal in the absence of external electric field is given by the number of those dipoles which are perpendicular to the c -axis, multiplied by a certain constant ϵ . The number of configurations with the same energy and the same polarization was calculated by an approximate method and a unique type of transition was predicted. His theory does not however, account for the anomaly of the piezoelectric effect which was also observed⁽⁵⁾ and was found to be very similar to the anomaly of the dielectric constants, although he considers it to be the primary cause of the λ -type broadening of the transition temperature. As a matter of fact, we must expect a close relationship between them. In the present paper, Slater's theory was modified in such a way as to include the deformation of the crystal and consequently to account for the anomalous piezoelectric effect. A change of the transition temperature of the crystal in going from the free state to the clamped state, and a spontaneous deformation of the crystal which occurs in passing from the upper to the lower modification are deduced from our theory. Further, the temperature dependence of the elastic constant can be satisfactorily explained.

2. Introduction of Strain into the Theory.

If the crystal undergoes a shearing strain x_y (a -, b -, c -axes being taken as

* A preliminary report of this work was read on 29th April, 1946 at the annual meeting of the Physical Society of Japan. Afterwards, we came aware of an important paper due to Mason (W. P. Mason: Phys. Rev. **69** (1946), 173) who measured the dielectric, piezoelectric and elastic constants of potassium and ammonium dihydrogen phosphates, and discussed especially the change of the transition temperature in going from a free crystal to a clamped one. We therefore referred his results in comparing our theory with experiment.

cartesian coordinate axes x, y, z , respectively), PO_4 groups will change their orientations and the configurational energy of the crystal will also be changed. To take this into consideration, we simply assume that the energy of an $(\text{H}_2\text{PO}_4)^-$ group, which is directed towards $+c$ or $-c$, relative to the energy of that which is perpendicular to this axis, is a linear function of the strain. Writing the energy of the group directed to the $+c$ -axis as ϵ_+ and that for the $-c$ -axis as ϵ_- , we thus put

$$\left. \begin{aligned} \epsilon_+ &= -\epsilon_0 - \beta x_y, \\ \epsilon_- &= -\epsilon_0 + \beta x_y, \end{aligned} \right\} \quad (1)$$

where ϵ_0 and β are constants. This assumption is compatible with the symmetry of the crystal, 11_2 : by inversion followed by a rotation through an angle of $\pi/2$ about the z -axis x_y is converted into $-x_y$, and ϵ_+ and ϵ_- are interchanged. A more general assumption which conforms this symmetry property would be to include terms $-\gamma z$ and $+\gamma z$ in ϵ_+ and ϵ_- , respectively, where z is the net fraction of the dipoles parallel to $+c$ (Slater's x). The inclusion of these terms, however, does not change any essential feature of the following theory, and, moreover, they may be perhaps unimportant in actuality. The third and the higher odd powers of x_y and z may also be added. However, we shall confine ourselves to the simplest assumption of (1). A complication due to the fact that, in a strained crystal, there are four essentially different orientations of $(\text{H}_2\text{PO}_4)^-$ groups which are perpendicular to the c -axis (two in an unstrained one) will also be neglected.

With the above assumption, the internal energy of the crystal may be written in the following form:

$$U = N_+ \epsilon_+ + N_- \epsilon_- - (N_+ - N_-) \mu E + N \frac{1}{2} a x_y^2 - \frac{1}{2} \chi_1 E^2 - \chi_1 x_y E, \quad (2)$$

where N_+ and N_- are the numbers of $(\text{H}_2\text{PO}_4)^-$ dipoles which are parallel to $+c$ and $-c$, respectively, N is the total number of the dipoles, a, χ_1, ζ_1 are constants. The third term of this expression is the potential energy of the dipoles (of moment μ) with respect to the external electric field E . The fourth term is the elastic energy of the medium, and $N a$ is the normal elastic constant, if N is referred to the unit volume of the crystal; this term is that part of the elastic energy which is independent of the hydrogen bonds configuration. The fifth and sixth terms are relatively unimportant; the fifth term is the energy of the crystal due to the polarization which is induced by the external field, independently of the hydrogen bonds configuration, and the sixth term is the interaction energy between that polarization which is produced by the strain x_y , independently of the hydrogen bonds configuration, and the external electric field E . The discussion of the last two terms will be postponed to the Appendix, together with the discussion of the nature of the dipole moment μ and the problem

of the local field. We shall show there that the introduction of the local field factor into the susceptibility, as was done by Slater, can not be justified. Here we shall mention only that the consideration of the local field is fully taken into account by such a simple expression as (2), apart from the Lorentz type inner field $\mp r\tau$ mentioned in connection with (1). Substituting (1) into (2), we have

$$U = N \left\{ -\beta x_y z + (\nu_0 - 1) \epsilon_0 - \mu z E + \frac{\alpha}{2} x_y^2 \right\} - \frac{1}{2} \chi_1 E^2 - \zeta_1 x_y E, \quad (3)$$

where $z = (N_+ - N_-)/N$ and $\nu_0 = N_0/N$, N_0 being the number of dipoles perpendicular to the c -axis.

For the number of configurations with the same values of z and ν_0 , we shall assume the formula given by Slater. It is

$$F(N_+, N_0, N_-) = \left\{ \left[\frac{(1+z)^2}{2(1+z-\nu_0)} \right]^{\frac{1+z-\nu_0}{2}} \left[\frac{1-z^2}{\nu_0} \right]^{\nu_0} \left[\frac{(1-z)^2}{2(1-z-\nu_0)} \right]^{\frac{1-z-\nu_0}{2}} \right\}^N. \quad (4)$$

The condition of the minimum of the free energy, $A = U - kT \ln F = \text{minimum}$, gives the equations

$$\nu_0 \exp [2\epsilon_0/kT] = 4 \{ (1-\nu_0)^2 - z^2 \} \quad (5)$$

and

$$(1+z)^2 (1-z-\nu_0) \exp [2(\beta x_y + \mu E)/kT] = (1-z)^2 (1+z-\nu_0), \quad (6)$$

corresponding to $\partial A/\partial \nu_0 = 0$ and $\partial A/\partial z = 0$, respectively. From these equations we have

$$\nu_0 = \frac{2(1-z^2)}{(1+z) \exp [(\epsilon_0 + \beta x_y + \mu E)/kT] + 2(1-z)}, \quad (7)$$

$$z = \frac{\sinh [(\beta x_y + \mu E)/kT]}{2 \exp [-\epsilon_0/kT] - \cosh [(\beta x_y + \mu E)/kT]}. \quad (8)$$

If we substitute these quantities into the expression of the free energy, it becomes a function of x_y , E and T . Referring N to the unit volume of the crystal, we then have for the polarization and stress component the following equations:

$$-\frac{\partial A}{\partial E} = P = N\mu z + \chi_1 E + \zeta_1 x_y, \quad (9)$$

$$-\frac{\partial A}{\partial x_y} = X_y = -Na x_y + N\beta z + \zeta_1 E. \quad (10)$$

3. The Phase Transition.

In this section we shall confine ourselves to the case that the crystal is free from stress and external electric field. In this case, our Helmholtz free energy

A is identical with the Gibbs free energy. From Eq. (10) we have $x_y/z = \beta/a$, and the expression (3) becomes*

$$U = N\{ -(\beta^2/2a)z^2 + (\nu_0 - 1)\epsilon_0 \}. \quad (11)$$

Further, Eq. (7) can be written in terms of z , and the free energy becomes a function of z and temperature.† Fig. 1 shows the curves of the free energy versus z for different temperatures; the parameter $\beta^2/2a$ is here chosen to be equal to ϵ_0 . It may be seen that at a certain temperature, T_c , given by

$$2 = \exp[\epsilon_0/kT_c] (2 \exp[\delta/kT_c] - 1), \quad (\delta = \beta^2/2a) \quad (12)$$

z changes from zero to 1, but the type of the transition is not the same as that given by Slater; there is some range of temperature in which a metastable state with $z=0$ or $z=\pm 1$ exists. The lower and the upper limit of this temperature range, T_1 and T_2 , respectively, are given by

$$2 = \exp[\epsilon_0/kT_1] \cdot (1 + 2\delta/kT_1) \quad \text{and} \quad 2 = \exp[(\epsilon_0 + 2\delta)/kT_2]. \quad (13)$$

The crystal not only polarizes spontaneously below the transition temperature, but also it deforms spontaneously, their magnitudes being given by

$N\mu + \zeta_1\beta/a$ and β/a , respectively.

Another remarkable result is the fact that the transition temperature exists even if ϵ_0 is zero or negative, so far as $\delta = \beta^2/2a > |\epsilon_0|$. In this case, there are either two values of T_1 (for $|\epsilon_0| > 0.465\delta$) or there is no such a temperature (for $|\epsilon_0| < 0.465\delta$). The essential features of the transition, however, remain the same.

The entropy change at the transition temperature is given by

$$\Delta S = -R\{ (1 - \nu_0^c) \ln 2 (1 - \nu_0^c) + \nu_0^c \ln \nu_0^c \}, \quad (14)$$

where

* $\beta^2/2a$ is replaced by $\beta^2/2a + \gamma$, if we include the term $\mp \gamma z$ in (1)

† The curve for $A(z)$ thus obtained is the envelope the curves of $A(z, x_y)$ versus z , x_y being considered as a parameter. It is easy to see that the envelope is the lower limit of the original set of curves.

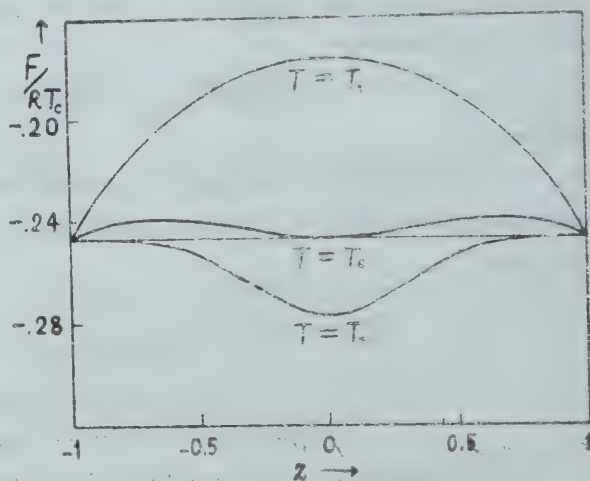


Fig. 1.

Curves of A/NkT_c for a special case of $\delta = \beta^2/2a = \epsilon_0$. In this case, T_c , T_1 , T_2 are given respectively by $\epsilon_0/kT_c = 0.248$, $\epsilon_0/kT_1 = 0.266$, $\epsilon_0/kT_2 = 0.231$.

$$\nu_0 = \frac{2}{\exp[\epsilon_0/kT_c] + 2} = 1 - \frac{1}{2} \exp[-\delta/kT_c] \quad (15)$$

is the value of ν_0 of the upper phase at the transition temperature. The plots of ΔS and ν_0^c against ϵ_0/δ is shown in Fig. 2; those values of ΔS and ν_0^c , which correspond to the Slater's theory, are their limiting values for $\epsilon_0/\delta \rightarrow \infty$. It seems hardly possible to determine ϵ_0/δ from these curves by comparing them with experiment. The experimental value of ΔS due to Stephenson is 0.74 ± 0.06 cal/deg. mol., which is nearly equal to the maximum value of our curve.

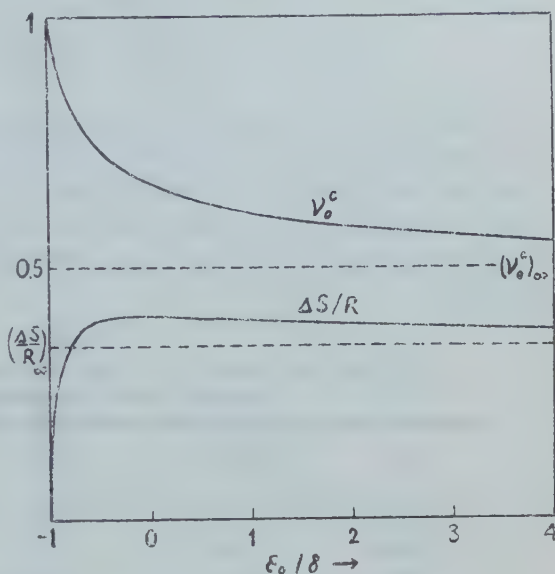


Fig. 2.

4. The Electric Susceptibility and the Piezoelectric Constant.

Above the transition point, the dipole polarization $N\mu\epsilon$ is small, and we can expand the Eq. (8) in powers of z , x_y , and E . Retaining only those terms which are linear in them, we have

$$z = \frac{1}{D} (\beta x_y + \mu E), \quad \text{where } D = kT (2 \exp[-\epsilon_0/kT] - 1). \quad (16)$$

Eliminating z and x_y from (9), (10) and (16), we have

$$P = \left\{ \frac{N(\mu + \beta\zeta_1/Nu)^2}{D - 2\delta} + \left(\chi_1 + \frac{\zeta_1^2}{Nu} \right) \right\} E - \left\{ \frac{\beta(\mu + \beta\zeta_1/Nu)}{a(D - 2\delta)} + \frac{\zeta_1}{Nu} \right\} X_y. \quad (17)$$

The coefficient of E is the susceptibility of the stress-free crystal and the coefficient of X_y is the piezoelectric constant. Writing them as χ and d_{36} , we have

$$\left. \begin{aligned} \chi &= \chi_0 + \chi', \\ \chi_0 &= \frac{N(\mu + \beta\zeta_1/Nu)^2}{D - 2\delta}, \quad \chi' = \chi_1 + \frac{\zeta_1^2}{Nu}, \end{aligned} \right\} \quad (18)$$

$$\left. \begin{aligned} d_{36} &= d_{36,0} + d_{36}', \\ d_{36,0} &= \frac{\beta}{a} \frac{\mu + \beta\zeta_1/Nu}{D - 2\delta}, \quad d_{36}' = \frac{\zeta_1}{Nu}. \end{aligned} \right\} \quad (19)$$

They consist of the temperature-dependent parts χ_0 and $d_{36,0}$, and the temperature-

independent parts χ' and χ'' . Further, the susceptibility of a clamped (i.e. strain-free) crystal can be derived from (18) by putting $\beta=\delta=\zeta_1=0$. Thus, it is given as

$$\chi^{cl}=\frac{N\mu^2}{D}+\chi_1=\chi_0'+\chi_1. \tag{20}$$

It follows that the stress-free susceptibility becomes infinite at $D=2\delta$, that is, at the temperature T_1 defined by the first of the Eqs. (13), whereas the strain-free susceptibility becomes infinite at the temperature for which $D=0$. Table I gives the three temperatures T_c , T_1 , and the transition temperature of the strain-free crystal, T_c' , which corresponds to $D=0$, for various values of the ratio δ/ϵ_0 .

Table I.

The last two columns are calculated for such a value of ϵ_0 that gives the observed transition temperature $T_c=122^\circ\text{K}$.

δ/ϵ_0	ϵ_0/kT_c	ϵ_0/kT_1	T_c-T_1	T_c-T_c'
1.000	0.248	0.266	3.2°	78°
0.100	0.578	0.583	0.9°	20°
0.050	0.624	0.625	0.2°	12°
0.025	0.658	0.658	0.05°	6.2°
0.020	0.665	0.665	0.01°	4.9°
0.015	0.672	0.672	0.00°	3.7°
0.010	0.679	0.679	—	2.5°
0.000	0.693	0.693	—	0.0°

Mason gives a value 3.5° for the difference between the two temperatures T_c and T_c' for which the susceptibility of the free and the clamped crystals, respectively, become infinite. The value 3.5 is somewhat small compared with the range of temperatures in which the spontaneous polarization and the specific heat vary markedly with temperature, namely $20^\circ\sim10^\circ$. Mason's value corresponds to $\delta/\epsilon_0=0.015$. Now, $N\alpha$ can be identified with the reciprocal of the elastic compliance s_{66} , if N is referred to the unit volume. According to Mason, $s_{66}=16.4\times10^{-12}$ c.g.s. at room temperature, and $N=1.04\times10^{23}\text{ cm}^{-3}$. With these values we have $\alpha=5.8\times10^{-12}$. Further, the values of ϵ_0 and δ can be taken from Table I, assuming $T_c=122^\circ\text{K}$. We thus have $\delta/\epsilon_0=0.015$, and

$$\frac{\beta^2}{\alpha^2}=\frac{2\delta}{\alpha}=\frac{2\delta}{\epsilon_0}\cdot\frac{\epsilon_0}{kT_c}\cdot\frac{kT_c}{\alpha}=0.03\times0.672\times\frac{1.37\times10^{-16}\times122}{5.8\times10^{-12}}=5.8\times10^{-5},$$
$$\frac{\beta}{\alpha}=7.6\times10^{-3}=26' \text{ in angle.}$$

Thus the crystal deforms spontaneously with an angle of $26'$ below the transition

temperature, if it is free from stress.

To see the temperature dependence of χ and d_{33} , the function $(D-2\delta)^{-1}$ will be conveniently expressed by an approximate formula

$$\frac{\epsilon_0}{D-2\delta} = \frac{a}{x-x_c} - b, \quad \text{where} \quad x = \frac{kT}{\epsilon_0}, \quad x_c = \frac{kT_c}{\epsilon_0}.$$

Numerical computations yield the following values of a , b :

$$\delta/\epsilon_0 = 0: \quad a = 1.39, \quad b = 0.110; \quad x_c = 1.443,$$

$$\delta/\epsilon_0 = 0.015: \quad a = 1.36, \quad b = 0.106; \quad x_c = 1.488.$$

With these values the approximate formula agrees with the exact one within errors of a few percents for $x = x_c \sim 3x_c$. Now, the experimental results for the susceptibility are as follows:

$$\chi = 248/(T-122) + 0.28, \quad (\text{Mason})$$

$$\chi = 306/(T-115) + 0.60. \quad (\text{Busch})$$

The second formula is that given by Slater. Similar formula can be given for the piezoelectric constant:

$$d_{33} = \{105/(T-122) + 0.07\} \times 10^{-6} \text{ e.s.u.}, \quad (\text{Mason})$$

$$d_{33} = \{150/(T-123) - 0.35\} \times 10^{-6} \text{ e.s.u.} \quad (\text{Lüdy})$$

Mason's data extend from the Curie point up to 100°C and were obtained by a dynamic method; the empirical formulas given above agree with the measured values within errors of a few percents for the temperature range from T to 40°C . Lüdy's measurements are due to static method and are given for the temperature range from T_c to 0°C .

That the temperature dependence of the measured χ and d_{33} are similar to each other is a verification of our theory. Comparison of our formula with experiment yields the following values of the parameters. If Mason's χ and d_{33} are used, we have (assuming $\delta/\epsilon_0 = 0.015$)

$$\mu' \equiv \mu + \beta\zeta_1/Na = \mu(1 + 0.0043) = 1.55 \times 10^{-18},$$

$$\beta/u = 6.8 \times 10^{-3} = 23',$$

$$\chi' = \chi_1(1 + 0.0029) = 0.52, \quad d_{33}' = 0.17 \times 10^{-6}.$$

If Busch's result and Lüdy's result are combined together, we have

$$\mu' = \mu(1 - 0.0062) = 1.72 \times 10^{-18}, \quad \beta/u = 8.8 \times 10^{-3} = 30',$$

$$\chi' = \chi_1(1 + 0.0030) = 0.89, \quad d_{33}' = -0.21 \times 10^{-6}.$$

Since the thermal expansion may result in changing the values of the parameters, the results obtained above for χ' and d_{33}' may not be very significant. We can

safely put $\mu' = \mu$ and $\chi' = \chi_1$. The very good agreement between the angle of deformation β/a calculated here (23' or 30') and that calculated previously (26') can be considered as another verification of our theory.

The dipole moment μ' determined above is somewhat greater than that can be obtained from the spontaneous polarization observed below the transition point. Busch's observation shows that the latter is not independent of temperature but increases from zero at the transition point to a constant value at and below nearly 100°K. The dipole moment derived from this constant value is $\mu' = 1.25 \times 10^{-18}$. According to Slater's idea, the gradual increase of the spontaneous polarization and the λ -type anomaly of the specific heat are due to the irregularly distributed stresses inside the crystal, caused by the spontaneous strain; they give rise to different transition temperatures for different domains. Accepting this idea, it is expected that in Busch's experiment not all the domains are directed to the applied electric field whose maximum strength was 3000 volt/cm. A simple calculation shows, by equating the electric potential energy of the spontaneously polarized piece of crystal placed in an applied field, with its elastic energy which corresponds to the spontaneous strain, that the field strength at which all the domains are directed to the field direction is of the order of 3×10^4 volt/cm.

Thus it is expected that, in Busch's experiment, some of the domains are still in the opposite direction to the applied field. The disagreement between the value of μ' obtained from the susceptibility and that obtained from the spontaneous polarization may be explained in this way, rather than introducing a local field factor, as Slater did.

5. The Elastic Constant.

The difference $T_c - T_c' = 3.5^\circ$ given by Mason is not a value observed directly, but it was calculated by deriving the clamped susceptibility from the observed dielectric, piezoelectric and elastic constants. We can follow the same process within our theory. Namely, eliminating z from Eqs. (9) and (10), we have

$$x_y = d_{33} E - s_{66} X_y, \quad s_{66} = \frac{\beta^2}{Na^2(D-2\zeta)} + \frac{1}{Na} = \left(\frac{2\delta}{D-2\delta} + 1 \right) \frac{1}{Na}, \quad (21)$$

and when this is combined with Eq. (17), i.e. $P = \chi E - d_{33} X_y$, we can have Eq. (20) for χ^{cl} and hence T_c^{cl} . (Our s_{66} is Mason's s_{66}^K .)

Since χ and d_{33} were proved to have the right temperature dependence and the parameters involved in them were adjusted so as to conform the experiment, there remains only the discussion of s_{66} . The cause of the very good agreement between the two values of β/a calculated in the preceding section by two independent methods must be sought in the right temperature dependence of the theoretical elastic constant. Fig. 3 shows the observed and calculated elastic constants. If we take Na to be independent of temperature, the theoretical curve

deviates from the experimental curve at the high temperature side (dotted curve). However, by taking

$$N\alpha = 7.04 \times 10^{10} \{1 - (T - T_c) \times 6.7 \times 10^{-4}\}, \quad (22)$$

an excellent agreement can be obtained (full curve). The theoretical curve is sensitive to the choice of δ/ϵ_0 , and the best fitting is obtained for $\delta/\epsilon_0 = 0.014$ (Fig. 3 is drawn for $\delta/\epsilon_0 = 0.015$). The temperature coefficient $6.7 \times 10^{-4} \text{ deg}^{-1}$ of the elastic constant is of the same order of magnitude as those of other substances which show no anomalous behavior.

Taking $N\alpha = 7.04 \times 10^{10}$ at the transition temperature, and combining this with $\beta/\epsilon_0 = 6.8 \times 10^{-3}$, which was determined from Mason's observed susceptibility and piezoelectric constant, we have $\delta = \beta^2/2\alpha = 1.57 \times 10^{-16}$. On the other hand, the observed transition temperature gives $\epsilon_0 = 1.22 \times 10^{-14}$ (taking $\delta/\epsilon_0 = 0.015$, refer Table I). Combining these two values, we have again $\delta/\epsilon_0 = 0.014$. It

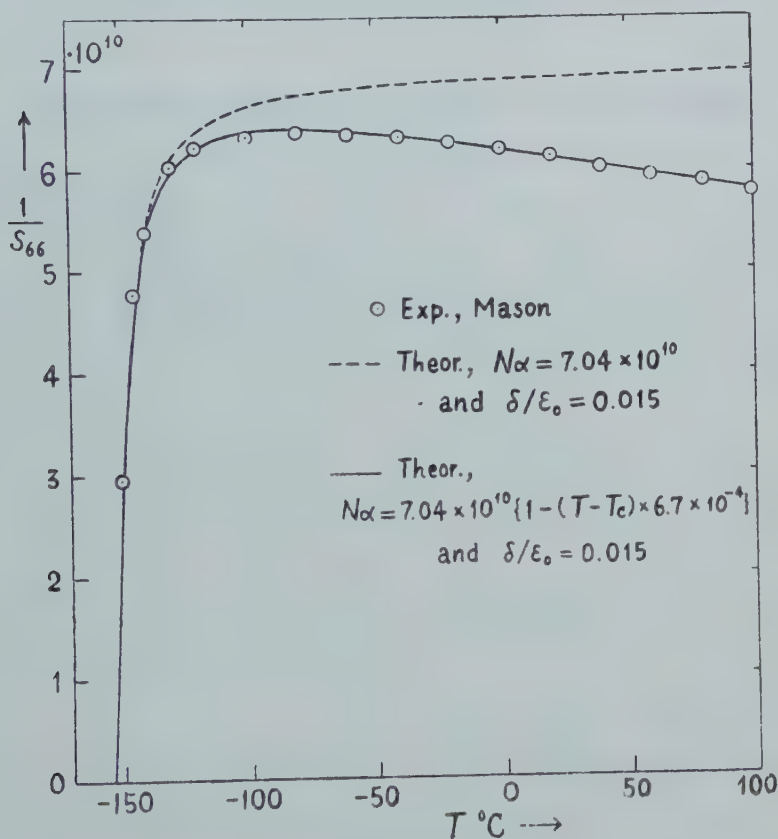


Fig. 3.

Elastic constant. Points are due to Mason. The broken curve is calculated by taking $N\alpha = 7.04 \times 10^{10}$, independent of temperature, and $\delta/\epsilon_0 = 0.015$, and the full curve is calculated by taking Eq. (22) and $\delta/\epsilon_0 = 0.015$.

corresponds just to $T_c - T_c' = 3.5^\circ$. Thus we see that all the observed data are consistently represented by our theory.

In spite of all the trials, we could not find a good fitting of our theoretical formulas to the experiment for the case of $\text{NH}_4\text{H}_2\text{PO}_4$.

6. Summary.

An extension of Slater's theory of KH_2PO_4 to the case of a strained crystal is given. Assuming that the energies of the $(\text{H}_2\text{PO}_4)^-$ groups orientated to $+c$ and $-c$, with respect to the energies of those orientated perpendicular to the c -axis, are given by $\epsilon_+ = -\epsilon_0 - \beta x_y$ and $\epsilon_- = -\epsilon_0 + \beta x_y$, respectively, and adding to the free energy expression a term which represents a normal elastic energy and other terms, formulas for the transition temperature, dielectric, piezoelectric and elastic constants, etc. are derived. It is shown that all the observed data, especially those given by Mason, are very well and consistently represented by these formulas.

Appendix. A Consideration on the Local Field Factor.

In Slater's theory, as in ours, the polarization of the crystal associated with the hydrogen bond system is written as $N\mu z$, where $z = (N_+ - N_-)/N$, N_+ and N_- being the number of $(\text{H}_2\text{PO}_4)^-$ groups directed towards $+c$ and $-c$, respectively. The dipole moment μ , however, cannot be identified with the dipole moment of an $(\text{H}_2\text{PO}_4)^-$ group. A change of the arrangement of protons in the hydrogen bonds system does not mean the rotation of these groups, but it means the sliding of protons along the lines which connect each two oxygens of the different PO_4 -groups. Since these lines are almost perpendicular to the c -axis (inclined about 4° with the horizontal (001)-plane), a change of the positions of protons does not produce an appreciable polarization in the direction of c . The dipole moment μ must thus be ascribed to the polarization induced in a PO_4 group by the formation of an $(\text{H}_2\text{PO}_4)^-$ group directed to $+c$ or $-c$. It may include in addition the dipole moments induced on a potassium ion and an $(\text{H}_2\text{PO}_4)^-$ by the electric field of their surroundings. The latter may change from ion to ion, but its mean value may approximately be proportional to z . (More strictly, it may be a function of z and ν_0 .)

In Slater's theory, a local field factor α was introduced into the susceptibility in the form $\chi = \alpha\chi_0 + \chi_1$, where χ_0 is the susceptibility due to the hydrogen bonds system and χ_1 is that due to other parts. To show the incorrectness of this assumption, let us take a simple system consisting of two polarizable dipoles. Let the magnitudes of the moments of these dipoles in the absence of external electric field be μ_1 and μ_2 , and let the magnitude they would have is the case that the interactions between them were out off be μ_1^0 and μ_2^0 . The difference $\mu_1 - \mu_1^0$ is of

course induced by the electric field due to the second dipole at the position of the first dipole. $\mu_2 - \mu_2^0$ can be interpreted similarly. If an external electric field E is imposed on this system, further polarization p_1 and p_2 will occur. The energy of the system then will be given by

$$U = -(\mu_1 + p_1)E - (\mu_2 + p_2)E + (\text{dipole-dipole interaction}) \\ + \frac{1}{2a_1}(\mu_1 + p_1 - \mu_1^0)^2 + \frac{1}{2a_2}(\mu_2 + p_2 - \mu_2^0)^2,$$

where a_1, a_2 are the polarizabilities of the dipoles. The second term is proportional to $(\mu_1 + p_1)(\mu_2 + p_2)$, and can be interpreted as the potential energy of the first dipole in the electric field of the second one, or, inversely, the potential energy of the second dipole in the electric field of the first one. We shall write it as

$$-a(\mu_1 + p_1)(\mu_2 + p_2).$$

On minimizing U with respect to $\mu_1 + p_1$, we have

$$\mu_1 + p_1 - \mu_1^0 = a_1(E + a(\mu_2 + p_2)) \\ = a_1(E + \text{electric field due to the second dipole}).$$

Similarly, we have

$$\mu_2 + p_2 - \mu_2^0 = a_2(E + a(\mu_1 + p_1)) \\ = a_2(E + \text{electric field due to the first dipole}).$$

Now, the quantities μ_1 and μ_2 can be determined by putting $E=0$ and $p_1=p_2=0$. We have

$$\mu_1 - \mu_1^0 = a_1 a \mu_2, \quad \mu_2 - \mu_2^0 = a_2 a \mu_1, \\ \mu_1 = \frac{\mu_1^0 + a_1 a \mu_2^0}{1 - a_1 a_2 a^2}, \quad \mu_2 = \frac{\mu_2^0 + a_2 a \mu_1^0}{1 - a_1 a_2 a^2}.$$

Hence,

$$p_1 = \frac{a_1 + a a_1 a_2}{1 - a_1 a_2 a^2} E, \quad p_2 = \frac{a_2 + a a_2 a_1}{1 - a_1 a_2 a^2} E.$$

Putting these formulas into U , we have

$$U = -\mu_1 E - \mu_2 E - \frac{1}{2} \chi_1 E^2 + \text{const.},$$

where the const. is the term independent of the field, and

$$\chi_1 = \frac{a_1 + a_2 + 2a a_1 a_2}{1 - a^2 a_1 a_2}.$$

The polarization of the system can be derived by differentiating U with respect

to E . We have

$$P = -\partial U / \partial E = \mu_1 + \mu_2 + p_1 + p_2 = \mu_1 + \mu_2 + \chi_1 E.$$

Thus it is the sum over the dipole moments and the induced polarizations; we see that the local field factor a disappears from U and P , and thus from $\chi = \partial P / \partial E$. Extension of our results to a system of more than two dipoles presents no difficulty.

In the case of KH_2PO_4 , the potential energy U is given by

$$U = -(N\mu z + P_1) E + \frac{1}{2} \chi_1 P_1^2,$$

if P_1 is the additional polarization of the system due to the external field and χ_1 is the susceptibility connected with the polarizabilities and the displacements of the ions. On minimizing U with respect to P_1 we have $P_1 = \chi_1 E$, and

$$U = -N\mu z E - \frac{1}{2} \chi_1 E^2.$$

In a strained crystal, a polarization proportional to the strain and independent of the hydrogen bonds configuration and of the external field may exist. Let this polarization be $\zeta_1 x_y$. The additional energy of the crystal due to it consists of the interaction between it and the external field, namely $-\zeta_1 x_y E$, the interaction between it and $N\mu z$ which can be absorbed into the term $-N\beta x_y z$, and the interaction between it and P_1 which can be absorbed into the term $-\zeta_1 x_y E$. Thus the total energy of the crystal can be written as Eq. (2).

The present work is aided by the research grant delivered from the Ministry of Education.

Note added in proof: We came aware of an experimental research by the method of X-ray analysis held by A. R. Ubbelohde and I. Woodward (Proc. Roy. Soc. **A188** (1949), 358). They show that the crystal deforms by an angle of $27'$ with the spontaneous polarization, in complete agreement with our theory ($23'$ to $30'$, § 4).

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Note on the Finite Extension of Electron.

Katurō SAWADA.

Department of Physics, Kyoto University

(Received April 10, 1949)

Since the fine analysis of self-energy problem by Weisskopf⁽¹⁾, the fact that the electron has finite extension over the region \hbar/mc has been used to calculate the reaction problem. But the positive proof for this conventional model is not yet given. In this paper we shall discuss the explicit form of electron.

By the way, the unique method of separation of self-energy which is independent of the external field applied on the electron will also be given.

§ 1. Introduction.

According to the analysis of Weisskopf⁽¹⁾, the self-energy of electron consists of two parts; one is the interaction of bounded field produced by the electron with itself, and the other is the interaction energy between the induced current by the forced oscillation of electron by zero-point amplitude of the radiation field and zero-point amplitude itself. The former interaction between bounded field and source can be treated in the alternative way; we now treat it as the interaction between emitted field and the current induced by the recoil force in the emission of radiation.

The emission and reabsorption of virtual photon can be interpreted as follows; there are zero-point amplitude of radiation fluctuation \mathbf{A} , this acts on the electron inducing forced oscillation ($m\ddot{\mathbf{x}} = -e\dot{\mathbf{A}}$ in classical theory), then this oscillating electron emits radiation whose strength is to be ruled by $\mathbf{E}' \propto \ddot{\mathbf{x}}$, so that the acting potential \mathbf{A} and potential of emitted radiation \mathbf{A}' is in phase, and since the system is stationary the amplitude of emitted and acting potential is to be the same ($\mathbf{A} \equiv \mathbf{A}'$). Then the electron suffers recoil force in the course of emission of radiation which is proportional to $\ddot{\mathbf{x}}$ in classical theory. So, if we denote the potential which produces this force by $\bar{\mathbf{A}}$, it has just $\pi/2$ shifted phase from \mathbf{A} and \mathbf{A}' . Thus the electron suffers two forces due to acting potential \mathbf{A} and recoil potential $\bar{\mathbf{A}}$, and the induced current by these forces interacts with \mathbf{A} and \mathbf{A}' respectively. But, since \mathbf{A} and \mathbf{A}' are the selfsame the current interacts as well with zero-point vacuum fluctuation of radiation. Though, with those emission and absorption the electron does not change its energy, but may change its inner variable. So that the diagonal part of this interaction density between induced current and zero-point amplitude gives the self-energy and non-diagonal part gives radiation correction

matrix element of elastic transition from one hyperbolic orbit of electron to another hyperbolic orbit with the same energy. After all, we can conclude that, in one electron theory, electron suffers two sorts of force.

In positron theory, however, there exist infinite electrons which just fill negative energy states, and this results in diminishing the freedom of electron. This result appears as a diminish of the force acting on the electron, since some sorts of forced oscillation of electron are forbidden by Pauli exclusion principle. This diminish of force is the same thing as the manifestation that the electron has some extension and acting force remains the same.

By analysing the self-energy problem as above, we can construct the equation of motion for electron which is oscillated by recoil force due to emission of radiation and force exerted by the zero-point amplitude of radiation fluctuation. And if the electron has any extension over space, the source function will appear in this equation in the interaction term; in fact, in one electron theory the extension is delta-functional form and in positron theory the form is that extend \hbar/mc around the center of the electron.

§ 2. Reformation of perturbation formula for radiation interaction.

It is not convenient to treat the self-energy problem of photon field in the usual form of separating static and dynamic self-energy, so we firstly rewrite it in the more convenient form firstly, the self-energy of the electron in state i or the elastic transition matrix element from state i to state f ($E_i = E_f$) is given by

$$2\pi e^2 \sum_{\nu} \sum_{\lambda} \sum_l \frac{1}{\nu_0(E_0 - E_l - \nu_0)} (f | (\alpha \mathbf{l}_{\lambda}) e^{i\nu x} | l) (l | (\alpha \mathbf{l}_{\lambda}) e^{-i\nu x} | i) + \frac{e^2}{2} \iint \sum_l \varphi_f^*(\mathbf{x}) \varphi_i(\mathbf{x}) \frac{1}{|\mathbf{x} - \mathbf{x}'|} \varphi_i^*(\mathbf{x}') \varphi_i(\mathbf{x}') d\mathbf{x} d\mathbf{x}' \quad (1)$$

in one electron theory; and in positron theory it is given by

$$2\pi e^2 \sum_{\nu} \sum_{\lambda} \sum_l^{(+)} \frac{1}{\nu_0(E_0 - E_l - \nu_0)} (f | (\alpha \mathbf{l}_{\lambda}) e^{i\nu x} | l^{(+)} (l^{(+)} | (\alpha \mathbf{l}_{\lambda}) e^{-i\nu x} | i) + 2\pi e^2 \sum_{\nu} \sum_{\lambda} \sum_l^{(-)} \frac{1}{\nu_0(E_0 - E_l - \nu_0)} (f | (\alpha \mathbf{l}_{\lambda}) e^{i\nu x} | l^{(-)} (l^{(-)} | (\alpha \mathbf{l}_{\lambda}) e^{-i\nu x} | i) + \frac{e^2}{2} \iint \left(\sum_l^{(+)} - \sum_l^{(-)} \right) \varphi_f^*(\mathbf{x}) \varphi_i(\mathbf{x}) \frac{1}{|\mathbf{x} - \mathbf{x}'|} \varphi_i^*(\mathbf{x}') \varphi_i(\mathbf{x}') d\mathbf{x} d\mathbf{x}' \quad (2)$$

where E_0 is the energy of state i and f , and $(m/F/n)$ is the matrix element between state m and n of electron, \mathbf{l}_{λ} is the polarization vector of photon which is perpendicular to ν . If we take $i=f$ in the above expression, we obtain the self-energy.

Now, following Weisskopf, we write down the one-electron self-energy as

$$W_{st} = 2\pi e^2 \sum_{\nu} \sum_i \frac{1}{\nu_0^2} (f | e^{i\nu x} | l) (l | e^{-i\nu x} | i)$$

$$W_{el} - W_{mag} = 2\pi e^2 \sum_{\nu} \sum_{\lambda} \sum_i \frac{1}{(E_0 - E_l)^2 - \nu_0^2} (f | (a l_{\lambda}) e^{i\nu x} | l) (l | (a l_{\lambda}) e^{-i\nu x} | i)$$

$$W_{nuct} = 2\pi e^2 \sum_{\nu} \sum_{\lambda} \sum_i \frac{1}{\nu_0} \frac{E_0 - E_l}{(E_0 - E_l)^2 - \nu_0^2} (f | (a l_{\lambda}) e^{i\nu x} | l) (l | (a l_{\lambda}) e^{-i\nu x} | i) \quad (3)$$

Taking into account the following relation

$$\sum_{\lambda} (f | (a l_{\lambda}) e^{i\nu x} | l) (l | (a l_{\lambda}) e^{-i\nu x} | i) = (f | a \cdot e^{i\nu x} | l) (l | a \cdot e^{-i\nu x} | i) -$$

$$- \frac{1}{\nu^2} (f | (a \nu) e^{i\nu x} | l) (l | (a \nu) e^{-i\nu x} | i)$$

$$= (f | a \cdot e^{i\nu x} | l) (l | a \cdot e^{-i\nu x} | i) + \frac{1}{\nu^2} (f | [H_0 + V, e^{i\nu x}]_- | l) (l | [H_0 + V, e^{-i\nu x}]_- | i)$$

$$= (f | a \cdot e^{i\nu x} | l) (l | a \cdot e^{-i\nu x} | i) - \left(1 + \frac{(E_0 - E_l)^2 - \nu^2}{\nu^2}\right) (f | e^{i\nu x} | l) (l | e^{-i\nu x} | i), \quad (4)$$

where $H_0 = a P + \beta m$, $V(x)$ is the arbitrary external field acting on the electron, we can rewrite (3) as

$$W_{st} + W_{el} - W_{mag} = 2\pi e^2 \sum_{\nu} \sum_i \frac{1}{(E_0 - E_l)^2 - \nu_0^2} \{ (f | a \cdot e^{i\nu x} | l) (l | a \cdot e^{-i\nu x} | i) -$$

$$- (f | e^{i\nu x} | l) (l | e^{-i\nu x} | i) \},$$

$$W_{nuct} = 2\pi e^2 \sum_{\nu} \sum_i \frac{1}{\nu_0} \frac{E_0 - E_l}{(E_0 - E_l)^2 - \nu_0^2} \{ (f | a \cdot e^{i\nu x} | l) (l | a \cdot e^{-i\nu x} | i) -$$

$$- (f | e^{i\nu x} | l) (l | e^{-i\nu x} | i) \} - 2\pi e^2 \sum_{\nu} \sum_i \frac{E_0 - E_l}{\nu_0^3} (f | e^{i\nu x} | l) (l | e^{-i\nu x} | i), \quad (5)$$

The second term of W_{nuct} vanishes since

$$- 2\pi e^2 \sum_{\nu} \sum_i \frac{E_0 - E_l}{\nu_0^3} (f | e^{i\nu x} | l) (l | e^{-i\nu x} | i)$$

$$= - 2\pi e^2 \sum_{\nu} \sum_i \frac{1}{\nu_0^3} (f | [H_0 + V, e^{i\nu x}]_- | l) (l | e^{-i\nu x} | i)$$

$$= - 2\pi e^2 \sum_{\nu} \frac{1}{\nu_0^3} (f | (a \nu) | i)$$

and the last expression vanishes after integration over all direction of ν .

So we obtain for self-energy the following expressions:

$$\begin{aligned}
 W_{st} + W_{el} - W_{mag} &= 2\pi e^2 \sum_{\nu} \sum_l \frac{1}{(E_0 - E_l)^2 - \nu_0^2} \{ (f | \alpha \cdot e^{i\nu z} | l) (l | \alpha \cdot e^{-i\nu z} | i) \\
 &\quad - (f | e^{i\nu z} | l) (l | e^{-i\nu z} | i) \}, \\
 W_{fluct} &= 2\pi e^2 \sum_{\nu} \sum_l \frac{1}{\nu_0} \frac{E_0 - E_l}{(E_0 - E_l)^2 - \nu_0^2} \{ (f | \alpha \cdot e^{i\nu z} | l) (l | \alpha \cdot e^{-i\nu z} | i) \\
 &\quad - (f | e^{i\nu z} | l) (l | e^{-i\nu z} | i) \}, \quad (6)
 \end{aligned}$$

$$\begin{aligned}
 W_{st} + W_{el} - W_{mag} &= 2\pi e^2 \sum_{\nu} \sum_l \frac{E_l}{|E_l|} \frac{1}{(E_0 - E_l)^2 - \nu_0^2} \{ (f | \alpha \cdot e^{i\nu z} | l) (l | \alpha \cdot e^{-i\nu z} | i) \\
 &\quad - (f | e^{i\nu z} | l) (l | e^{-i\nu z} | i) \}, \\
 W_{fluct} &= 2\pi e^2 \sum_{\nu} \sum_l \frac{1}{\nu_0} \frac{E_0 - E_l}{(E_0 - E_l)^2 - \nu_0^2} \{ (f | \alpha \cdot e^{i\nu z} | l) (l | \alpha \cdot e^{-i\nu z} | i) \\
 &\quad - (f | e^{i\nu z} | l) (l | e^{-i\nu z} | i) \}. \quad (7)
 \end{aligned}$$

in one electron and in positron theory respectively.

§ 3. Derivation of the form of the electron.

We start with one electron theory. As has been shown by Weisskopf the part $W_{st} + W_{el} - W_{mag}$ can be interpreted as due to the interaction between produced field and its source, this can be seen by integrating (6) over ν ;

$$\ll -\frac{1}{4} \int \{ [\rho, A_0'']_+ - [j, A'']_+ \} d\mathbf{x} \gg_{\text{time average}} \quad (3.1)$$

where A_0'' and A'' is given by;

$$\square A_0'' = 4\pi \rho, \quad \square A'' = 4\pi j \quad (3.2)$$

The part W_{fluct} is the interaction energy between induced current by zero-point amplitude of radiation field and zero-point amplitude itself.

But as mentioned in the introduction, we can treat the first part in different way, that is the interaction of recoil current induced by the emission of radiation and the emitted field. In this form, the energy can be written in the following form instead of (3.1):

$$\ll \frac{1}{4} \int \{ [\rho'', A_0]_+ - [j'', A]_+ \} d\mathbf{x} \gg_{\text{time average}} \quad (3.3)$$

since the emitted radiation is in phase with acting zero-point amplitude and has the same amplitude, we do not distinguish between emitted and acting field and write them as A_0 and A , as mentioned in the introduction. The charge-current density ρ'' and j'' is the recoiled charge-current in the emission of radiation. Then we can determine whether or not the form of electron is delta-

functional type by deducing the equation of motion of recoiled electron wave; this will be given in the form:

$$(\alpha p + \beta m + V(x) + e \int U(x-x') (\bar{A}_0(x', x_0) - \alpha \bar{A}(x', x_0)) dx') \varphi'' = i \frac{\partial}{\partial x_0} \varphi'' \quad (3.4)$$

where $U(x-x')$ is the extension function of the electron for recoil of radiation and \bar{A}_0 and \bar{A} are recoil potentials and have $\pi/2$ shifted phase to A_0 and A , as mentioned in the introduction, and will be imaginary quantity because the recoil acts to damp the oscillation of electron. We now proceed to calculate $U(x-x')$ and determine the forms of \bar{A}_0 and \bar{A} .

We return to (6) of §. 2 and first show that

$$\begin{aligned} W_{st} + W_{el} - W_{mag} &= 2\pi e^2 \sum_{\nu} \sum_l \frac{1}{2\nu_0} \left(\frac{1}{E_0 - E_l - \nu_0} - \frac{1}{E_0 - E_l + \nu_0} \right) \\ &\quad \{ (f | \alpha \cdot e^{i\nu x} | l) (l | \alpha \cdot e^{-i\nu x} | i) - (f | e^{i\nu x} | l) (l | e^{-i\nu x} | i) \}, \\ &= \frac{1}{4} \sum_{\nu} \sum_l \frac{2\pi e^2}{\nu_0} \left\{ (f | \alpha \cdot e^{i\nu x} | l) (l | \frac{\alpha \cdot e^{-i\nu x}}{E_0 - E_l - \nu_0} - \frac{\alpha \cdot e^{-i\nu x}}{E_0 - E_l + \nu_0} | i) \right. \\ &\quad + (f | \frac{\alpha \cdot e^{i\nu x}}{E_0 - E_l - \nu_0} - \frac{\alpha \cdot e^{i\nu x}}{E_0 - E_l + \nu_0} | l) (l | \alpha \cdot e^{-i\nu x} | i) \\ &\quad - (f | e^{i\nu x} | l) (l | \frac{e^{-i\nu x}}{E_0 - E_l - \nu_0} - \frac{e^{-i\nu x}}{E_0 - E_l + \nu_0} | i) \\ &\quad \left. - (f | \frac{e^{i\nu x}}{E_0 - E_l - \nu_0} - \frac{e^{i\nu x}}{E_0 - E_l + \nu_0} | l) (l | e^{-i\nu x} | i) \right\}, \end{aligned}$$

This form allows us to write it in the following form, by taking Schwinger's conventional definition of vacuum for radiation field, $A_0^{(-)} \Psi = A^{(-)} \Psi = 0$;

$A_0(x, x_0) = \sum (A_{0\nu}^{(+)} e^{-i\nu x + i\nu_0 x_0} + A_{0\nu}^{(-)} e^{i\nu x - i\nu_0 x_0})$, similar expansion for A .

$$[A_{0\nu}^{(+)}, A_{0\nu}^{(-)}] = [A_{\nu}^{(-)}, A_{\nu}^{(+)}] = \frac{2\pi}{\nu_0}$$

$$\begin{aligned} W_{st} + W_{el} - W_{mag} &= \frac{1}{4} \sum_l \left\{ [(f | e A_0 - e A A | l), (l | \sum_{\nu} \frac{(e A_{0\nu}^{(+)} - e A A_{\nu}^{(+)})}{E_0 - E_l - \nu_0} e^{-i\nu x} \right. \\ &\quad \left. - \sum_{\nu} \frac{(e A_{0\nu}^{(-)} - e A A_{\nu}^{(-)})}{E_0 - E_l + \nu_0} e^{i\nu x} | i)]_+ + \text{Hermitian Conjugate} \right\}, \end{aligned}$$

because

$$= \frac{1}{4} \sum_l \sum_{\nu} \frac{2\pi e^2}{\nu_0} \left\{ - (f | e^{i\nu x} | l) (l | \frac{e^{-i\nu x}}{E_0 - E_l - \nu_0} | i) + (f | e^{-i\nu x} | l) (l | \frac{e^{i\nu x}}{E_0 - E_l + \nu_0} | i) \right\}$$

$$\begin{aligned}
& + (f | \frac{e^{-i\nu x}}{E_0 - E_i + \nu_0} | l) (l | e^{i\nu x} | i) - (f | \frac{e^{i\nu x}}{E_0 - E_i - \nu_0} | l) (l | e^{-i\nu x} | i) \\
& + (f | \alpha \cdot e^{i\nu x} | l) (l | \frac{\alpha \cdot e^{-i\nu x}}{E_0 - E_i - \nu_0} | i) - (f | \alpha \cdot e^{-i\nu x} | l) (l | \frac{e^{i\nu x}}{E_0 - E_i + \nu_0} | i) \\
& - (f | \frac{\alpha \cdot e^{-i\nu x}}{E_0 - E_i + \nu_0} | l) (l | \alpha \cdot e^{i\nu x} | i) + (f | \frac{\alpha \cdot e^{i\nu x}}{E_0 - E_i - \nu_0} | l) (l | \alpha \cdot e^{-i\nu x} | i) \Big\},
\end{aligned}$$

Then this is the time average of the following expression; for the state one electron in state i transitions to state f with same energy and no radiation field present; (for $i=f$, this is a part for the self-energy)

$$\ll \frac{1}{4} \int \{ [\rho'', A_0]_+ - [j'', \mathbf{A}]_+ \} dx \gg \text{time average} \quad (3.5)$$

where

$$\rho'' = e(\varphi^* \varphi'' + \varphi^{*''} \varphi), \quad j'' = e(\varphi^* \alpha \varphi'' + \varphi^{*''} \alpha \varphi)$$

$$\varphi'' = \sum_m a_m \varphi_m''; \quad a_m a_n^* + a_n^* a_m = \delta_{mn}$$

$$(\alpha p + \beta m + V(x)) \varphi_m = i \frac{\partial}{\partial x_0} \varphi_m$$

$$\varphi_m''(x, x_0) = \varphi_m(x, x_0) + \sum_l \varphi_l(x) \cdot e^{-iK_m x_0}.$$

$$\cdot \left(l | \sum \frac{e A_{0\nu}^{(+)} - e \alpha \mathbf{A}_\nu^{(+)}}{E_m - E_l - \nu_0} e^{-i\nu x + i\nu_0 x_0} - \sum_\nu \frac{e A_{0\nu}^{(-)} - e \alpha \mathbf{A}_\nu^{(-)}}{E_m - E_l + \nu_0} e^{i\nu x - i\nu_0 x_0} | m \right), \quad (3.6)$$

$$\begin{aligned}
\varphi_m^{*''}(x, x_0) &= \varphi_m^*(x, x_0) + \sum_l (m | - \sum_\nu \frac{e A_{0\nu}^{(+)} - e \alpha \mathbf{A}_\nu^{(+)}}{E_m - E_l + \nu_0} e^{-i\nu x + i\nu_0 x_0} \\
&+ \sum_\nu \frac{e A_{0\nu}^{(-)} - e \alpha \mathbf{A}_\nu^{(-)}}{E_m - E_l - \nu_0} e^{i\nu x - i\nu_0 x_0} | l) \cdot e^{iK_m x_0} \varphi_l^*(x). \quad (3.6')
\end{aligned}$$

This is the recoiled electron wave, which is the first order solution of the following equation of motion for electron;

$$(\alpha p + \beta m + V(x) + e(\bar{A}_0(x, x_0) - \alpha \bar{\mathbf{A}}(x, x_0))) \varphi_m'' = i \frac{\partial}{\partial x_0} \varphi_m'' \quad (3.7)$$

$$\varphi_m^{*''}(\alpha p + \beta m + V(x) - e(\bar{A}_0(x, x_0) - \alpha \bar{\mathbf{A}}(x, x_0))) = -i \frac{\partial}{\partial x_0} \varphi_m^{*''} \quad (3.7')$$

where A_0 and $\bar{\mathbf{A}}$ is given by;

$$\begin{aligned}
\bar{A}_0(x, x_0) &= \sum_\nu (A_{0\nu}^{(+)} e^{-i\nu x + i\nu_0 x_0} - A_{0\nu}^{(-)} e^{i\nu x - i\nu_0 x_0}); \quad \bar{A}_0^* = -\bar{A}_0 \\
\bar{\mathbf{A}}(x, x_0) &= \sum_\nu (\mathbf{A}_\nu^{(+)} e^{-i\nu x + i\nu_0 x_0} - \mathbf{A}_\nu^{(-)} e^{i\nu x - i\nu_0 x_0}); \quad \bar{\mathbf{A}}^* = -\bar{\mathbf{A}} \quad (3.8)
\end{aligned}$$

(3.8) is the definition of recoil wave field acting on the electron, it is out of

phase just $\pi/2$ from A_0 and \mathbf{A} and is imaginary quantity, the reason is stated before.

Secondly, W_{direct} is given, analogous to Weisskopf's result, by the time average of the following expression;

$$\ll -\frac{1}{4} \int \{ [\rho', A_0]_+ - [\mathbf{j}', \mathbf{A}]_+ \} d\mathbf{x} \gg \quad (3.9)$$

$$\rho' = e(\varphi^* \varphi' + \varphi'^* \varphi); \quad \mathbf{j}' = e(\varphi^* \mathbf{a} \varphi' + \varphi'^* \mathbf{a} \varphi),$$

$$\varphi' = \sum_m a_m \varphi_m'; \quad a_m a_n^* + a_n^* a_m = \delta_{mn}$$

$$\varphi_m'(\mathbf{x}, x_0) = \varphi_m(\mathbf{x}, x_0) + \sum_l \varphi_l(\mathbf{x}) \cdot e^{-iE_m x_0}.$$

$$\cdot (l | \sum_v \frac{eA_{0v}^{(+)} - e\mathbf{a}\mathbf{A}_v^{(+)}}{E_m - E_l - \nu_0} e^{-i\nu x + i\nu_0 x_0} + \sum_v \frac{eA_{0v}^{(-)} - e\mathbf{a}\mathbf{A}_v^{(-)}}{E_m - E_l + \nu_0} e^{i\nu x - i\nu_0 x_0} | m), \quad (3.10)$$

φ_m' is the first order solution under the force exerted by zero-point fluctuation;

$$(\alpha p + \beta m + V(\mathbf{x}) + e(A_0(\mathbf{x}, x_0) - \mathbf{a}\mathbf{A}(\mathbf{x}, x_0))) \varphi_m' = i \frac{\partial}{\partial x_0} \varphi_m' \quad (3.11)$$

Collecting the above results, the self-energy is given by the interaction of induced and recoil charge-current density with zero-point amplitude of radiation field;

$$W_{\text{self}} = \ll -\frac{1}{4} \int \{ [\bar{\rho}, A_0]_+ - [\bar{\mathbf{j}}, \mathbf{A}]_+ \} d\mathbf{x} \gg \quad (3.12)$$

$$\bar{\rho} = e(\varphi^* \bar{\varphi} + \bar{\varphi}^* \varphi), \quad \bar{\mathbf{j}} = e(\varphi^* \bar{\mathbf{a}} \varphi + \bar{\varphi}^* \mathbf{a} \varphi) \quad (3.13)$$

$$\bar{\varphi} = \sum_m a_m \bar{\varphi}_m; \quad a_m a_n^* + a_n^* a_m = \delta_{mn} \quad (3.14)$$

The equation of motion for $\bar{\varphi}_m$ is given by;

$$(\alpha p + \beta m + V(\mathbf{x}) + e(\bar{A}_0(\mathbf{x}, x_0) - \mathbf{a}\bar{\mathbf{A}}(\mathbf{x}, x_0))) \bar{\varphi}_m = i \frac{\partial}{\partial x_0} \bar{\varphi}_m \quad (3.15)$$

From this equation we can conclude that the electron has delta-functional form in one-electron theory.

In positron theory, we can rewrite (7) of §2 as follows:

$$\begin{aligned} W_{\text{st}} + W_{\text{el}} - W_{\text{mag}} = & 2\pi e \sum_v \sum_l \frac{E_0}{|E_l|} \frac{1}{(E_0 - E_l)^2 - \nu_0^2} \{ (f | \mathbf{a} \cdot e^{i\nu x} | l) (l | \mathbf{a} \cdot e^{-i\nu x} | i) \\ & - (f | e^{i\nu x} | l) (l | e^{-i\nu x} | i) \} - \\ & - 2\pi e^2 \sum_v \sum_l \frac{1}{|E_l|} \frac{E_0 - E_l}{(E_0 - E_l)^2 - \nu_0^2} \{ (f | \mathbf{a} \cdot e^{i\nu x} | l) (l | \mathbf{a} \cdot e^{-i\nu x} | i) \\ & - (f | e^{i\nu x} | l) (l | e^{-i\nu x} | i) \}, \end{aligned}$$

$$W_{\text{net}} = 2\pi e^2 \sum_{\nu} \sum_i \frac{1}{\nu_0} \frac{E_0 - E_i}{(E_0 - E_i)^2 - \nu_0^2} \{ (f | \mathbf{a} \cdot e^{i\nu \mathbf{r}} | l) (l | \mathbf{a} \cdot e^{-i\nu \mathbf{r}} | i) - (f | e^{i\nu \mathbf{r}} | l) (l | e^{-i\nu \mathbf{r}} | i) \}. \quad (3.16)$$

The first term is of the same form as in one electron theory except the appearance of the factor $E_0/|E_i|$, which is due to the limitation of freedom of recoil oscillation by the presence of negative energy electron and gives extension to the electron which was given by Weisskopf.¹⁾ On the contrary, the second term is of the form W'_{net} with density of virtual photon replaced by the density of virtual electron $1/|E_i|$ and sign is reversed to W_{net} , which denotes the diminish in freedom of forced oscillation of electron under zero-point amplitude. Now, writing in the same way as in one-electron treatment;

$$\begin{aligned} W_{\text{st}} + W_{\text{el}} - W_{\text{mag}} = & \frac{1}{4} \sum_i \left\{ [(f | eA_0 - e\mathbf{a}\mathbf{A} | l), (l | \sum_{\nu} \frac{E_0}{|E_i|} \left(\frac{eA_{0\nu}^{(+)} - e\mathbf{a}\mathbf{A}_{\nu}^{(-)}}{E_0 - E_i - \nu_0} e^{-i\nu \mathbf{r}} \right. \right. \\ & \left. \left. - \frac{eA_{0\nu}^{(-)} - e\mathbf{a}\mathbf{A}_{\nu}^{(+)}}{E_0 - E_i + \nu_0} e^{i\nu \mathbf{r}} \right) | i)]_+ + \text{Hermitian conjugate} - \right. \\ & - [(f | eA_0 - e\mathbf{a}\mathbf{A} | l), (l | \sum_{\nu} \frac{\nu_0}{E_i} \left(\frac{eA_{0\nu}^{(+)} - e\mathbf{a}\mathbf{A}_{\nu}^{(+)}}{E_0 - E_i - \nu_0} e^{-i\nu \mathbf{r}} + \right. \\ & \left. \left. + \frac{eA_{0\nu}^{(-)} - e\mathbf{a}\mathbf{A}_{\nu}^{(-)}}{E_0 - E_i + \nu_0} e^{i\nu \mathbf{r}} \right) | i)]_+ + \text{Hermitian conjugate} \right\}, \\ W_{\text{net}} = & \frac{1}{4} \sum_i \left\{ [(f | eA_0 - e\mathbf{a}\mathbf{A} | l), (l | \sum_i \left(\frac{eA_{0\nu}^{(+)} - e\mathbf{a}\mathbf{A}_{\nu}^{(+)}}{E_0 - E_i - \nu_0} e^{-i\nu \mathbf{r}} \right. \right. \\ & \left. \left. + \frac{eA_{0\nu}^{(-)} - e\mathbf{a}\mathbf{A}_{\nu}^{(-)}}{E_0 - E_i + \nu_0} e^{i\nu \mathbf{r}} \right) | i)]_+ + \text{Hermitian conjugate} \right\}. \quad (3.17) \end{aligned}$$

For the free electron only $i=f$ remains and then $E_i=E_{i-\nu}$, and for the bounded electron or slow electron whose energy is negligibly small compared to rest energy of electron E_i can be replaced safely E_i with negligible error since for $\nu_0 \ll m$ this is effectively constant (rest-energy) and for ν_0 sufficiently higher than the ionization energy of electron considered the state l which contributes to matrix element (l/i) is to be considered as perfectly free. Further if we confine our attention to slow electron i and f , $E_0=m$ may be allowed.

Now (3.17) can be written as follows by introducing the function $U^{(w)}(\mathbf{x})$ and $U^{(s)}(\mathbf{x})$;

$$\begin{aligned} \frac{E_0}{|E_i|} e^{-i\nu \mathbf{r}} & \approx \frac{m}{\sqrt{m^2 + \nu^2}} e^{-i\nu \mathbf{r}} = \int U^{(w)}(\mathbf{x} - \mathbf{x}') e^{-i\nu \mathbf{x}'} d\mathbf{x}' \\ U^{(w)}(\mathbf{x} - \mathbf{x}') & = \frac{1}{(2\pi)^3} \int \frac{m}{\sqrt{m^2 + \nu^2}} e^{-i\nu(\mathbf{x} - \mathbf{x}')} d\nu \quad (3.18) \end{aligned}$$

$$\frac{|E_t| - \nu_0}{|E_t|} e^{-i\nu x} \approx \frac{\sqrt{m^2 + \nu^2} - \nu}{\sqrt{m^2 + \nu^2}} e^{-i\nu x} = \int U^{(s)}(x - x') e^{-i\nu x'} dx' \\ U^{(s)}(x - x') = \frac{1}{(2\pi)^3} \int \frac{\sqrt{m^2 + \nu^2} - \nu}{\sqrt{m^2 + \nu^2}} e^{-i\nu(x - x')} d\nu \quad (3.19)$$

$$W_{\text{self}} = -\frac{1}{4} \sum_i \left\{ \left[(f | eA_0 - e\alpha A | l), (l | \int U^{(w)}(x - x') \cdot \sum_{\nu} \left(\frac{eA_{0\nu}^{(+)} - e\alpha A_{\nu}^{(+)} }{E_0 - E_l - \nu_0} e^{-i\nu x'} \right. \right. \right. \\ \left. \left. \left. - \frac{eA_{0\nu}^{(-)} - e\alpha A_{\nu}^{(-)} }{E_0 - E_l + \nu_0} e^{i\nu x'} \right) dx' | i) \right]_+ + \text{Hermitian conjugate} \right. \\ \left. + \left[(f | eA_0 - e\alpha A | l), (l | \int U^{(s)}(x - x') \cdot \sum_{\nu} \left(\frac{eA_{0\nu}^{(+)} - e\alpha A_{\nu}^{(+)} }{E_0 - E_l - \nu_0} e^{-i\nu x'} \right. \right. \right. \right. \\ \left. \left. \left. + \frac{eA_{0\nu}^{(-)} - e\alpha A_{\nu}^{(-)} }{E_0 - E_l + \nu_0} e^{i\nu x'} \right) dx' | i) \right]_+ + \text{Hermitian conjugate} \right\}. \quad (3.20)$$

which is the time average of the following expression:

$$\ll \frac{1}{4} \int \{ [\bar{p}, A_0]_+ - [\bar{j}, A]_+ \} dx \gg, \quad (3.21)$$

$$\bar{p} = e(\varphi^* \bar{\varphi} + \bar{\varphi}^* \varphi); \quad \bar{j} = e(\varphi^* \alpha \bar{\varphi} + \bar{\varphi}^* \alpha \varphi),$$

$$\bar{\varphi} = \sum_m a_m \bar{\varphi}_m, \quad a_m a_n^* + a_n^* a_m = \delta_{mn}$$

$$(ap + \beta m + V(x)) \varphi = i \frac{\partial}{\partial x_0} \varphi$$

The equation of motion for $\bar{\varphi}_m$ is

$$(ap + \beta m + V(x) + e \int U^{(w)}(x - x') (\bar{A}_0(x', x_0) - \alpha \bar{A}(x', x_0)) dx' \\ + e \int U^{(s)}(x - x') (A_0(x', x_0) - \alpha A(x', x_0)) dx') \bar{\varphi}_m = i \frac{\partial}{\partial x_0} \bar{\varphi}_m. \quad (3.22)$$

This equation of motion shows that the electron has finite extension over space in the hole theory, but it is of two sorts; the one is $U^{(w)}(x)$ which is the extension of electron when acted by the damping force of the recoil of emission of radiation and the other is $U^{(s)}(x)$ which is the extension for forced oscillation by zero-point fluctuation of radiation field. The former extension was given by Weisskopf calculating the static self-energy of an electron.

Note that if we take, for free electron, the form function rigorously, it depends on the zero state momentum of the electron as follows:

$$U^{(w)}(x - x') = \frac{1}{(2\pi)^3} \int \frac{\sqrt{m^2 + p_0^2}}{\sqrt{m^2 + (p_0 - \nu)^2}} e^{-i\nu(x - x')} d\nu, \quad (3.23)$$

$$U^{(s)}(x-x')=\frac{1}{(2\pi)^3}\int\frac{\sqrt{m^2+(\mathbf{p}_0-\boldsymbol{\nu})^2}-\nu_0}{\sqrt{m^2+(\mathbf{p}_0-\boldsymbol{\nu})^2}}e^{-i\nu(x-x')}d\nu. \tag{3.24}$$

So that, if the electron moves, its form contracts, and when momentum becomes extremely high the form approaches to delta-functional type.

Further it is also to be noted that the form $U^{(w)}(x)$ is universal, since it is independent of the interacting field, but $U^{(s)}(x)$ changes for field with different mass.

In general, if we denote the interaction Hamilton density as follows :

$$f\psi^\dagger\gamma_{\dots}\chi_{\dots}\psi \qquad (\psi^\dagger=\psi^*\gamma^4) \tag{3.25}$$

where $((\square-\mu^2)\chi_{\dots}=0 \quad \text{for free case.} \tag{3.26}$

and γ_{\dots} is Dirac matrix so chosen that $\gamma_{\dots}\chi_{\dots}$ is scalar ; then the self-energy is given by the time average of the following expression ;

$$\ll\frac{1}{4}\int f[\psi^\dagger\gamma_{\dots}\bar{\psi}+\bar{\psi}^\dagger\gamma_{\dots}\psi,\chi_{\dots}]_+d\mathbf{x}\gg \tag{3.27}$$

and $\bar{\psi}$ satisfies the following equation of motion ;

$$\begin{aligned} (\gamma^\lambda\frac{\partial}{\partial x_\lambda}+m+\gamma^4V(x)+f\int U^{(w)}(x-x')\gamma_{\dots}\chi_{\dots}(x',x_0)d\mathbf{x}' \\ +f\int U^{(s)}(x-x')\gamma_{\dots}\chi_{\dots}(x',x_0)d\mathbf{x}')\bar{\psi}=0 \end{aligned} \tag{3.28}$$

where $U^{(w)}(x)$ is the same as (3.18) or (3.23) and $U^{(s)}(x)$ is the following ;

$$U^{(s)}(x-x')=\frac{1}{(2\pi)^3}\int\frac{\sqrt{m^2+(\mathbf{p}_0-\mathbf{K})^2}-\sqrt{\mu^2+\mathbf{K}^2}}{\sqrt{m^2+(\mathbf{p}_0-\mathbf{K})^2}}e^{-ix(x-x')}d\mathbf{K} \tag{3.29}$$

which depends on the mass μ of the field.

It is interesting to calculate the "radius" of the electron by taking ;

$$2\pi^2\int\frac{U'(\xi)}{|\xi|}d\xi^2=\frac{1}{a}$$

Then the radius of electron for recoil interaction tends to zero inverse logarithmically ;

$$\frac{1}{a^{(w)}}=\int_0^\infty\frac{m}{\sqrt{m^2+\nu^2}}d\nu=\lim_{\nu\rightarrow\infty}m\cdot\log\left(\frac{\sqrt{m^2+\nu^2}+\nu}{m}\right) \tag{3.30}$$

and the radius for forced oscillation is just Compton wave length of the electron in photon case ;

$$\frac{1}{a^{(s)}}=\int_0^\infty\frac{\sqrt{m^2+\nu^2}-\nu}{\sqrt{m^2+\nu^2}}d\nu=m. \tag{3.31}$$

The former extension gives a part of the logarithmic divergence of self-energy

and the latter extension gives finite correction for reaction problem, for example the finite level-shift of Hydrogen atom.

§ 4. Separation of self-energy which is independent of external field.

We now perform the separation of self-energy which has the same matrix element for free and bounded electrons. (the separation of mass type scattering matrix is obtained in the parallel way by taking final state as f);

By taking as the wave equation of electron (3.22);

$$(H + V(x) + U(x)) \bar{\varphi} = i \frac{\partial}{\partial x_0} \bar{\varphi} \quad (4.1)$$

where $H_0 = \alpha p + \beta m$ is the Hamiltonian, for free electrons $V(x)$ is the external field acting on the electron, and $U(x)$ denotes the interaction between radiation field (see Eq. (3.22)).

Now performing canonical transformation successively;

$$\bar{\varphi} = e^{iH_0 x_0} \bar{\varphi}' \quad (4.2)$$

$$(V(x(x_0)) + U(x(x_0), x_0)) \bar{\varphi}' = i \frac{\partial}{\partial x_0} \bar{\varphi}' \quad (4.3)$$

$$\bar{\varphi}' = e^{-i \int_{-\infty}^{x_0} U(x(x_0'), x_0') dx_0'} \chi \quad (4.4)$$

$$(V(x(x_0)) + i \int_{-\infty}^{x_0} [U(x(x_0'), x_0'), V(x, x_0)] dx_0' + O(U^2)) \chi = i \frac{\partial}{\partial x_0} \chi. \quad (4.5)$$

Then returning back to

$$\chi = e^{iH_0 x_0} \bar{\phi} \quad (4.6)$$

We get

$$(H_0 + V(x) + [i \int_{-\infty}^{x_0} [U(x(x_0') - x_0), x_0'] dx_0', V(x)] + O(U^2)) \bar{\phi} = i \frac{\partial}{\partial x_0} \bar{\phi} \quad (4.7)$$

Then the solution of equation of motion (4.1) can be written as follows:

$$\bar{\varphi} = e^{-iH_0 x_0} \cdot e^{-i \int_{-\infty}^{x_0} U(x(x_0'), x_0') dx_0'} \cdot e^{iH_0 x_0} \bar{\phi} \quad (4.8)$$

We need only to take up the first order solution in e (interaction constant contained in $U(x)$), so that by expanding the middle term of (4.8) and denoting

$$(\alpha p + \beta m + V(x)) \varphi = i \frac{\partial}{\partial x_0} \varphi \quad (4.9)$$

We have the solution;

$$\bar{\varphi} = \bar{\phi} - i \int_{-\infty}^{x_0} U(x(x'_0 - x_0), x'_0) dx'_0 \cdot \varphi. \quad (4.10)$$

The second part of this solution gives the matrix element which is independent of $V(x)$ i.e. which has the same matrix element for free and bounded electrons;

$$W_{\text{unres}} = \ll \frac{1}{4} \int \{ [\varphi^* \cdot (-i) \int_{-\infty}^{x_0} U(x(x'_0 - x_0), x'_0) dx'_0 \cdot \bar{\varphi}, eA_0 - e\alpha A]_+ \\ + \text{Hermitian conjugate} \} dx \gg. \quad (4.11)$$

The remainder contains $\bar{\phi}$, and this satisfies (4.7), so this part gives the combined effect of radiation reaction with external field $V(x)$ and hence reactive corrections.

§ 5. Concluding remarks.

We have thus shown that the electron has finite extension in hole theory which contracts when electron moves, the form is different for recoil of emission of radiation and for forced oscillation by zero-point amplitude of radiation field. The former extension is universal but the latter is different for fields with different masses.

By taking non-relativistic approximation in (4.7) and solving $\bar{\phi}$, we have the level shift of hydrogen atom which turns out

$$\frac{e^2}{3\pi} \frac{1}{m^2} \langle \Delta V(x) \rangle = \int_{\epsilon}^{\infty} \frac{V(m^2 + \nu^2) - \nu}{\nu \sqrt{m^2 + \nu^2}} d\nu = \frac{e^2}{3\pi} \frac{1}{m^2} \langle \Delta V(x) \rangle_0 \log \left(\frac{m}{2\epsilon} \right) \quad (5.1)$$

which just coincides with the exact calculation by French and Weisskopf⁽²⁾ and Schwinger⁽³⁾ apart from the relativistic correction and polarization part.

Related problems are now being examined.

In conclusion I say much thanks to Prof. M. Kobayasi for his kind interest in this work.

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Cosmic-Ray Underground I.

Satio HAYAKAWA and Sin-itiirô TOMONAGA

Central Meteorological Observatory and Tokyo Bunrika Daigaku

(Received April 3, 1949)

§ 1. Introduction.

Behaviour of elementary particles in the cosmic-ray in moderate high energy region can be inferred by the analysis of cosmic-ray phenomena underground, but few such attempts have ever been made and all of them have failed in giving a satisfactory understanding of the problem. Several years ago one of us (S.T.) tried without success to explain the remarkable bend of the intensity-depth curve on the basis of our knowledge at that time: on the assumption that this bend is caused by the energy loss of the cosmic-ray particles due to radiation and pair creation. Now later developments of cosmic-ray physics, i.e. the discovery of two kinds of mesons, made it possible to remove this unsolved difficulty by explaining this bend as due to the π -mu decay¹⁾. Thus the existence of the bend is no longer a puzzling fact, but it provides an important clue to determine the life of π -mesons. In this situation it seems to be of importance to get forth more comprehensive analysis of the underground phenomena in order to certify the hypothesis about the model of mesons as well as about their interactions with matter.

We shall henceforth discuss how far one can account for the phenomena on the basis of our current picture without introducing extra hypothetical processes. According to our picture primary protons or heavy nuclei produce π -mesons on colliding with air nuclei; the produced π -mesons disintegrate into μ -mesons on falling down through the atmosphere entirely or partly according as the agent π -meson has lower or higher energy. Thus at the ground surface there are π - and μ -mesons mixed, but π -mesons as well as nucleons can not penetrate deep into underground because of their strong interaction with matter nuclei. Electronic components falling on the ground surface are so highly absorbable that they are, of course, negligible constituents of the deep rays. Only component surviving down to the great depth is μ -mesons including their secondary products, which are in equilibrium with the μ -mesons. The main constituents of these secondary particles are electrons and photons produced by electromagnetic interactions of the μ -mesons. Besides these particles weakly interacting particles such as neutrini amount more and more with increasing depth, but their effects may be too small to be detectable in the depth under consideration. Therefore we may assume that only μ -mesons and their electromagnetic interactions play a

rôle in the phenomena underground.

Electromagnetic interactions of charged particles in high energy region are classified into the following three: collision with atomic electrons or ionization, radiation by acceleration in nuclear Coulomb field and creation of electron-positron pair in nuclear Coulomb field. The first and the second processes depend strongly on the spin and the magnetic moment of the particle in the relativistic region, but the third process, the pair creation, is almost independent on the spin and the magnetic moment²⁾. The frequency of bursts at sea level as well as the fact that the decay electron of mu-meson has a continuous energy spectrum seems to show that this will have spin $1/2$ and no intrinsic magnetic moment, though spin 0 may not be definitively ruled out. The possibility of spin 1 should be excluded because this would result in a much larger burst frequency; but this does not affect the result considerably as far as the energy loss is concerned. The energy loss is mainly due to ionization as has been hitherto considered by many authors, but it is found that in deepest region one must take into account the energy loss caused by the emission of bremsstrahlung and the creation of pair. We shall first estimate the magnitude of these three effects (§2), and then go over to the calculation of the range of the mu-meson as function of its energy (§3). We calculate, on the other hand, the energy spectrum of mu-mesons at sea level according to our picture: the mu-mesons are decay products of pi-mesons which are produced in the atmosphere by the primary particles, presumably by protons or eventually by some heavier nuclei (§4). If we thus find the range-energy relation of mu-mesons and their spectrum at sea level, we obtain the intensity-depth curve which can be compared with experiments (§5). In deriving the spectrum of mu-mesons we first neglect the absorption of pi-mesons in the atmosphere because we know at present only very little about how pi-mesons traverse the atmosphere. It is certain that this absorption is not negligible so that we next discuss to what extent this absorption affects the result (§6).

§2. Elementary processes and energy loss.

As for the first two of the electromagnetic interactions mentioned above, detailed accounts are summarized in Rossi and Greisen's article²⁾, so that we may here refer only to the necessary results. As for the pair creation no remark has hitherto been given so that some mentions seem to be necessary. The differential cross-section for the creation of a pair by an incident charged particle in a nuclear Coulomb field was calculated by various authors. We refer here to the result obtained by Bhabha¹⁾ according to which the differential cross-section for the creation of electron and positron with energy between ϵ , $\epsilon + d\epsilon$ and ϵ' , $\epsilon' + d\epsilon'$ respectively is given by

$$dQ_1 = \frac{8}{\pi} (aZ)^2 r_0^2 \frac{\epsilon^2 + \epsilon'^2 + \frac{2}{3} \epsilon \epsilon'}{(\epsilon + \epsilon')^4} \ln \left(\frac{k \epsilon \epsilon'}{(\epsilon + \epsilon') m c^2} \right) \ln \left(\frac{k' \hat{\xi} m c^2}{\epsilon + \epsilon'} \right) d\epsilon d\epsilon' \\ \text{for } m c^2 < \epsilon, \epsilon' \text{ and } \epsilon \epsilon' / (\epsilon + \epsilon') < \frac{1}{2} \eta m c^2, \quad (1a)$$

$$dQ_2 = \frac{8}{\pi} (aZ)^2 r_0^2 \frac{\epsilon^2 + \epsilon'^2 + \frac{2}{3} \epsilon \epsilon'}{(\epsilon + \epsilon')^4} \ln (k\eta) \ln \left(\frac{k' \hat{\xi} m c^2}{\epsilon + \epsilon'} \right) d\epsilon d\epsilon' \\ \text{for } \frac{1}{2} \eta m c^2 < \epsilon + \epsilon' \text{ and } \epsilon, \epsilon' < \hat{\xi} m c^2, \quad (1b)$$

$$dQ_3 = \frac{8}{\pi} (aZ)^2 r_0^2 \frac{(\hat{\xi} m c^2)^2}{(\epsilon + \epsilon')^4} \ln \left(\frac{k\eta(\epsilon + \epsilon')}{\hat{\xi} m c^2} \right) d\epsilon d\epsilon' \\ \text{for } \hat{\xi} m c^2 < \epsilon, \epsilon' \text{ and } \epsilon + \epsilon' < 2\hat{\xi}^2 m c^2 / \eta, \quad (1c)$$

$$dQ_4 = \frac{8}{\pi} (aZ)^2 r_0^2 \frac{(\hat{\xi} m c^2)^2}{(\epsilon + \epsilon')^4} \ln (2k\hat{\xi}) d\epsilon d\epsilon' \\ \text{for } 2\hat{\xi}^2 m c^2 / \eta < \epsilon + \epsilon' < \hat{\xi} m c^2 \quad (1d)$$

for each specified energy region. Here μ is the mass and $\hat{\xi}$ the Lorentz factor of the incident charged particle, a the fine structure constant, r_0 the classical electron radius, and η the shielding radius of the collided atom: $\eta = 183Z^{-1/3}$, Z being its atomic number. Further, k and k' are numerical constants of the order of unity; the remaining symbols need not be explained. As mentioned above this result holds roughly independently of the spin of the incident particle.

For the sake of simplicity we put

$$\epsilon \epsilon' / (\epsilon + \epsilon') = (\epsilon + \epsilon') / 4 \quad (2)$$

approximately. Then Bhabha's formulae can be reduced to

$$dQ_1 = \frac{8}{\pi} (aZ)^2 r_0^2 \frac{7}{9} \frac{1}{(\epsilon + \epsilon')^2} \ln \left(\frac{k(\epsilon + \epsilon')}{m c^2} \right) \ln \left(\frac{k' \hat{\xi} m c^2}{\epsilon + \epsilon'} \right) d\epsilon d\epsilon' \\ \text{for } m c^2 < \epsilon + \epsilon' < 2\eta m c^2, \quad (1a')$$

$$dQ_2 = \frac{8}{\pi} (aZ)^2 r_0^2 \frac{7}{9} \frac{1}{(\epsilon + \epsilon')^2} \ln (k\eta) \ln \left(\frac{k' \hat{\xi} m c^2}{\epsilon + \epsilon'} \right) d\epsilon d\epsilon' \\ \text{for } 2\eta m c^2 < \epsilon + \epsilon' < \hat{\xi} m c^2, \quad (1b')$$

$$dQ_3 = \frac{8}{\pi} (aZ)^2 r_0^2 \frac{(\hat{\xi} m c^2)^2}{(\epsilon + \epsilon')^2} \ln \left(\frac{k\eta(\epsilon + \epsilon')}{\hat{\xi} m c^2} \right) d\epsilon d\epsilon' \\ \text{for } \hat{\xi} m c^2 < \epsilon + \epsilon' < 2\hat{\xi}^2 m c^2 / \eta, \quad (1c')$$

$$dQ_4 = \frac{8}{\pi} (aZ)^2 r_0^2 \frac{(\xi mc^2)^2}{(\epsilon + \epsilon')^4} \ln(2k\xi) d\epsilon d\epsilon' \quad (1d')$$

for $2\xi^2 mc^2/\eta < \epsilon + \epsilon' < \xi \mu c^2$.

Expressed in this form we can immediately find the cross-section for the production of a pair whose energy lies between ϵ and $\epsilon + d\epsilon$ irrespective to their partition on the electron and the positron:

$$dQ_1' = \frac{8}{\pi} (aZ)^2 r_0^2 \frac{7}{9} \ln\left(\frac{k\epsilon}{mc^2}\right) \ln\left(\frac{k'\xi mc^2}{\epsilon}\right) \frac{d\epsilon}{\epsilon} \quad (3a)$$

for $mc^2 < \epsilon < 2\eta mc^2$,

$$dQ_2' = \frac{8}{\pi} (aZ)^2 r_0^2 \frac{7}{9} \ln(k\eta) \ln\left(\frac{k'\xi mc^2}{\epsilon}\right) \frac{d\epsilon}{\epsilon} \quad (3b)$$

for $2\eta mc^2 < \epsilon < \xi mc^2$,

$$dQ_3' = \frac{8}{\pi} (aZ)^2 r_0^2 (\xi mc^2)^2 \ln\left(\frac{k\eta\epsilon}{\xi mc^2}\right) \frac{d\epsilon}{\epsilon} \quad (3c)$$

for $2\xi mc^2 < \epsilon < 2\xi^2 mc^2/\eta$,

$$dQ_4' = \frac{8}{\pi} (aZ)^2 r_0^2 (\xi mc^2)^2 \ln(2k\xi) \frac{d\epsilon}{\epsilon} \quad (3d)$$

for $2\xi^2 mc^2/\eta < \epsilon < \xi \mu c^2$,

Multiplying each of (3a) to (3d) by ϵ , integrating with respect to ϵ over each specified region and adding the results, we find that the energy loss of the incident particle by the pair creation is given by

$$\left(-\frac{dE}{dx}\right)_{\text{pair}} \approx \frac{8}{\pi} (aZ)^2 r_0^2 \frac{V}{A} \left[\xi mc^2 \left\{ \frac{9}{16} \ln \eta + 1 - \frac{m}{\mu} \ln(2\xi) \right\} - \frac{14}{9} \eta mc^2 \left\{ \ln \xi - \ln(2\eta) + \frac{37}{18} \right\} \right] \quad (4)$$

where V is the Loschmidt number and A the atomic weight of the collided atom. Substituting the numerical values for each constant, and assuming $\mu = 217m$, $Z = 10$ and $A = 20$, (4) gives numerically

$$(-dE/dx)_{\text{pair}} = 3.77 \times 10^{-4} [\xi mc^2 - 24.8 mc^2 \log \xi]. \quad (5)$$

Next we consider other two effects. The ionization is the most important effect over the considerable range of depth. It may be put nearly constant because its logarithmic increasing will be much weakened by Fermi's density effect.⁵⁾ Since the atomic weight and the density of rock and soil are similar to those of carbon, we may use the value for the ionization energy loss for carbon appeared in the paper of Halpern and Hall⁶⁾; the energy of the incident particle

is thereby assumed to be $10^{11} eV$;

$$(-dE/dx)_{\text{ion}} = a = 2.5 \times 10^6 eV \text{ per } g \text{ cm}^{-2}. \quad (6)$$

The radiation loss of a meson is given in the case where the screening is not yet completed (the complete screening sets in at about $5 \times 10^{11} eV$).

$$\left(-\frac{dE}{dx}\right)_{\text{rad}} = 4aZ^2r_0^2 \frac{N}{A} \xi \mu^2 \left[\ln \left(\frac{12\xi}{5Z^{1/3}} \right) - \frac{1}{3} \right]. \quad (7)$$

Numerically it is

$$(-dE/dx)_{\text{rad}} = 3.45 \times 10^{-7} E [\log \xi - 0.10] eV \text{ per } g \text{ cm}^{-2}. \quad (8)$$

§ 3. Range of energetic meson.

Summing up the above three energy losses we obtain the total energy loss

$$(-dE/dx)_{\text{total}} = (-dE/dx)_{\text{ion}} + (-dE/dx)_{\text{rad}} + (-dE/dx)_{\text{pair}}. \quad (9)$$

Then the range of a meson with energy E is calculated immediately by means of

$$x(E) = \int_0^E dE / \left(-\frac{dE}{dx} \right)_{\text{total}} \quad (10)$$

If we approximate (5) and (8) by

$$(-dE/dx)_{\text{pair}} = pE \text{ eV per } g \text{ cm}^{-2} \quad (5')$$

$$\text{with } p = 1.6 \times 10^{-6}$$

and

$$(-dE/dx)_{\text{rad}} = rE \text{ eV per } g \text{ cm}^{-2} \quad (8')$$

$$\text{with } r = 1.0 \times 10^{-6}$$

respectively, the range (10) can be expressed analytically by

$$x(E) = \frac{1}{p+r} \ln \left[1 + \frac{p+r}{a} E \right]. \quad (11)$$

If the energy is so small that $(p+r)E \ll a$, it simplifies into

$$x(E) = E/a, \quad (11')$$

which evidently means that only ionization is effective in lower energy region as usually assumed. The energy for which the radiation and the pair creation losses become important lies at about $10^{11} eV$.

The energy of a meson with the residual range x is given by solving (11):

$$E = \frac{a}{p+r} [\exp \{ (p+r)x \} - 1]. \quad (12)$$

If the integral energy spectrum of mu-mesons at sea level is known, we can immediately obtain the intensity-depth relation for the mu-mesons: Let this spectrum be $F(E)$, then the function of x obtained by substituting E in $F(E)$ by (12) gives just the required intensity-depth relation.

§ 4. Energy spectrum of mu-mesons.

If we assume that $F(E)$ will be represented by a smooth function (e.g. a power law) obtained by an extrapolation from lower energy part, we can not reproduce the bend of the experimental intensity-depth curve, even if the energy loss caused by the radiation and the pair creation is taken into consideration. This is because these effects are still too small at the depth where the bend appears to account for this remarkable phenomenon. Thus the bend of the intensity-depth curve must be attributed to the bend of the energy spectrum itself at sea level. Such a bend will not exist in the primary spectrum because the frequency of large air showers can be well explained by extrapolating the primary spectrum at lower energy to the region of extremely high energy. It is highly improbable that the production of pi-mesons from their primaries is performed in such a way that their spectrum has a remarkable bend while the spectrum of the primaries is of a smooth form. So we must conclude that the bend is the result of the pi-mu decay. The temperature effect and the angular distribution of cosmic ray in deeper region give powerful supports for this interpretation, though Barnothy and Forro attributed these facts to the effect of neutrini⁹⁾.

The energy spectrum of mu-mesons produced by the decay of pi-mesons can be calculated as follows. In the energy region under consideration we may neglect the ionization loss for both kind of mesons as well as the disintegration of mu-mesons. We further neglect the absorption of pi-mesons in the atmosphere, though this is certainly not allowed, leaving the discussion about this effect in the next section.

The primary ray which decreases as $\exp(-l/A)$, A being the mean free path of the primary particles, produces pi-mesons with a probability proportional to $\exp(-l/A) dl$. The pi-meson produced at the atmospheric depth l' is transformed into mu-mesons at the depth l with the probability

$$\begin{cases} 1 - (l'/A)^{1/E_0}, \\ B = (\mu_\pi c / \tau_\pi) (l/\rho) = 3.4 \times 10^2 \text{ BeV per g cm}^{-2}, \end{cases} \quad (13)$$

where μ_π is the mass and τ_π the life of the pi-meson, its energy being denoted by E_0 , and ρ means the average density of air. Multiplying (13) by $\exp(l'/A) dl'$ and integrating over l' , we obtain the probability of existing of a mu-meson at the depth l produced somewhere by a pi-meson of the energy E_0 . The result is

$$L(E_0) = 1 - \exp(-l/\Lambda) - (\Lambda/l)^{B/E_0+1} \Gamma(B/E_0+1, l/\Lambda). \quad (14)^*$$

We now assume that the energy spectrum of pi-mesons when they are produced is

$$g(E_0) dE_0 = \text{const. } E_0^{-\tau-1} dE_0, \\ \tau = 1.8. ** \quad (15)$$

The energy spectrum of mu-mesons at sea level is then given by

$$f(E) dE = dE \int_E^\infty L(E_0) g(E_0) dE_0 / E_0. \quad (16)$$

Since we have, in deriving (16), neglected the energy loss of both kinds of mesons and the decay of mu-mesons, the low energy part of this $f(E)$ is not correct. In fact it shows a deviation from the experimental results. So we take for the low energy part of $f(E)$ the values summarized by Rossi⁹⁾. We use Rossi's curve up to the energy of $3 \times 10^{10} \text{ eV}$ and determine the const. in (16) in such a way that (16) coincides with Rossi's curve at that point. The integral spectrum

$$F(E) = \int_E^\infty f(E') dE' \quad (17)$$

obtained in this way is represented in Fig. 1. The values of mass and life for the pi-meson used are

$$\mu_\pi = 286m, \quad \tau_\pi = 1 \times 10^{-8} \text{ sec.},$$

which were determined at Berkeley for the mesons produced artificially by the cyclotron¹⁰⁾. As for Λ little is known, but we may assume that $\Lambda = 125 \text{ g cm}^{-2.9}$. This value is taken from the experimental data for the altitude dependence of the frequency of stars, bursts and penetrating showers, and may be regarded tentatively as the mean free path of the nucleon component of cosmic-ray, which are the agents of the pi-mesons.

§ 5. Intensity-depth curve.

Substituting (12) into (17) we obtain the intensity-depth curves:

$$I(x) = F(E(x)). \quad (18)$$

The result calculated numerically is plotted in the upper curve of Fig. 2. The

* The formula in the previous letter of one of us (S.H.) must be corrected as above. The life of pi-meson concluded there is erroneous because the energy spectra of pi- and mu-mesons are not properly considered.

** The value of τ may be taken as 2.0, as emphasized by Mr. Nishimura on the basis of the analysis of meson production at the upper atmosphere. If we accept this figure, the intensity-depth curve can be explained by assuming $\tau_\pi = 8.5 \times 10^{-9} \text{ sec.}$ as recently informed by Dr. Gardner to Dr. Taketani.

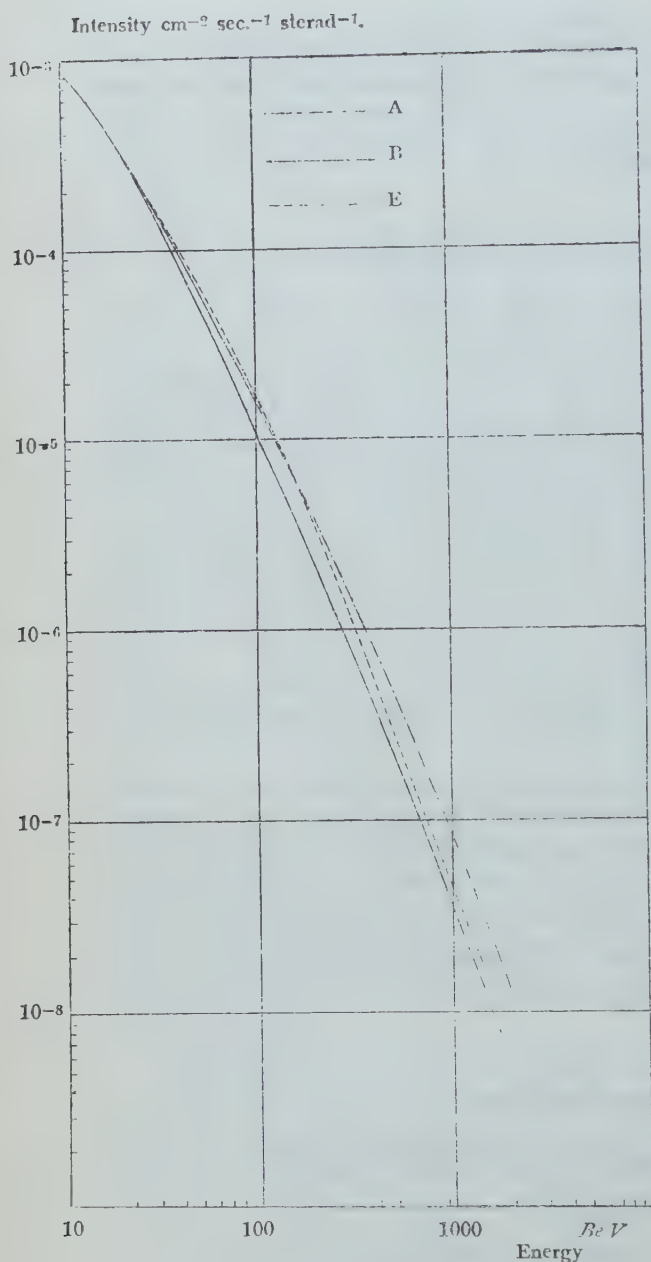


Fig. 1. Integral energy spectrum at sea level.
 A: Calculated by (16) and (17); B: Calculated by (22) and (17). E: Converted by (12) from the experimental intensity-depth relation.

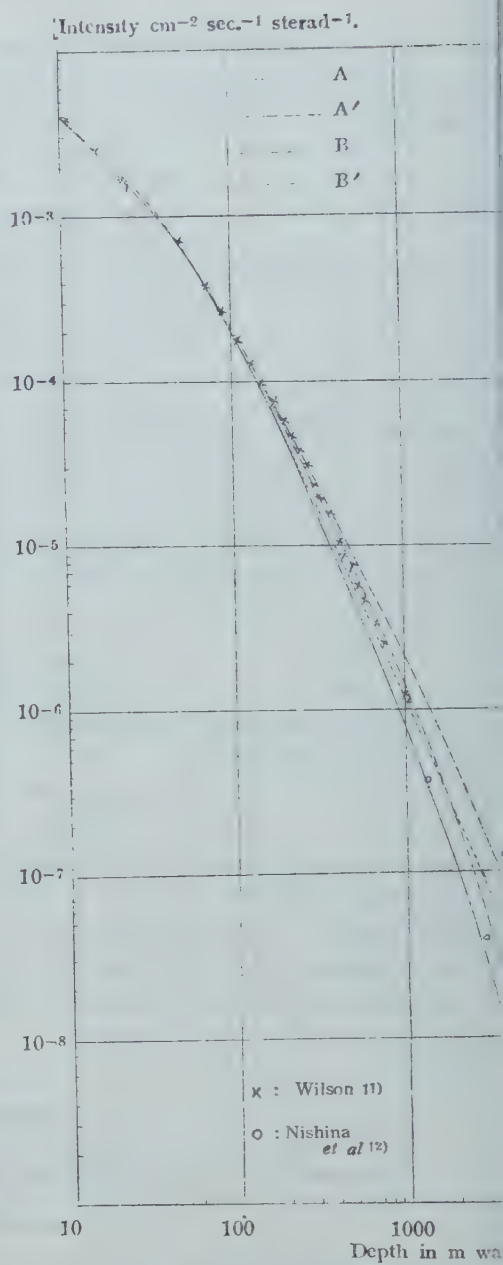


Fig. 2. Intensity-depth relation.
 A: Calculated by (16), (17), (11), and (18). A': Calculated by (22), (17), (11), and (18). B: Calculated by (22), (17), (11), and (18). B': Calculated by (22), (17), (11), and (18). D: Calculated by (22), (17), (11), and (18). D': Calculated by (22), (17), (11), and (18). Experimental points marked by x and o.

experimental points of Wilson¹¹⁾ are also marked on the figure by \times . If we did not take into account the energy losses due to radiation and pair creation, the intensity-depth relation would be of the form represented by the dotted curve, which lies too high showing clearly the importance of these energy losses. This fact is shown most markedly by the experiment carried out by Nishina *et al* at still greater depth (at Shimizu tunnel, 1400 and 3000 m water equivalent). The points marked by \circ represent the result of the latter experiment.

§ 6. Effect of absorption of pi-mesons in the atmosphere.

Before entering into the discussion of our result we must notice that our formula (14) is derived without considering the absorption of pi-mesons in the atmosphere. It is certain that this absorption is appreciable because pi-mesons interact very strongly with nucleons. Unfortunately only little is known about this absorption but it is plausible that pi-mesons will be as absorbable as nucleon component of cosmic rays. So we assume tentatively that the absorption coefficient of pi-mesons in the atmosphere is approximately so large as that of the nucleons. Then the spectrum of pi-meson satisfies the diffusion equation

$$\frac{\partial g(E_0, l)}{\partial l} = -\frac{B}{E_0 l} g(E_0, l) - \frac{1}{A} g(E_0, l) + g_0(E_0, l), \quad (19)$$

$g(E_0, l)$ being the required spectrum, and $g_0(E_0, l)$ the source function for pi-mesons. As mentioned above we assume that the absorption coefficient $1/A$ of the pi-meson is the same as that of the nucleons;

$$A = 125 \text{ g cm}^{-2}, \text{ and } B \text{ is given by (13).}$$

The diffusion equation (19) can be integrated immediately if we make plausible assumption as to the source function $g_0(E_0, l)$:

$$g_0(E_0, l) dE_0 = \text{const. } E_0^{-\tau-1} dE_0 \exp(-l/A). \quad (20)$$

The solution is

$$g(E_0, l) = \text{const. } l \exp(-l/A) E_0^{-\tau-1} (1 + B/E_0). \quad (21)$$

The endrgy spectrum of mu-mesons at the depth l is then found to be

$$\begin{aligned} f(E, l) dE &= dE \int_E^\infty dE_0/E_0 \int_0^l (B/E_0 l') g(E_0, l') dl' \\ &= \text{const. } dE (1 - \exp(-l/A)) \int_E^\infty E_0^{-\tau-2} (1 + E_0/B)^{-1} dE_0. \end{aligned} \quad (22)$$

The spectrum at sea level is given in Fig. 1. Using (22) instead of (17) we obtain the lower curve in Fig. 2.

§ 7. Discussions.

It is seen that the experimental intensity-depth curve lies between the two theoretical curves, one calculated neglecting the absorption of pi-mesons and the other assuming for their absorption coefficient the same value as that for nucleons. The life of pi-mesons was thereby taken from Berkeley experiment. In view of the uncertainty of our knowledge about the production and absorption of pi-mesons as well as of the statistical inaccuracy of their lives, we can not yet draw any decisive conclusion from our result, but our result seems to show that the life of pi-mesons measured at Berkeley for the artificially produced mesons does not contradict with the cosmic-ray data so that we may identify the cosmic-ray mesons with mesons produced at Berkeley.

Our result seems also to show that no extra processes other than the known electromagnetic ones are needed to explain the behaviour of deep rays so far as the present status of our experimental knowledge is concerned. As for the question whether or not some non-electromagnetic processes are necessary to explain the phenomena underground, it will be decisively answered, if we examine whether the spectrum of mu-mesons at sea level required to reproduce the observed intensity-depth curve is able to account for the data concerning the butsts under thick absorber at sea level⁽¹⁰⁾. Also the experimental results for the shower frequency underground provide a clue for this question. Such discussions will be given in later papers. Further the temperature effect and angular distribution of the deep rays will be treated in a separate paper.

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A Remark on Relativistically Invariant Formulation of the Quantum Field Theory.

Suteo KANESAWA and Zirô KOBAYASHI.

(Received April 18, 1949)

1. Introduction and summary.

In a series of papers¹⁾²⁾ Tomonaga and others developed a new formalism of the quantum field theory which was more satisfactory than the current one from the stand-point of the relativistic invariance. The theory was formulated in terms of only concepts possessed of invariant space-time meaning while in the current theory one had to use non-relativistic concepts borrowed from the ordinary quantum mechanics. In the process of derivation of this theory, however, one employed canonical formalism in which time variable was unnecessarily distinguished from space variables. It is the purpose of the present paper to show that one can reach the same formalism without referring to the canonical formulation.

According to the results obtained by applying our theory to several actual cases, the generalized Schrödinger equation in each physical system has the Lorentz invariant form despite of its non-relativistic way of derivation, and moreover the interaction Hamilton density coincides with the interaction Lagrange density except the sign if we neglect its surface-dependent part. These facts suggest to connect directly the interaction Lagrange density to the generalized Schrödinger equation.

In section 2 we show that the same result as the original formalism is obtained by this method without using the canonical momenta. In the fundamental equation of our new formalism

$$\left\{ L_P[C] - \frac{\hbar}{i} \frac{\delta}{\delta C_P} \right\} \Psi[C] = 0, \quad (1.1)$$

the generalized interaction Lagrange density $L_P[C]$ appears instead of the generalized Hamilton density $H_P[C]$ which is related to the former by means of

$$H_P[C] = -L_P[C], \quad (1.2)$$

this function $L_P[C]$ is directly determined from the interaction Lagrange density $L(P)$ as a special solution of the equation for the integrability of (1.1).

$$\left[L_X[C] - \frac{\hbar}{i} \frac{\delta}{\delta C_X}, L_{X'}[C] - \frac{\hbar}{i} \frac{\delta}{\delta C_{X'}} \right] = 0$$

or

$$\frac{\delta L_X[C]}{\delta C_{X'}} - \frac{\delta L_{X'}[C]}{\delta C_X} = -\frac{i}{\hbar} [L_X[C], L_{X'}[C]] \quad (1.3)$$

where X and X' are the two world points which lie in a space-like direction with respect to each other. A general prescription for constructing this special solution $L_P[C]$ is formulated in section 3.

In section 4 we give a proof of the equivalence of our new method of formulation to the old one when the system under consideration is describable in a canonical form. Because the method proposed in section 3 is applicable also when the canonical description is impossible, our method can describe more general interactions than those occurring in the current one. In section 5 we shall give a simple illustration how one can formulate the quantum field theory in such a case.

Strictly speaking there is yet no justification of applying our formalism to this general case, since all we have proved is that our formalism coincides with the ordinary one when the latter is possible, as was kindly pointed out by Dr. Watanabe³⁾. Indeed there can be no criterion with regard to cases when the canonical formalism fails except agreement with experimental facts. But we may be allowed to hope that our method of generalization, whose guiding principle is the integrability of the Tomonaga-Schwinger equation, is perhaps a correct one, because according to our several experiences this condition is an essential feature of the current quantum field theory, so that it will hold in general as far as one is concerned only with a complicated type of coupling and not with those problematic aspects of the theory such as universal length or ultraviolet divergencies.

Further we must emphasize that our special solution obtained by the prescription described in section 3 is not the unique one which makes the fundamental equation (1.1) integrable.⁴⁾ Thus the aim of the present paper lies rather in demonstrating the possibility of including more general kind of interactions in the field theory than those which allow the canonical description. There arise then the questions how the energy-momentum law will be formulated and what the generalization of the stress-energy tensor in this general case. Such questions will be discussed in a later occasion.

2. Some examples for the case where the canonical formulation is possible.

Before entering into individual cases we give the common assumptions which are made throughout all examples. They are as follows: the field variables contained in the interaction Lagrange density $L(X)$ are solutions of the free field equations, consequently, satisfy the same commutation relations with those between free ones. Now, as was shown by Pauli,⁵⁾ the four-dimensional commutation relations between the vacuum field variables can be determined by the requirement that the energy and momentum of the field are quantized so that the

field shows a particle nature.

Example 1. Case of electron field interacting with electro-magnetic field.⁶⁾

The interaction Lagrange density in our case is given by

$$L(X) = e (\phi^* a \phi A), \quad (2.1)$$

As was shown in the paper II, the equation

$$[L(X), L(X')] = 0 \quad (2.2)$$

holds when the world points X and X' lie space-like to each other. According to our general prescription for constructing the $L_P[C]$ which will be given in the next section, it is found that in such a case we can use $L(P)$ itself as $L_P[C]$, so that

$$\left\{ L(P) - \frac{\hbar}{i} \frac{\delta}{\delta C_P} \right\} \Psi[C] = 0. \quad (2.3)$$

In this case it is obvious that

$$L(P) = -H(P) \quad (2.4)$$

where $H(P)$ means the interaction Hamilton density of the fields. Of course the equation (2.3) coincides with the equation (III) in II.

Example 2. Case of scalar-(pseudoscalar-) meson field interacting with electro-magnetic field.⁷⁾

In this case we have the following interaction Lagrange density

$$\begin{cases} L(X) \equiv e L^{(1)(0)}(X) + e^2 L^{(1)(1)}(X), \\ L^{(1)(0)}(X) = -\frac{1}{4\pi} \frac{i}{\hbar c} A_\mu \left(\phi^* \frac{\partial \phi}{\partial x_\mu} - \frac{\partial \phi^*}{\partial x_\mu} \phi \right), \\ L^{(1)(1)}(X) = \frac{1}{4\pi} \left(\frac{i}{\hbar c} \right)^2 A_\mu A_\mu \phi^* \phi. \end{cases} \quad (2.5)$$

According to the same discussion as in III, one obtains the following relation .

$$\begin{cases} [L^{(1)(0)}(X), L^{(1)(0)}(X')] = -\frac{1}{4\pi} \left(\frac{i}{\hbar c} \right) \left\{ A_\lambda A_\mu \phi^* \phi \frac{\partial^2 D_{II}(X-X')}{\partial x_\mu \partial x_\lambda} \right. \\ \quad \left. - A_\lambda' A_\mu' \phi'^* \phi' \frac{\partial^2 D_{II}(X'-X)}{\partial x_\mu' \partial x_\lambda'} \right\} \\ [L^{(1)(0)}(X), L^{(1)(1)}(X')] = [L^{(1)(1)}(X), L^{(1)(0)}(X')] = [L^{(1)(1)}(X), L^{(1)(1)}(X')] = 0. \end{cases} \quad (2.6)$$

when the world point X' lies outside the light-cone whose vertex is X . Now, according to our prescription, we assume that $L_X[C]$ is determined so as to satisfy the integrability condition (1.3), putting

$$L_X[C] = L(X) + A_X[C], \quad (2.7)$$

then a functional differential equation of the unknown quantity $A_X[C]$ is given by

$$\frac{\delta A_X[C]}{\delta C_{X'}} - \frac{\delta A_{X'}[C]}{\delta C_X} = -\frac{i}{\hbar} \{ [L(X), L(X')] + [A_X[C], L(X')] + [L(X), A_{X'}[C]] + [A_X[C], A_{X'}[C]] \}. \quad (2.8)$$

By the method described in section 3, the solution of the equation (2.8) is found such that it depends only on the upper variable surface-element in the space-time world; the result being

$$A_X[C] = \frac{1}{4\pi} \left(\frac{ie}{\hbar c} \right)^2 (A_\alpha N_\alpha)^2 \phi^* \phi. \quad (2.9)$$

In this way we find the generalized interaction Lagrange density of our system;

$$L_X[C] = -\frac{1}{4\pi} \left[\left(\frac{ie}{\hbar c} A_\mu \left(\phi^* \frac{\partial \phi}{\partial x_\mu} - \frac{\partial \phi^*}{\partial x_\mu} \phi \right) - \left(\frac{ie}{\hbar c} \right)^2 \phi^* \phi \{ (A_\alpha A_\alpha) + (A_\alpha N_\alpha)^2 \} \right]. \quad (2.10)$$

Since this expression is identical with that of (3.11) in III except the sign, it is clear that our generalized Schrödinger equation is equivalent to that of the case (A) in III.

Example 3. Case of vector-(pseudovector-) meson field interacting with electro-magnetic field.⁷⁾

The interaction Lagrange density is here of the form:

$$\begin{cases} L(X) \equiv L^{(1)(0)}(X) + e^2 L^{(1)(1)}(X), \\ L^{(1)(0)}(X) = -\frac{1}{4\pi} \left(\frac{i}{\hbar c} \right) A_\lambda (\phi_\mu^* \chi_{\lambda\mu} - \chi_{\lambda\mu}^* \phi_\mu), \\ L^{(1)(1)}(X) = \frac{1}{4\pi} \left(\frac{i}{\hbar c} \right)^2 A_\lambda (A_\lambda \phi_\mu^* - A_\mu \phi_\lambda^*) \phi_\mu. \end{cases} \quad (2.11)$$

Further, one obtains the following relations by the analogous calculations made in the case B of III:

$$\begin{aligned} [L^{(1)(0)}(X), L^{(1)(0)}(X')] &= -\frac{1}{4\pi} \left(\frac{i}{\hbar c} \right) \left[\{ (A_\lambda \phi_\mu^* - A_\mu \phi_\lambda^*) (A_\lambda \phi_\beta - A_\beta \phi_\lambda) + \right. \\ &+ \frac{1}{x^2} A_\lambda A_\alpha \chi_{\lambda\mu}^* \chi_{\alpha\beta} \} \frac{\partial^2 D_{II}(X-X')}{\partial x_\mu \partial x_\beta} - \{ (A_\lambda' \phi_\mu^* - A_\mu' \phi_\lambda^*) (A_\lambda' \phi_\beta' - A_\beta' \phi_\lambda') \\ &+ \frac{1}{x'^2} A_\lambda' A_\alpha' \chi_{\lambda\mu}^* \chi_{\alpha\beta}' \} \frac{\partial^2 D_{II}(X'-X)}{\partial x'_\mu \partial x'_\beta} \Big], \\ [L^{(1)(0)}(X), L^{(1)(1)}(X')] &+ [L^{(1)(1)}(X), L^{(1)(0)}(X')] = -\frac{1}{16\pi} \left(\frac{i}{\hbar c} \right)^2 \frac{1}{x^2} \times \end{aligned}$$

$$\begin{aligned} & \times [A_\lambda A_\alpha \{A_\alpha \phi_\beta^* - A_\beta \phi_\alpha^*\} \chi_{\lambda\mu} - \chi_{\lambda\mu}^* (A_\alpha \phi_\beta - A_\beta \phi_\alpha) \} \frac{\partial^2 D_{II}(X-X')}{\partial x_\mu \partial x'_\beta} - \\ & - A_\lambda' A_\alpha' \{A_\alpha' \phi_\beta'^* - A_\beta' \phi_\alpha'^*\} \chi'_{\lambda\mu} - \\ & \chi_{\lambda\mu}^* (A_\alpha' \phi_\beta' - A_\beta' \phi_\alpha') \} \frac{\partial^2 D_{II}(X'-X)}{\partial x'_\mu \partial x_\beta} \Big], \end{aligned} \quad (2.12)$$

$$\begin{aligned} [L^{(1)(1)}(X), L^{(1)(1)}(X')] = & \frac{1}{4\pi x^2} \left(\frac{i}{\hbar c} \right)^2 \times \\ & \times \left\{ A_\lambda A_\alpha (A_\lambda \phi_\mu^* - A_\mu \phi_\lambda^*) (A_\alpha \phi_\beta - A_\beta \phi_\alpha) \frac{\partial^2 D_{II}(X-X')}{\partial x_\mu \partial x'_\beta} - \right. \\ & \left. - A_\lambda' A_\alpha' (A_\lambda' \phi_\mu'^* - A_\mu' \phi_\lambda'^*) (A_\alpha' \phi_\beta' - A_\beta' \phi_\alpha') \frac{\partial^2 D_{II}(X'-X)}{\partial x'_\mu \partial x_\beta} \right\} \end{aligned}$$

provided that two world points X and X' lie outside each other's light-cone. On account of the relation (2.12), we can prove that the integrability condition (1.3) is satisfied by the function

$$\begin{aligned} L_X[C] = & -\frac{1}{4\pi} \left[\left(\frac{ie}{\hbar c} \right) A_\lambda (\phi_\mu^* \chi_{\lambda\mu} - \chi_{\lambda\mu}^* \phi_\mu) - \left(\frac{ie}{\hbar c} \right)^2 \left\{ A_\lambda (A_\lambda \phi_\mu^* - A_\mu \phi_\lambda^*) \phi_\mu + \right. \right. \\ & \left. \left. + (A_\lambda \phi_\mu^* - A_\mu \phi_\lambda^*) (A_\lambda \phi_\rho - A_\rho \phi_\lambda) N_\mu N_\rho + \frac{1}{x^2} A_\lambda A_\nu \chi_{\lambda\mu}^* \chi_{\nu\rho} N_\mu N_\rho \right\} \right] \end{aligned} \quad (2.13)$$

which is just the minus of the function $H_X[C]$ appeared in the case B of III:

$$L_X[C] = -H_X[C], \quad (2.14)$$

Thus, also here we have furnished a justification of the equivalence of our new formulation to the old one.

In the above mentioned examples we have considered only the cases containing one kind of coupling constant e between two fields, but analogous statements can be extended easily to the cases where more than two coupling constants exist between the two interacting fields, for example to the case of nucleon field interacting with meson field.⁸⁾

3. The prescription for constructing the required special solution of the equation (1.3)

Thus far we have considered only the system which is describable in canonical form, but our formalism can also be extended to the case where canonical variables can not be defined in the ordinary sense. Let us suppose for simplicity that there are only two fields interacting with each other, and the total Lagrange density is given by

$$L_{\text{total}} = L_I + L_{II} + L_{I\ II} \quad (3.1)$$

where L_I and L_{II} mean respectively the Lagrange densities of the first and the second fields, $L_{I II}$ the interaction one. We assume here that the free field variables U_α° and $U_\alpha^{\circ*}$ satisfy the following second order wave equation:

$$\begin{cases} (\square - x^2) U_\alpha^\circ = 0 \\ (\square - x^2) U_\alpha^{\circ*} = 0, \end{cases} \quad (3.2)$$

x being a constant. The interaction Lagrange density $L_{I II}$ may contain derivatives of the free field variables of any order and is a Lorentz-invariant function consisting of polynomial or power series of U_α , $U_{\alpha;\mu}$, $U_{\alpha;\mu\nu}$, ..., and U_α^* , $U_{\alpha;\mu}^*$, $U_{\alpha;\mu\nu}^*$, Of course L_I and L_{II} are also Lorentz-invariant. Moreover L_I , L_{II} are all real.

Now we assume that the quantum mechanical system corresponding to the above stated classical one is described by the equation

$$\left\{ L_P[C] - \frac{\hbar}{i} \frac{\partial}{\partial C_P} \right\} \Psi[C] = 0, \quad (I)$$

where the field variables contained in $L_P[C]$ are solutions of the free field equations, and satisfy the same four-dimensional commutation relations as the free field ones. As will be shown in the following, the operator in (I) can be determined from the interaction Lagrange density $L_{I II}(P)$ such that (i) it satisfies the integrability condition

$$\frac{\delta L_X[C]}{\delta C_{X'}} - \frac{\delta L_{X'}[C]}{\delta C_X} = -\frac{i}{\hbar} [L_X[C], L_{X'}[C]], \quad (3.3)$$

where X and X' are space-like to each other and (ii) the equation (I) becomes equivalent to the ordinary one when the canonical formulation is possible. The formalism proposed here will be considered as a natural quantum mechanical generalization of the way of describing a system to which canonical description is not applicable.

We must here determine the unknown quantity $L_X[C]$. Let us suppose $L_{I II}$ to be the M th order polynomial of the coupling constant. The case where more than two kinds of interaction constants are involved can also be treated in the same way. According to the assumption made above, $L_{I II}$ has the form

$$L_{I II}(X) \equiv L(X) = \sum_{j=0}^{M-1} e^{j+1} L^{(j)}(X), \quad (3.4)$$

where the coefficients $L^{(j)}(X)$ of e^{j+1} ($j=0, 1, 2, \dots, M-1$) are all known functions. Let us suppose that $L_X[C]$ was obtained, and expand it into the power series of the parameter e :

$$L_X[C] = \sum_{n=1} \sum_{j=0} e^{n+j} L_X^{(n,j)}[C] = \sum_{m=1} e^m \left\{ \sum_{n=1}^m L_X^{(n,m-n)}[C] \right\}. \quad (3.5)$$

Among the coefficients of e^m , ($m=1, 2, \dots$) on the right-hand side of (3.5),

especially those for $n=1$ are just the known functions $L^{(1)(m-1)}(X)$, ($m=1, 2, \dots, \dots, m$), themselves. Further the number of terms appearing in the right-hand side of the equation (3.5) are not always finite. If one substitutes (3.5) into (3.3) one obtains the following first order functional differential equation expanded in the parameter ϵ :

$$\sum_{m=2}^{\infty} \sum_{n=2}^m \epsilon^m \left\{ \frac{\partial L_X^{(n)(m-n)}[C]}{\partial C_{X'}} - \frac{\partial L_{X'}^{(n)(m-n)}[C]}{\partial C_X} \right\} \\ = \sum_{m=2}^{\infty} \sum_{n=2}^m \sum_{r=1}^{m-1} \sum_{k=0}^{m-n} \epsilon^m \left(-\frac{i}{\hbar} \right) [L_X^{(r)(k)}[C], L_{X'}^{(n-r)(m-n-k)}[C]]. \quad (3.6)$$

It is thus required that the coefficients of the the same order terms in ϵ on both sides of the equation (3.6) must be equal:

$$\frac{\partial L_X^{(n)(m-n)}[C]}{\partial C_{X'}} - \frac{\partial L_{X'}^{(n)(m-n)}[C]}{\partial C_X} = -\frac{i}{\hbar} \sum_{r=1}^{n-1} \sum_{k=0}^{m-n} [L_X^{(r)(k)}[C], L_{X'}^{(n-r)(m-n-k)}[C]] \\ (n=2, 3, \dots, m; m=2, 3, 4, \dots). \quad (3.7)$$

In order to solve the equation (3.6), we use the following relation which is valid when X and X' are space-like with respect to each other:

$$\sum_{r=1}^{n-1} \sum_{k=0}^{m-n} [L_X^{(r)(k)}[C], L_{X'}^{(n-r)(m-n-k)}[C]] = \sum_{J=I, II} \left[\{ F_{\lambda\mu}^{(2)(n)(m-n)}[X; N] + \right. \\ \left. + F_{\lambda\mu\nu}^{(2)(n)(m-n)}[X; N](x'_\nu - x_\nu) + \dots \} \frac{\partial^2 D_J(X-X')}{\partial x_\mu \partial x'_\lambda} + \{ F_{\lambda\mu\nu}^{(3)(n)(m-n)}[X; N] + \right. \\ \left. + F_{\lambda\mu\nu\sigma}^{(3)(n)(m-n)}[X; N](x'_\sigma - x_\sigma) + \dots \} \frac{\partial^3 D_J(X-X')}{\partial x_\nu \partial x_\mu \partial x'_\lambda} + \dots \right] - \sum_{J=I, II} [X', X], \quad (3.8)$$

where $F_{\lambda\mu}[X; N]$, $F_{\lambda\mu\nu}[X; N]$, mean respectively the second-, third-, rank tensors composed of the field quantities U_α , $U_{\alpha;\mu}$, $U_{\alpha;\mu\nu}$,, U_α^* , $U_{\alpha;\mu}^*$, $U_{\alpha;\mu\nu}^*$, at the point X . The notation $\sum_{J=I, II} [X', X]$ on the right-hand side of the equation (3.8) was used for the expression obtained by exchanging X and X' in the first term of the same equation.

Now we can prove the relation (3.8) as follows. First let us notice that the left-hand side of the equation (3.8) can also be written as

$$\sum_{r=1}^{n-1} \sum_{k=0}^{m-n} [L_X^{(r)(k)}[C], L_{X'}^{(n-r)(m-n-k)}[C]] = \\ \left(\sum_{r=1}^{\frac{n-1}{2}} \left[\sum_{k=0}^{\frac{m-n}{2}-1} \{ ([L_X^{(r)(k)}[C], L_{X'}^{(n-r)(m-n-k)}[C]] - [L_{X'}^{(r)(k)}[C], L_X^{(n-r)(m-n-k)}[C]]) + \right. \right. \\ \left. + ([L_X^{(n-r)(k)}[C], L_{X'}^{(r)(m-n-k)}[C]] - [L_{X'}^{(n-r)(k)}[C], L_X^{(r)(m-n-k)}[C]]) \} + \right. \\ \left. + ([L_X^{(r)(\frac{m-n}{2})}[C], L_{X'}^{(n-r)(\frac{m-n}{2})}[C]] - [L_{X'}^{(r)(\frac{m-n}{2})}[C], L_X^{(n-r)(\frac{m-n}{2})}[C]]) \right], \\ \text{(for odd } m, n)$$

$$\begin{aligned}
& \sum_{r=1}^{\frac{n-2}{2}} \left[\sum_{k=0}^{\frac{m-n}{2}-1} \{ ([L_X^{(r)(k)}[C], L_{X'}^{(n-r)(m-n-k)}[C]] - [L_{X'}^{(r)(k)}[C], L_X^{(n-r)(m-n-k)}[C]]) + \right. \\
& \quad + (L_X^{(n-r)(k)}[C], L_{X'}^{(r)(m-n-k)}[C]) - [L_{X'}^{(n-r)(k)}[C], L_X^{(r)(m-n-k)}[C]] \} + \\
& \quad + ([L_X^{(r)(\frac{m-n}{2})}[C], L_{X'}^{(n-r)(\frac{m-n}{2})}[C]) - [L_{X'}^{(r)(\frac{m-n}{2})}[C], L_X^{(n-r)(\frac{m-n}{2})}[C]] \} + \\
& \quad + \sum_{k=0}^{\frac{m-n}{2}-1} \{ [L_X^{(\frac{n}{2})(k)}[C], L_{X'}^{(\frac{n}{2})(m-n-k)}[C]] - [L_{X'}^{(\frac{n}{2})(k)}[C], L_X^{(\frac{n}{2})(m-n-k)}[C]] \} + \\
& \quad + \frac{1}{2} \{ [L_X^{(\frac{n}{2})(\frac{m-n}{2})}[C], L_{X'}^{(\frac{n}{2})(\frac{m-n}{2})}[C]] - [L_{X'}^{(\frac{n}{2})(\frac{m-n}{2})}[C], L_X^{(\frac{n}{2})(\frac{m-n}{2})}[C]] \}, \\
& \qquad \qquad \qquad \text{(for even } m, n) \tag{3.9} \\
& \sum_{r=1}^{\frac{n-1}{2}} \left[\sum_{k=0}^{\frac{m-n}{2}} \{ ([L_X^{(r)(k)}[C], L_{X'}^{(n-r)(m-n-k)}[C]] - [L_{X'}^{(r)(k)}[C], L_X^{(n-r)(m-n-k)}[C]]) + \right. \\
& \quad + ([L_X^{(n-r)(k)}[C], L_{X'}^{(r)(m-n-k)}[C]] - [L_{X'}^{(n-r)(k)}[C], L_X^{(r)(m-n-k)}[C]]) \}, \\
& \qquad \qquad \qquad \text{(for even } m, \text{ odd } n) \\
& \sum_{r=1}^{\frac{n-2}{2}} \left(\frac{m-n}{2} \right) \left[\sum_{k=0}^{\frac{m-n}{2}} \{ ([L_X^{(r)(k)}[C], L_{X'}^{(n-r)(m-n-k)}[C]] - [L_{X'}^{(r)(k)}[C], L_X^{(n-r)(m-n-k)}[C]]) + \right. \\
& \quad + ([L_X^{(n-r)(k)}[C], L_{X'}^{(r)(m-n-k)}[C]] - [L_{X'}^{(n-r)(k)}[C], L_X^{(r)(m-n-k)}[C]]) \} - \\
& \quad + \sum_{k=0}^{\frac{m-n}{2}} \{ [L_X^{(\frac{n}{2})(k)}[C], L_{X'}^{(\frac{n}{2})(m-n-k)}[C]] - [L_{X'}^{(\frac{n}{2})(k)}[C], L_X^{(\frac{n}{2})(m-n-k)}[C]] \}. \\
& \qquad \qquad \qquad \text{(for odd } m, \text{ even } n)
\end{aligned}$$

Now it is to be noticed that the expressions $L_X^{(n)(m-n-1)}[C]$ ($n=2, 3, \dots, m-1$) appearing in the right-hand side of (3.9) are Lorentz-invariant functionals of the given function $L^{(1)(j)}(X)$ ($j=0, 1, 2, \dots, m-1$) and of the unit normal N_α and its derivatives $N_{\alpha;\mu}, N_{\alpha;\mu\nu}, \dots$ at the world point X and moreover they are polynomials or power series with respect to the field quantities $U_\alpha, U_{\alpha;\mu}, \dots, U_\alpha^*, U_{\alpha;\mu}^*, \dots$, because these assumptions, as will be shown at the end of this section, can be derived from the assumptions made on $L^{(1)(j)}(X)$ ($j=0, 1, 2, \dots, M-1$) by mathematical induction. In the following we shall separately treat cases where both fields are of integer or half odd integer spin.

(I) Case where both fields are of integral spin.

Since the commutation relations between field variables for the integral spin are combinations of even number of derivatives of the D-function, it is clear that the left-hand of the equation (3.8) is given by combination of D-function and its successive derivatives and moreover is Lorentz-invariant.

First of all we see that the term containing D-function itself must vanish, even if it appears in (3.8), because $D(X-X')$ is zero for any space-like pair

X and X' . Further the term with the first derivative of D-function is zero on account of the properties of D-function. Therefore the left-hand side of (3.8) can be written as the following by noticing the relation of (3.9):

$$\begin{aligned} \sum_{j=1}^{n-1} \sum_{l=0}^{n-j} [L_X^{(n)(k)}[C], L_{X'}^{(n-j)(m-n-k)}[C]] = \\ = \sum_{j=1, \text{ II}} \left\{ {}_j F_{\lambda\mu}^{(2)(n)(m-n)}[X, X'; N, N'] \frac{\partial^2 D_J(X-X')}{\partial x_\mu \partial x_{\lambda'}} + \right. \\ \left. + {}_j F_{\lambda\mu\nu}^{(3)(n)(m-n)}[X, X'; N, N'] \frac{\partial^3 D_J(X-X')}{\partial x_\nu \partial x_\mu \partial x_{\lambda'}} + \dots \right\} \\ - \sum_{j=1, \text{ II}} \{X', X; N', N\}, \end{aligned} \quad (3.10)$$

where ${}_j F_{\lambda\mu}^{(2)(n)(m-n)}[X, X'; N, N']$, ${}_j F_{\lambda\mu\nu}^{(3)(n)(m-n)}[X, X'; N, N']$, are respectively the second-, third-, rank tensors composed of the field quantities and of unit normal and its successive derivatives at the points X and X' ; the notation $\sum_{j=1, \text{ II}} \{X', X; N', N\}$ means expression obtained by exchanging X and X' in the first term of the right-hand side of (3.10).

Let us now expand the functionals ${}_j F_{\lambda\mu}^{(2)(n)(m-n)}[X, X'; N, N']$, ${}_j F_{\lambda\mu\nu}^{(3)(n)(m-n)}[X, X'; N, N']$, and ${}_j F_{\lambda\mu}^{(2)(n)(m-n)}[X', X; N', N]$, ${}_j F_{\lambda\mu\nu}^{(3)(n)(m-n)}[X', X; N', N]$, respectively at the point X or X' , then the required relation (3.8) is obtained at once.

(II) Case where both fields are of half odd integral spin.

In this case the anti-commutators of field components with its own ajoin at the same or different world point are given and, as is well known, they are combination of odd number of derivatives of D-function. The essential difference in the present case from the case of integral spin lies in the fact that one has to calculate the commutators of the form of (3.9) in terms of anti-commutators of field variables. But this makes no difficulties and the proof of the relation (3.8) in this case is nearly the same as in the case of integral spin.

The case of Bose field interacting with Fermi field reduces to either the case of (I) or (II). In this way we have arrived at the required result.

Now we see that the equation (3.7) is satisfied if the following equation is fulfilled:

$$\begin{aligned} \frac{\partial L_X^{(n)(m-n)}[C]}{\partial C_{X'}} = - \frac{i}{\hbar} \sum_{j=1, \text{ II}} \left[\left\{ {}_j F_{\lambda\mu}^{(2)(n)(m-n)}[X; N] + {}_j F_{\lambda\mu\nu}^{(3)(n)(m-n)}[X; N](x'_\nu - x_\nu) + \dots \right\} \times \right. \\ \left. \times \frac{\partial^2 D_J(X-X')}{\partial x_\mu \partial x_{\lambda'}} + \left\{ {}_j F_{\lambda\mu\nu}^{(3)(n)(m-n)}[X'; N] + {}_j F_{\lambda\mu\sigma\tau}^{(4)(n)(m-n)}[X'; N](x'_\sigma - x_\sigma) + \dots \right\} \times \right. \\ \left. \times \frac{\partial^3 D_J(X-X')}{\partial x_\nu \partial x_\mu \partial x_{\lambda'}} + \dots \right], \end{aligned} \quad (3.13)$$

$$\frac{\delta L_X^{(n)(m-n)}[C]}{\delta C_{X'}} = -\frac{i}{\hbar} \frac{\partial}{\partial x_{\lambda'}} G_{\lambda}^{(n)(m-n)}[X, X'; N] \quad (3.14)$$

with the abbreviation :

$$\begin{aligned} G_{\lambda}^{(n)(m-n)}[X, X'; N] = & \sum_{J=I, II} \left[\{ {}^{(2)}F_{\lambda\mu}^{(n)(m-n)}[X; N] + {}^{(2)}F_{\lambda\mu\nu}^{(n)(m-n)}[X; N](x_{\nu}' - x_{\nu}) + \right. \\ & + {}^{(2)}F_{\lambda\mu\nu\sigma}^{(n)(m-n)}[X; N](x_{\nu}' - x_{\nu})(x_{\sigma}' - x_{\sigma}) + \dots \} \frac{\partial D_J(X - X')}{\partial x_{\mu}} + \\ & + \{ {}^{(2)}F_{\mu\lambda\nu}^{(n)(m-n)}[X; N] \delta_{\mu\lambda} + {}^{(2)}F_{\mu\lambda\nu\sigma}^{(n)(m-n)}[X; N](\delta_{\mu\sigma} \cdot \overline{x_{\nu}' - x_{\nu}} + \delta_{\mu\nu} \cdot \overline{x_{\sigma}' - x_{\sigma}}) + \\ & + \dots \} D_J(X - X') + \{ {}^{(3)}F_{\lambda\mu\nu}^{(n)(m-n)}[X; N] + {}^{(3)}F_{\lambda\mu\nu\sigma}^{(n)(m-n)}[X; N](x_{\sigma}' - x_{\sigma}) + \\ & + {}^{(3)}F_{\lambda\mu\nu\sigma\rho}^{(n)(m-n)}[X; N](x_{\sigma}' - x_{\sigma})(x_{\rho}' - x_{\rho}) + \dots \} \frac{\partial^2 D_J(X - X')}{\partial x_{\nu} \partial x_{\mu}} + \\ & + \{ {}^{(3)}F_{\mu\lambda\nu\sigma}^{(n)(m-n)}[X; N] \delta_{\mu\sigma} + {}^{(3)}F_{\mu\lambda\nu\sigma\rho}^{(n)(m-n)}[X; N](\delta_{\mu\rho} \cdot \overline{x_{\sigma}' - x_{\sigma}} + \delta_{\mu\sigma} \cdot \overline{x_{\rho}' - x_{\rho}}) + \dots \} \times \\ & \frac{\partial D_J(X - X')}{\partial x_{\nu}} + \{ {}^{(3)}F_{\nu\mu\lambda\sigma\rho}^{(n)(m-n)}[X; N](\delta_{\mu\sigma} \delta_{\nu\rho} + \delta_{\mu\rho} \delta_{\nu\sigma}) + \\ & + \dots \} D_J(X - X') + \dots \Big]. \end{aligned} \quad (3.15)$$

Now a special solution of the equation (3.14) can be obtained as a surface integral on the variable surface element C :

$$\begin{aligned} L_X^{(n)(m-n)}[C] = & -\frac{i}{\hbar C} \sum_{J=I, II} \left[(N_{\lambda} N_{\mu}) {}^{(2)}F_{\lambda\mu}^{(n)(m-n)}[X; N] + \right. \\ & + \left\{ \left(N_{\nu} \frac{\partial N_{\lambda}}{\partial x_{\mu}} + N_{\mu} \frac{\partial N_{\lambda}}{\partial x_{\nu}} \right) {}^{(3)}F_{\lambda\mu\nu}^{(n)(m-n)}[X; N] + (N_{\mu} N_{\nu} \delta_{\lambda\sigma} + \right. \\ & + N_{\nu} N_{\lambda} \delta_{\mu\sigma} + N_{\lambda} N_{\mu} \delta_{\nu\sigma}) {}^{(3)}F_{\lambda\mu\nu\sigma}^{(n)(m-n)}[X; N] + \dots \Big\} \Big], \\ & (n=2, 3, \dots, m; m=2, 3, 4, \dots). \end{aligned} \quad (3.16)$$

As is seen from (3.16), $L_X^{(n)(m-n)}[C]$ ($n=2, 3, \dots, m$) are also Lorentz-invariant functions which are the polynomials or power series of U_{α} , $U_{\alpha;\mu}$, ..., U_{α}^{**} , $U_{\alpha;\mu}^{**}$, ... and N_{α} , $N_{\alpha;\mu}$, Since $L_X^{(2)(0)}[C]$ ($m=n=2$) is a solution of the equation

$$\begin{aligned} \frac{\delta L_X^{(2)(0)}[C]}{\delta C_{X'}} - \frac{\delta L_{X'}^{(2)(0)}[C]}{\delta C_X} = & -\frac{i}{2\hbar} \left\{ [L^{(1)(0)}(X), L^{(1)(0)}(X')] - \right. \\ & \left. [L^{(1)(0)}(X'), L^{(1)(0)}(X)] \right\}, \end{aligned} \quad (3.17)$$

one sees at once that it has the same form as the expression (3.16). Therefore by mathematical induction we know all the solutions of the equation (3.7) have the forms of (3.16). In this way $L_X[C]$ is given by

$$L_X[C] = L(X) -$$

$$\begin{aligned} & - \frac{i}{\hbar c} \sum_{m=2} e^m \left[\sum_{n=2}^m \sum_{j=1}^{\Pi} \left[(N_1 N_2) F_{12}^{(2)(m-n)}[X; N] + \left\{ \left(N_3 \frac{\partial N_1}{\partial x_2} + \right. \right. \right. \right. \\ & + N_2 \frac{\partial N_1}{\partial x_3} \left. \left. \left. \right) F_{123}^{(3)(m-n)}[X; N] + (N_2 N_3 \delta_{14} + N_3 N_1 \delta_{34} + \right. \right. \\ & \left. \left. + N_1 N_2 \delta_{34} \right) F_{1234}^{(3)(m-n)}[X; N] \right\} + \dots \right] \Big], \quad (II) \end{aligned}$$

where for simplicity 1, 2, are used instead of λ, μ, \dots . We now postulate that just this solution should be used in the fundamental equation (I) of the quantum field theory. It is to be noticed that $L_P[C]$ obtained in this way contains, in general, not only the unit normal of the surface C but its derivatives, so that it depends not only on the inclination of the surface C at the point P but on curvature etc. of C at this point.

4. Proof of equality between the canonical formalism and ours.

In this section we prove that our new formalism is perfectly equivalent to the old theory if the physical system under consideration can be written in a canonical form. For this purpose it is sufficient to prove the following relation:

$$L_X[C_0] = -H(X) \quad (4.1)$$

where $L_X[C]$ means the generalized interaction Lagrange density which is derived from the interaction Lagrange density $L(X)$ by the prescription described in section 3 so that (1.1) satisfies the integrability condition (1.3), and C_0 is a special surface of C which is a plane parallel to the xyz-plane.

Now, for the first step of the proof of the equation (4.1) we will summarize the outline of the canonical formalism in the one-time theory. If we denote the variables of the free fields by the capital letter U with components U_α , their derivatives with respect to the space-time coordinates x_μ by $U_{\alpha;\mu}$, our system has a canonical form when the following two conditions are satisfied: (I) the Lagrange density functions of free fields are of bilinear forms of U_α , $U_{\alpha;\mu}$ and their adjoints, (II) the interaction Lagrange density of the fields contains $U_{\alpha;4}$ and $U_{\alpha;4}^*$ at most linearly.

Put

$$L_{\text{total}} = L^0 + L \quad (4.2)$$

where L^0 and L mean respectively the free and interaction Lagrange densities. Then by the assumption they are of the form

$$\begin{cases} L^0 = \Gamma_{\alpha\beta}^0 U_{\beta;\mu}^* U_{\alpha;\mu} + A_{\alpha\mu}^0 U_{\alpha;\mu} + U_{\alpha;\mu}^* A_{\alpha\mu}^0 \\ L = \Xi_{\alpha\mu} U_{\alpha;\mu} + U_{\alpha;\mu}^* \Xi_{\alpha\mu}^* + \theta \end{cases} \quad (4.2')$$

where Γ^0 , Λ^0 , Ξ and θ do not contain $U_{\alpha;4}$ and $U_{\alpha;4}^*$, and hence not $U_{\alpha;i}$ and $U_{\alpha;i}^*$, $i=1,2,3$, by relativistic symmetry. The total Hamilton density of our system is then given by

$$H_{\text{total}} = U_{\alpha}^+ U_{\alpha;4} + \text{adjoint} - L^0(U_{\alpha}, U_{\alpha;\mu}, \text{adjoints}) - L(U_{\alpha}, U_{\alpha;\mu}, \text{adjoints}) \quad (4.3)$$

with

$$U_{\alpha}^+ = \frac{\partial L^0}{\partial U_{\alpha;4}} + \frac{\partial L}{\partial U_{\alpha;4}} = \Gamma_{\alpha\beta}^0 U_{\beta 4}^* + \Lambda_{\alpha 4}^0 + \Xi_{\alpha 4}. \quad (4.4)$$

Of course, since we assume determinant of $\Gamma_{\alpha\beta}^0$ is not zero, we can solve the equation (4.4) with respect to $U_{\beta 4}^*$. Let the result be

$$U_{\beta 4}^* = \mathcal{A}_{\beta\gamma}^0 (U_{\gamma}^+ + \Lambda_{\gamma 4}^0 - \Xi_{\gamma 4}). \quad (4.5)$$

As is well known, $\mathcal{A}_{\beta\gamma}^0$ is connected $\Gamma_{\alpha\beta}^0$ by the relation

$$\Gamma_{\alpha\beta}^0 \mathcal{A}_{\beta\gamma}^0 = \delta_{\alpha\gamma}. \quad (4.6)$$

Further (4.3) reduces to

$$\begin{aligned} H_{\text{total}} = & U_{\alpha}^+ \mathcal{A}_{\alpha\beta}^{0*} (U_{\beta}^{*+} - \Lambda_{\beta 4}^{0*} - \Xi_{\beta 4}^*) + \text{adjoint} - \\ & - L^0(U_{\alpha}, U_{\alpha;4}, \mathcal{A}_{\alpha\beta}^{0*} (U_{\beta}^{*+} - \Lambda_{\beta 4}^{0*} - \Xi_{\beta 4}^*), \text{adjoints}) - \\ & - L(U_{\alpha}, U_{\alpha;4}, \mathcal{A}_{\alpha\beta}^{0*} (U_{\beta}^{*+} - \Lambda_{\beta 4}^{0*} - \Xi_{\beta 4}^*), \text{adjoints}) \end{aligned} \quad (4.7)$$

when one eliminates the time derivatives of field variables from (4.3) by means of the equation (4.4). Now the interaction part H among H_{total} is separated from (4.7) as follows:

$$H = \frac{1}{2} \Xi_{\alpha 4} \mathcal{A}_{\alpha\beta}^{0*} \Xi_{\beta 4}^* + \text{adjoints} - L(U_{\alpha}, U_{\alpha;4}, \mathcal{A}_{\alpha\beta}^{0*} (U_{\beta}^{*+} - \Lambda_{\beta 4}^{0*} - \Xi_{\beta 4}^*), \text{adjoints}). \quad (4.8)$$

The interaction part of the Lagrangian on the other hand is given by

$$L = \Xi_{\alpha 4} U_{\alpha;4} + \Xi_{\alpha 4} \mathcal{A}_{\alpha\beta}^{0*} (U_{\beta}^{*+} - \Lambda_{\beta 4}^{0*} - \Xi_{\beta 4}^*) + \text{adjoints} + \theta. \quad (4.8')$$

Here we introduce the unitary transformation of the field variables such that the free Hamilton density is eliminated from the Schrödinger equation, then we see, as was shown in J, the transformed variables become the solutions of the free field equations, therefore the expressions (4.8), (4.8') are reduced to

$$H = \frac{1}{2} \Xi_{\alpha 4} \mathcal{A}_{\alpha\beta}^{0*} \Xi_{\beta 4}^* + \text{adjoint} - L(U_{\alpha}, U_{\alpha;4}, U_{\alpha;4}, \text{adjoints}), \quad (4.9)$$

$$L = \Xi_{\alpha 4} U_{\alpha;4} + \Xi_{\alpha 4} \mathcal{A}_{\alpha\beta}^{0*} (U_{\beta}^{*+} - \Lambda_{\beta 4}^{0*}) + \text{adjoints} + \theta, \quad (4.9')$$

where we used for the transformed variables the same letters as the old ones.

The next task is to seek for the generalized interaction Lagrange density by the method described in section 3. For this purpose we must first calculate

$$[L(X), L(X')]. \quad (4.10)$$

In order to calculate the commutator (4.10), we transform the Lorentz frame in such a way that X and X' refer to the same instant of time. In this system

we can calculate the commutator by using the ordinary commutation relations referring to the variables at the same instant of time. If one notices that Ξ , \mathcal{A}^0 , \mathcal{A}^0 and θ do not contain any derivatives of the field variables, one can easily verify that the required commutator has the form

$$\begin{aligned} [L(X), L(X')]_{t=t'} = & \frac{i\hbar}{2} \left\{ \Xi_{\alpha j} \mathcal{A}_{\beta\delta}^0 \Xi_{\beta\delta}^* \delta_{\alpha\delta} \frac{\partial\delta(x-x')}{\partial x_j} + \right. \\ & + \Xi'_{\gamma k} \mathcal{A}_{\alpha\beta}^0 \Xi_{\alpha\beta}^* \delta_{\beta\gamma} \frac{\partial\delta(x'-x)}{\partial x'_k} + \text{adjoints} + \\ & \left. + \text{terms containing } \delta\text{-functions but not its derivatives} \right\}. \quad (4.11) \end{aligned}$$

Let us return to the original coordinate system. Then noticing the fact

$$\mathcal{A}_{\alpha\beta}^0 = \mathcal{A}_{\beta\alpha}^{0*}, \quad (4.11) \text{ becomes}$$

$$\begin{aligned} [L(X), L(X')] = & i\hbar \left\{ \Xi_{\alpha\mu} \mathcal{A}_{\alpha\beta}^{0*} \Xi_{\beta\nu} \frac{\partial^2 D(X-X')}{\partial x_\mu' \partial x_\nu} + \text{adjoint} + \right. \\ & \left. + (\text{terms containing a D-function or its first derivative}) \right\}. \end{aligned}$$

According to the prescription described in section 3, we now calculate $L_X^{(2)}[C]$. Then we find

$$L_X^{(2)}[C] = \frac{1}{2} \Xi_{\alpha\mu} \mathcal{A}_{\alpha\beta}^{0*} \Xi_{\beta\nu}^* N_\mu N_\nu + \text{adjoint}. \quad (4.12)$$

Finally let C in the expression (4.12) be parallel to the xyz-plane, then we obtain

$$L_X^{(2)}[C_0] = -\frac{1}{2} \{ \Xi_{\alpha 4} \mathcal{A}_{\alpha\beta}^{0*} \Xi_{\beta 4}^* + \text{adjoint} \}. \quad (4.13)$$

Further, the fact the Ξ , \mathcal{A} , θ do not contain any derivative of U_α results in

$$[L_X^{(2)}[C], L(X')] = 0,$$

whence the polynomial (3.5) contains no terms higher than (4.13). We thus find

$$L_X[C_0] = L + L^{(2)} = -\frac{1}{2} \{ \Xi_{\alpha 4} \mathcal{A}_{\alpha\beta}^{0*} \Xi_{\beta 4}^* + \text{adjoint} \} + L \quad (4.14)$$

which is just equal to the minus of the expression (4.9). In this way we have furnished the proof of the equation (4.1).

5. An example.

We consider here the modified theory of β -decay introduced by Konopinski and Uhlenbeck⁹⁾ as an example of our generalized formulation. It is the essential feature of the $K-U$ theory that a space-time gradient of the wave function of light particle is contained in the interaction energy of the system consisting of nucleon and light particle. Therefore it is sufficient to consider the following

simplified problem though it does not correspond to any existing system. There coexist two kinds of material fields with rest masses M and m and their interaction Lagrange density is given by

$$L = \frac{g}{x_2} J_\mu S_{;\mu} \quad (5.1)$$

with

$$\begin{cases} J_\mu = \Phi^\dagger \gamma_\mu \Phi \\ S_{;\mu} = \frac{\partial S}{\partial x_\mu} \\ S = -i\varphi^\dagger \varphi \end{cases} \quad (5.2)$$

where Φ and φ are respectively the field variables having the rest masses M and m and satisfy the commutation relations as follows:

$$\begin{cases} [\Phi_\sigma(X), \Phi_\rho^\dagger(X')]_+ = -\left\{ \gamma_\mu \frac{\partial}{\partial x_\mu} - x_1 \right\}_{\sigma\rho} D_1(X-X') \\ [\varphi_\sigma(X), \varphi_\rho^\dagger(X')]_+ = -\left\{ \gamma_\mu \frac{\partial}{\partial x_\mu} - x_2 \right\}_{\sigma\rho} D_2(X-X') \\ [\Phi_\sigma(X), \Phi_\rho(X')]_+ = [\Phi_\sigma^\dagger(X), \Phi_\rho^\dagger(X')]_+ = [\varphi_\sigma(X), \varphi_\rho(X')]_+ \\ = [\varphi_\sigma^\dagger(X), \varphi_\rho^\dagger(X')]_+ = 0 \end{cases} \quad (5.3)$$

in which x_1 and x_2 mean respectively $\frac{Mc}{\hbar}$ and $\frac{mc}{\hbar}$. g is a constant which measures the strength of interaction between the fields.

Now the commutator $[L(X), L(X')]$ composed of the interaction Lagrange density at the two points X and X' which is a space-like pair is easily calculated and gives

$$\begin{aligned} [L(X), L(X')] = & \left(\frac{g}{x_2} \right)^2 \frac{\partial}{\partial x_\sigma'} \left[J_\lambda J_\sigma' \frac{\partial}{\partial x_\lambda} \left\{ (\varphi^\dagger \gamma_\mu \varphi' - \varphi'^\dagger \gamma_\mu \varphi) \frac{\partial D_2(X-X')}{\partial x_\mu} \right. \right. \\ & \left. \left. - x_2 (\varphi^\dagger \varphi' + \varphi'^\dagger \varphi) D_2(X-X') \right\} \right]. \end{aligned} \quad (5.4)$$

We get, therefore, according to our prescription by performing the four-dimensional volume integration of the equation (5.4) the following expression:

$$L_X^{(2)}[C] = 2 \left(\frac{i}{\hbar c} \right) \left(\frac{g}{x_2} \right)^2 (N_\alpha J_\alpha)^2 (\varphi^\dagger \varphi). \quad (5.5)$$

Moreover we know

$$\int_V [L_X^{(2)}[C], L(X')] dv' = 0.$$

In this way we have the generalized Schrödinger equation of our system as follows:

$$\left\{ L_p[C] - \frac{\hbar}{i} \frac{\partial}{\partial C_p} \right\} \Psi[C] = 0 \quad (5.6)$$

with abbreviation of

$$L_p[C] = -i \left(\frac{g}{x_2} \right) J_\alpha \frac{\partial(\varphi^\dagger \varphi)}{\partial x_\alpha} + 2 \left(\frac{i}{\hbar c} \right) \left(\frac{g}{x_2} \right)^2 (N_\alpha J_\alpha)^2 (\varphi^\dagger \varphi). \quad (5.7)$$

In this case the situation was especially simple so that the generalized Lagrange density depends only on the inclination of C but not on its curvature etc. although the system under consideration does not allow the ordinary canonical description.

In conclusion we express our cordial thanks to Professor S. Tomonaga for his kind guidance and encouragement.

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Intervals of Cl I, A II, K III, Ca IV 2P .

Wataro WATARI.

Department of Industrial Chemistry, Kyoto University.

(Received Apr. 18, 1949)

The energy level of the atom with an electron-hole in its atomic core is the inverted doublet. F I, Ne II, Mg IV and Al V atoms have such an electron-configuration, i.e. $1s^2 2s^2 2p^5$, in their normal state. Their radial functions were obtained according to the method of self-consistent field. These tabular functions can be analytically expressed in a sufficient accuracy because their principal quantum numbers are small. Such analytical expressions were obtained and, by making use of these results, the doublet intervals were analytically computed by the present author¹⁾.

In this paper the result of the similar calculations for inverted $^2P's$ of Cl I, A II, K III and Ca IV will be reported. The calculation is carried out using the self-consistent tabular radial functions of Cl, A, K, Ca $1s^2 2s^2 2p^6 3s^2 3p^6$ which were calculated by D. R. Hartree and W. Hartree²⁾. The analytical expression of the $3s$ function can not be determined in a sufficient accuracy because the nodes of the function are too many. Therefore we calculate all parameters and intervals according to the method of numerical integration.

The results are shown in Table I. The meaning of the notations are the same as in I except for s' and s'' . The corrected screening constants s' or s''

Table I.

Z	Cl I	A II	K III	Ca IV
	17	18	19	20
$\zeta_{10,31}$	0.619	0.616	0.608	0.612
$\tilde{\zeta}_{10,31}$	0.210	0.209	0.211	0.213
$\zeta_{20,31}$	0.098	0.096	0.094	0.092
$\tilde{\zeta}_{20,31}$	-0.016	-0.016	-0.015	-0.015
$\zeta_{21,31}$	0.110	0.109	0.107	0.105
$\tilde{\zeta}_{21,31}$	0.028	0.028	0.028	0.028
$\zeta_{30,31}$	0.021	0.021	0.020	0.020
$\tilde{\zeta}_{30,31}$	0.004	0.003	0.002	0.001
$\zeta_{31,31}$	0.015	0.016	0.017	0.018
$\tilde{\zeta}_{31,31}$	0.007	0.008	0.009	0.009

σ	4.04	4.04	4.04	4.05
s	9.26	9.02	8.82	8.63
$\overline{r^{-3}}$ (a.u.)	5.73	8.95	13.02	18.15
$(^2P_{1/2}-^2P_{3/2})$ (cm ⁻¹)	-649	-1091	-1701	-2529
s'	8.71	8.49	8.30	8.12
$\overline{(r^{-3})}'$ (a.u.)	7.03	10.63	15.12	20.69
$(^2P_{1/2}-^2P_{3/2})'$ (cm ⁻¹)	-796	-1296	-1975	-2883
s''	8.77	8.55	8.37	8.19
$\overline{(r^{-3})}''$ (a.u.)	6.87	10.42	14.85	20.35
$(^2P_{1/2}-^2P_{3/2})''$ (cm ⁻¹)	-778	-1270	-1940	-2838
observed ⁴⁾ $^2P_{1/2}-^2P_{3/2}$ (cm ⁻¹)	-881	-1431	-2164	-3130

are now given by $s'=16s/17$ or $s''=(14s+2)/15$.

We see that the calculated values of the intervals are smaller than observed ones.⁴⁾ The same fact was seen in I and Tutihasi's results³⁾ too. Its reason was considered in I. The parameters $\zeta_{n'l'nl}$, $\tilde{\zeta}_{n'l'nl}$ or σ are almost constant for the atoms with the same configuration and the large atomic number.

In conclusion the author wishes to express his hearty thanks to Prof. G. Araki for the interest he has taken in this work.

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Triplet Intervals of Beryllium and Calcium.

Wataro WATARI.

Department of Industrial Chemistry, Kyoto University.

(Received Apr. 18, 1949)

Introduction.

The 3P intervals of a beryllium atom with an electron configuration $1s^2 2s 2p$ were theoretically calculated by G. Araki¹⁾. He computed the parameters involved in the interval formula using hydrogen-like radial functions except a parameter r^{-3} . He calculated the last parameter using a self-consistent radial function. Since the principal quantum numbers of one-electron functions are small in this configuration, Slater's method²⁾ is applicable in order to obtain the analytical expressions of the self-consistent functions. In this paper these analytical expressions will be obtained and the calculation of the above mentioned intervals will be repeated, by making use of them, according to the general formula of Araki¹⁾.

Further the calculation will be extended to the triplet $1s^2 2s^2 2p^6 3s^2 3p^6 4s 4p$ 3P of a calcium atom. In this case Slater's method can not be applied because the one-electron radial functions with large principal quantum numbers have too many nodes to determine accurately their analytical form. Therefore the calculation will be carried out by the method of numerical integrations.

It will be seen that our result is unfortunately worse than Araki's for beryllium and as for calcium the calculated intervals are far from agreeing with observed ones.

§ 1. Analytical Self-Consistent Functions of Be $1s^2 2s 2p$.

The self-consistent radial functions of a beryllium atom with an electron-configuration $1s^2 2s 2p$ were computed in tabular forms by D. R. Hartree and W. Hartree³⁾ according to Fock's method which takes into account an effect of an exchange. We obtain the analytical expressions of these tabular functions by Slater's method.

One-electron functions are written in the form $R_{nl}(r) Y_{lm}(\theta, \varphi)$, where Y_{lm} is a normalized spherical surface harmonic of l -th degree, and n, l and m are the principal, azimuthal and orbital magnetic quantum numbers of the one-electron state respectively. The result is given by

$$R_{10}(r) = 13.661 \exp(-3.60r) \quad (1.1)$$

$$R_{20}(r) = 2.505 \exp(-3.07r) - 1.294r \exp(-1.031r) \quad (1.2)$$

$$R_{21}(r) = r[0.784 \exp(-2.71r) + 0.698 \exp(-0.826r)] \quad (1.3)$$

where r is measured in atomic units and the radial functions are normalized. The $2s$ -function is slightly modified so that this is orthogonal to the $1s$ -function. Hartree's $P_N(nl)$ is equal to our $rR_{nl}(r)$.

§ 2. Triplet Intervals of Energy Levels for Be I and Ca I.

The atoms with n_0s and nl electrons outside their cores consisting of closed shells have triplet terms. The general formula which represents intervals of such triplets was derived by G. Araki⁽¹⁾. This formula is given by

$$\begin{aligned} {}^3L_L - {}^3L_{L+1} &= (L+1) \{ Z - \sigma - \zeta + 3(2L-1)\eta \} \mu^2 r^{-3} \\ {}^3L_{L-1} - {}^3L_L &= L \{ Z - \sigma - \zeta - 3(2L+3)\eta \} \mu^2 r^{-3} \end{aligned} \quad (2.1)$$

where Z is the atomic number, L is the azimuthal quantum number of the atom, μ is the Bohr magneton, and σ is a screening constant due to the spin-orbit interactions between the nl -electron and the core electrons. The parameters ζ and η are due to the spin-orbit and spin-spin interactions between the n_0s and nl electrons respectively. These parameters are given as follows:

$$r^{-3} = \int_0^\infty \frac{1}{r^3} [R_{nl}(r) r]^2 dr \quad (2.2)$$

$$\sigma = \sum_{n'l'} 2(2l'+1) (\zeta_{n'l', nl} + 3 \tilde{\zeta}_{n'l', nl}) \quad (2.3)$$

$$\zeta = 3 (\zeta_{n_0, nl} + 2 \tilde{\zeta}_{n_0, nl}) \quad (2.4)$$

$$\eta = \frac{2}{(2l-1)(2l+3)} \zeta_{n_0, nl} \quad (2.5)$$

$$\zeta_{n'l', nl} = -\frac{1}{r^{-3}} \int_0^\infty \int_0^\infty \frac{1}{r_2} \frac{\partial a_0}{\partial r_2} [R_{n'l'}(1) R_{nl}(2) r_1 r_2]^2 dr_1 dr_2 \quad (2.6)$$

$$\tilde{\zeta}_{n'l', nl} = -\frac{1}{(2l+1)r^{-3}} \int_0^\infty \int_0^\infty \frac{a_l}{r_2} R_{n'l'}(1) \frac{dR_{n'l'}(2)}{dr_2} R_{nl}(1) R_{nl}(2) r_1^2 r_2^2 dr_1 dr_2 \quad (2.7)$$

$$\begin{aligned} \tilde{\zeta}_{n'l', nl} = & -\frac{1}{(2l+1)r^{-3}} \int_0^\infty \int_0^\infty \left[\left(-\frac{1}{2l-1} \frac{\partial a_{l-1}}{\partial r_2} + \frac{1}{2l+3} \frac{\partial a_{l+1}}{\partial r_2} \right) \frac{R_{n'l'}(1) R_{n'l'}(2)}{r_2} \right. \\ & \left. + \left(\frac{l-1}{2l-1} a_{l-1} + \frac{l+2}{2l+3} a_{l+1} \right) \frac{R_{n'l'}(1)}{r_2} \frac{dR_{n'l'}(2)}{dr_2} \right] R_{nl}(1) R_{nl}(2) r_1^2 r_2^2 dr_1 dr_2 \end{aligned} \quad (2.8)$$

$$\begin{aligned} a_v &= r_1^v / r_2^{v+1} \quad \text{for } r_1 \leq r_2 \\ &= r_2^v / r_1^{v+1} \quad \text{for } r_2 < r_1 \end{aligned} \quad (2.9)$$

where the summation in σ is to be extended over all orbitals in the atomic core.

According to these general formula, we calculate the intervals of Be $1s^2 2s 2p^3 P$ and Ca $1s^2 2s^2 2p^6 3s^2 3p^6 4s 4p^3 P$. The calculation of the former triplet is

carried out using the radial functions obtained in §1, whereas the latter is computed by the numerical integration using the tabular radial functions given by D. R. Hartree and W. Hartree⁴⁾, because the nodes of $3s, 4s$ and $4p$ functions are too many to determine their analytical expressions in a sufficient accuracy. The value of the physical constant which is adopted in the present calculations is $a^2 R \text{ cm}^{-1} = 5.822 \text{ cm}^{-1}$. The results are shown in Table I and II together with the observed values⁵⁾.

Table I Values of parameters

$$\begin{aligned} \text{Be. } Z=4 \quad & \zeta_{10, 21} = 0.67194 \quad \tilde{\zeta}_{10, 21} = 0.18452 \\ & \zeta_{20, 21} = 0.10299 \quad \tilde{\zeta}_{20, 21} = 0.011479 \\ & \bar{r}^{-3} = 0.28666 \text{ (a.u.)}, \zeta = 0.37784, \eta = 0.041196, \sigma = 2.4510. \end{aligned}$$

$$\text{Ca. } Z=20$$

$$\begin{aligned} \zeta_{10, 41} = 0.719, \zeta_{20, 41} = 0.098, \zeta_{31, 41} = 0.112, \zeta_{30, 41} = 0.033, \zeta_{31, 41} = 0.032, \zeta_{40, 41} = 0.009, \\ \tilde{\zeta}_{10, 41} = 0.210, \tilde{\zeta}_{20, 41} = -0.014, \tilde{\zeta}_{21, 41} = 0.026, \tilde{\zeta}_{30, 41} = -0.004, \tilde{\zeta}_{31, 41} = 0.008, \\ \tilde{\zeta}_{40, 41} = 0.005, \bar{r}^{-3} = 0.719 \text{ (a.u.)}, \zeta = 0.054, \eta = 0.004, \sigma = 4.10. \end{aligned}$$

Table II Triplet intervals.

		${}^3P_1 - {}^3P_2$ (cm^{-1})	${}^3P_0 - {}^3P_1$ (cm^{-1})	${}^3P_1 - {}^3P_2$ ${}^3P_0 - {}^3P_1$
Be $Z=4$	Calc.	2.1604	0.4615	4.681
	Obs.	2.35	0.68	3.46
Ca $Z=20$	Calc.	66.4	33.1	2.01
	Obs.	105.9	52.3	2.02

We see that the calculated values of the intervals are smaller than observed ones. Especially the agreement with experiment is worse for calcium. The theoretical value of the ratio $({}^3P_1 - {}^3P_2)/({}^3P_0 - {}^3P_1)$ is larger than the observed one for beryllium, whereas they agree with each other for calcium. This means that \bar{r}^{-3} is too small in the case of calcium, and therefore its $4p$ function is too diffused.

In conclusion the author wishes to express his hearty thanks to Professor G. Araki for the kind interest he has taken in the present work.

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Intervals of Si IV 2P levels.

Simpei TUTIHASI

Department of Industrial Chemistry, Kyoto University.

(Received Apr. 18, 1949)

Alkali-like atoms (or ions) have a valence electron outside its core consisting of closed shells, and their energy levels are doublets. The general formula for their intervals was derived theoretically by G. Araki.⁽¹⁾ It is given by

$$^2L_{L-\frac{1}{2}} - ^2L_{L+\frac{1}{2}} = (2L+1)(Z-\sigma)\mu^2\overline{r^{-3}},$$

where L is the azimuthal quantum number, Z is the atomic number, μ is the Bohr magneton, and σ is a screening constant due to the spin-orbit interactions between the valence electron and the core electrons. In my previous paper⁽²⁾ the intervals of Li I, B I, and O IV 2P were calculated according to this formula.

In the present paper the result of the similar calculation for the intervals of Si IV will be reported. Self-consistent one-electron radial functions were computed by W. Hartree, D. R. Hartree, and M. F. Manning⁽³⁾ for electron-configurations $1s^2 2s^2 2p^6 3p$, $1s^2 2s^2 2p^6 4p$, and $1s^2 2s^2 2p^6 5p$.

The analytical expressions of the above mentioned tabular functions are obtained by the Slater's method⁽⁴⁾ for the first configuration. All functions are normalized, $P(1s)$ and $P(2s)$ are made to be orthogonal to each other, and $P(2p)$ and $P(3p)$ too.

$$P(1s) : 93.96r \exp(-13.02r)$$

$$P(2s) : 27.72r \exp(-12.05r) - 57.04r^2 \exp(-4.64r)$$

$$P(2p) : r^2 \{ 52.45 \exp(-5.17r) + 6.44 \exp(-3.33r) \}$$

$$P(3p) : 26.39r^2 \exp(-4.50r) - r^3 \{ 64.76 \exp(-6.46r) + 2.91 \exp(-1.71r) \}.$$

The interval of $1s^2 2s^2 2p^6 3p^2 P$ is computed by using these analytical functions. The other intervals are calculated by numerical integrations, because the radial function has too many nodes to determine its analytical expression in a sufficient accuracy when its principal quantum number is large. The results are shown in the following tables.* $\zeta_{n'l'nl}$'s or $\tilde{\zeta}_{n'l'nl}$'s with the same n' , l' are nearly the same for $3p$, $4p$, and $5p$. $\tilde{\zeta}_{20nl}$'s and $\tilde{\zeta}_{21nl}$'s have negative signs, whereas $\tilde{\zeta}_{10nl}$'s have positive signs. σ is nearly independent of the principal quantum number of

* The exact meaning of the notations was accounted for in the reference (2).

Table I. Calculated values of ζ 's and $\tilde{\zeta}$'s.

ζ_{1031}	0.59844	ζ_{2031}	0.08806	ζ_{2131}	0.08860
ζ_{1041}	0.624	ζ_{2041}	0.085	ζ_{2141}	0.095
ζ_{1051}	0.621	ζ_{2051}	0.082	ζ_{2151}	0.092
$\tilde{\zeta}_{1031}$	0.20522	$\tilde{\zeta}_{2031}$	-0.01362	$\tilde{\zeta}_{2131}$	-0.01685
$\tilde{\zeta}_{1041}$	0.205	$\tilde{\zeta}_{2041}$	-0.016	$\tilde{\zeta}_{2141}$	-0.016
$\tilde{\zeta}_{1051}$	0.206	$\tilde{\zeta}_{2051}$	-0.016	$\tilde{\zeta}_{2151}$	-0.016

Table II. Calculated values of σ , s and r^{-3} .
(s is the screening constant calculated from the value of r^{-3} .)

Configuration	$1s^2 2s^2 2p^6 3p$	$1s^2 2s^2 2p^6 4p$	$1s^2 2s^2 2p^6 5p$
σ	2.7509	2.83	2.81
s	7.1866	7.52	7.71
r^{-3}	3.9048 a.u.	1.42 a.u.	0.66 a.u.

Table III. Doublet intervals of Si IV, $Z=14$: $^2P_{1/2} - ^2P_{1/2}$

Configuration	$1s^2 2s^2 2p^6 3p$	$1s^2 2s^2 2p^6 4p$	$1s^2 2s^2 2p^6 5p$
Calc. value	333.6 cm^{-1}	138 cm^{-1}	65 cm^{-1}
Obs. value ⁽⁶⁾	460 cm^{-1}	162 cm^{-1}	75 cm^{-1}

the valence electron. The calculated intervals are slightly smaller than the observed ones. This was the general tendency⁽²⁾⁽⁵⁾ of the result calculated by using the self-consistent functions.

In conclusion the author wishes to express his hearty thanks to Prof. G. Araki for the interest he has taken in this work.

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Semi-Classical Treatment of the Reactive Corrections. I.

— *The Anomalous Magnetic Moment of the Electron* —

Zirô KOBA.

*Institute of Physics, Faculty of Science, Tokyo University.**

(Received April 22, 1949)

Resumo. La metodo de Welton, kiu sukcesis je derivado de la nivelsoviĝo de hidrogena atomo en intuicia maniero, estas plibonigata kaj tiel sin montras, ke duonklasika komprenebligo valoras eĉ en la demando de nenormala magneta momento de elektrono. Plue ĉi tia modelo estas komparata kun la preciza kvantumteoria kalkulado kaj la regiono de ĝia valoreco estas diskutata.

§ 1. Introduction.

Recently Welton¹⁾ succeeded in explaining semi-classically the experimental results of Lamb and Retherford²⁾ by calculating the mean extension of the position of an electron which is caused by the coupling of the latter with zero-point fluctuations of electromagnetic field. (The same idea was once suggested by Nambu and Ono³⁾.) This method seems to us quite remarkable, because it offers an intuitive foothold to the elaborate quantum mechanical calculation of Bethe, Tomonaga, Schwinger⁶⁾ and others⁷⁾ concerning on the reaction of radiation field.

Welton's simple model, however, failed completely in the problem of the anomalous magnetic moment of the electron. Thus the relative correction for the magnetic moment derived by him turns out, in the natural unit $\hbar=c=1$ which will be used throughout this paper,

$$\delta_1 = -\frac{e^2}{2\pi} \quad (1.1)$$

which is of the equal order of magnitude to the experimental results obtained by Kusch and Foley⁸⁾, but, to our regret, with the wrong sign. The self-consistent subtraction (Tomonaga-Schwinger theory), on the other hand, has given a result which agrees with the experiment.

Since in this case the vacuum polarization effect, or virtual pair creation and destruction, plays an essential role, one might be inclined to think that such a problem is by nature beyond the reach of a classical treatment. Nevertheless we should like to propose here a new semi-classical model, which, as we hope, may indicate that it is still possible to take into account the effect of vacuum

* Transferred to Department of Physics, Osaka University.

electrons in an intuitive one-body treatment.

Our model is based on the so-called Zitterbewegung of Dirac electron discussed about 20 years ago by Schrödinger⁹⁾. It will be seen that this concept of Zitterbewegung is quite useful for intuitive understanding of those properties of Dirac electron which are due to its virtual transition to negative energy states. In fact we shall see that, when one properly combines Welton's fluctuation with Schrödinger's Zitterbewegung, one is able to obtain a right semi-quantitative result for the magnetic moment of the electron.

In §2 we briefly describe the essential points of Welton's idea and analyze what points are overlooked in his model. Then in §3 we epitomize Schrödinger's theory of Zitterbewegung and apply it to some typical problems. Based on these considerations we propose in §4 a new model for obtaining radiative corrections, calculate in this way the anomalous magnetic moment of the electron and also give an intuitive image to our method of derivation. In §5 we investigate the correspondence of semi-classical method of Welton and us to the rigorous calculation of Tomonaga-Schwinger formalism and discuss the limits of the validity of the former.

§2. Criticism of Welton's model.

Welton has treated the radiative correction for the magnetic moment of the electron in the following way. Suppose a magnetic dipole with a definite moment μ lying in a homogeneous magnetic field B , the interaction energy being of course $-\mu B$. When, however, this dipole is coupled with zero-point fluctuation of the radiation field it will swing its head to and fro, so that in the average it will make a certain angle θ with the direction of the magnetic field and the magnitude of the energy will be reduced by a factor $\cos \theta$, in other words the effective magnetic moment will become $\mu \cos \theta$. (Fig. 1).



Fig. 1.

Thus it is from the beginning obvious that in this way one will obtain a negative correction to the magnetic moment of the electron.

In deriving his conclusion Welton has carried out his calculation classically and cut off a quadratically divergent integral at the Compton frequency, a procedure which may appear at first somewhat unreasonable. But one can easily verify that a more rigorous treatment yields a similar result as long as one makes use of this model. In fact, when one calculates the expectation value of the spin operator (σ_z),

$$(\sigma_z) = \int \psi^* \sigma_z \psi dv, \quad (2.1)$$

where ψ , ψ^* are quantized spinors for electron field, employing for σ_z the usual

representation

$$\sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (2.2)$$

one finds of course for an unperturbed state Ψ_0 with only one electron corresponding to the single electron wave function

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} L^{-3/4} \quad (2.3)$$

$$(\Psi_0^*, (\int \phi^* \sigma_z \phi dv) \Psi_0) = 1, \quad (2.4)$$

while for the state perturbed by the coupling with vacuum electromagnetic field

$$\Psi = \Psi_0 + e\Psi_1 + e^2\Psi_2 + \dots, \quad (2.5)$$

that expectation value turns out

$$(\Psi^*, (\int \phi^* \sigma_z \phi dv) \Psi) = 1 - \frac{2e^2}{3\pi} \int_0^\infty \frac{k dk}{(E_k + k) E_k}, \quad (2.6)$$

$$E_k = \sqrt{k^2 + m^2}. \quad (2.7)$$

If we assume $k \ll m$ in the above expression, we have a relative correction

$$-\frac{2e^2}{3\pi m^2} \int k dk, \quad (2.8)$$

which differs from Welton's expression only by a factor 2/3. The integral diverges for large k , to be sure, but by virtue of the positron-theoretical subtraction and Tomonaga-Schwinger renormalization—these are not taken into account in (2.6)—we can cancel the diverging part, so that the cut off at $k \sim m$ is justified if one aims only at a semi-quantitative conclusion. Thus one has to seek for the reason of the wrong result not in the procedure of the calculation, but in the used model itself.

In order to make clear what points are overlooked in Welton's model, we decompose the interaction energy of an electron in an external field,

$$e \int (\psi^* \alpha \psi, \hat{\mathbf{A}}) dv, \quad (2.9)$$

where $\hat{\mathbf{A}}$ denotes the external potential, which, for simplicity, shall represent a pure magnetic field, i.e., we put $\text{div } \hat{\mathbf{A}} = 0$ and $\dot{\hat{\mathbf{A}}} = 0$. Making use of the projection operator

$$A_{\pm} = \frac{E \pm (\alpha, \mathbf{p}) \pm m\beta}{2E}, \quad (2.10)$$

we can write (2.9) as

$$e \int (\psi^* \alpha \psi, \dot{\mathbf{A}}) dv = \int \psi^* \left\{ e A^- (\alpha \dot{\mathbf{A}}) + e (\alpha \dot{\mathbf{A}}) A^- + \frac{e(\mathbf{p}, \dot{\mathbf{A}})}{E} + \frac{e}{2E} (\boldsymbol{\sigma}, \text{rot } \dot{\mathbf{A}}) \right\} \psi dv \quad (2.11)$$

In this form one can easily recognize that Welton has considered only the last term and has replaced $2E$ by $2m$ (non-relativistic approximation).

A simple calculation shows that the third term in (2.11) does not contribute to the additional magnetic energy, so that we may conclude that the positive correction which compensates the negative effect found by Welton and yields the positive correction given by Tomonaga-Schwinger formalism, should originate in the first two terms. As is evident from their form, they represent that part of the coupling energy which is concerned with transition to negative energy states. Thus the defect of the Welton model consists in the neglect of negative energy states, as was already pointed out by Welton himself, who described somewhat vaguely that the positive correction found by Schwinger is due to the magnetic interaction with vacuum electrons.

§ 3. Schrödinger's Zitterbewegung and its applications.

i) As is well known Schrödinger clarified the characteristic behavior of a free Dirac electron most vividly in terms of the "Zitterbewegung." According to him the velocity operator α and the position operator x can be divided into two parts, the one which commutes with the Hamiltonian operator H and the other which anticommutes with it. Such a decomposition is indeed possible and uniquely determined when the operator considered commutes with the square of the Hamiltonian*, what is the case with α ; and x can be derived from α .

$$\begin{aligned} a_j &= \bar{a}_j + \tilde{a}_j, \\ x_j &= \bar{x}_j + \tilde{x}_j, \end{aligned} \quad (j=1, 2, 3) \quad (3.1)$$

where $-$ denotes the commutable part and \sim the anticommutable part. Explicitly written these are given by

$$\begin{aligned} \bar{a}_j &= H^{-1} p_j, \\ \tilde{a}_j &= \tilde{a}_j(0) e^{-2iHt} = e^{2iHt} \tilde{a}_j(0). \end{aligned} \quad (3.2)$$

* When for an operator Q , $[H^2, Q]=0$ holds, we have $Q=\bar{Q}+\tilde{Q}$, where

$$\bar{Q} = \frac{1}{2} \cdot (Q + H^{-1} Q H) = \frac{1}{2} (Q + H Q H^{-1}), \quad \tilde{Q} = \frac{1}{2} (Q - H^{-1} Q H) = \frac{1}{2} (Q - H Q H^{-1}).$$

In this paper we deal mainly with such operators.

The former is the macroscopic velocity; in the latter $\tilde{a}_j(0)$ is the value of \tilde{a}_j at $t=0$ and of course anticommutes with H . And

$$\begin{aligned}\bar{x}_j &= \bar{x}_j(0) + H^{-1} p_j t \\ \tilde{x}_j &= -\frac{1}{2i} \tilde{a}_j(0) H^{-1} e^{-2im} = -\frac{1}{2i} \tilde{a}_j H^{-1} = \frac{1}{2i} H^{-1} \tilde{a}_j,\end{aligned}\quad (3.3)$$

where $\bar{x}_j(0)$ is the initial value of the macroscopic position and can be put equal to zero without losing generality.

The magnitude of the anticommutable parts are

$$\begin{aligned}\tilde{a}_j^2 &= 1 - H^{-2} p_j^2, \\ \tilde{x}_j^2 &= \frac{1}{4} H^{-2} (1 - H^{-2} p_j^2); \end{aligned}\quad (3.4)$$

so that for a very slow electron

$$\begin{aligned}|a_j| &\approx 1, \\ |\tilde{x}_j| &\approx \frac{1}{2m}.\end{aligned}\quad (3.5)$$

One sees that a rest electron (more exactly, an electron with momentum zero) has no macroscopic velocity, to be sure, but performs an oscillation with frequency $2m$ and amplitude $1/2m$. One may notice that this amplitude, when multiplied by e , is nothing but the magnitude of the magnetic moment.

ii) In an eigenstate of the Hamiltonian operator for an electron, the anticommutable part \tilde{Q} of an operator Q has of course the vanishing expectation value. On the other hand, to the matrix element of Q between a positive and a negative state contributes only \tilde{Q} and not \bar{Q} . Thus \tilde{Q} is that part of Q which is concerned with pair creation and annihilation.

Now we turn to the expectation value of an operator R which is the product of two operators S and T ,

$$R = S \cdot T = (\bar{S} + \tilde{S})(\bar{T} + \tilde{T}). \quad (3.6)$$

The commutable part of R is given by

$$\bar{R} = \bar{S} \cdot \bar{T} + \tilde{S} \cdot \tilde{T}, \quad (3.7)$$

since the product of two anticommutable operators is commutable. In this way the anticommutable parts of the individual operators may yield an observable effect. Two examples for such a phenomenon will be described in the following.

iii) *Scattering cross-section.* When a charged particle of momentum \mathbf{p} without spin is elastically scattered by a fixed external potential V into momentum \mathbf{q} ($|\mathbf{q}| = |\mathbf{p}|$), the differential cross-section is given by

$$d\sigma \sim \left| \int e^{-i(\mathbf{q} \cdot \mathbf{x})} \cdot V \cdot e^{i(\mathbf{p} \cdot \mathbf{x})} d\mathbf{v} \right|^2. \quad (3.8)$$

But if the particle obeys Dirac equation (as is the case with the electron) one must multiply the above expression by an angle-dependent factor

$$\frac{1}{2} S \not{p} A_p^+ A_q^+ = 1 - \frac{\not{p}^2}{2E^2} (1 - \cos \theta), \quad (3.9)$$

θ : angle between \mathbf{p} and \mathbf{q} ,

i.e. for a slow electron we have the correction factor

$$1 - \frac{\not{p}^2}{2m^2} (1 - \cos \theta) \quad (3.10)$$

As is well known this factor (3.9) is due to the circumstance that the incident electron is in a positive energy state with a certain spin direction and the scattered electron is found in a positive energy state with an arbitrary spin direction. But heretofore no intuitive explanation has been given to this correction.

Now this factor can be derived as a consequence of the Zitterbewegung: When we substitute (3.1) and (3.3) into the initial and the final plane waves respectively we have

$$e^{i(\mathbf{p}, \bar{\mathbf{x}} + \tilde{\mathbf{x}})}, \quad e^{i(\mathbf{q}, \bar{\mathbf{x}} + \tilde{\mathbf{x}})} \quad (3.11)$$

and, since the parts $e^{i(\mathbf{p}, \bar{\mathbf{x}})}$ and $e^{i(\mathbf{q}, \bar{\mathbf{x}})}$ behave just the same as when the particle is spinless, the characteristic correction factor for the scattering cross-section of a Dirac electron turns out

$$|e^{-i(\mathbf{q}, \tilde{\mathbf{x}})} \cdot e^{i(\mathbf{p}, \tilde{\mathbf{x}})}|^2 \quad (3.12)$$

From this expression one has to pick up commutable terms according to the prescription of (3.7). In the case of a slow electron $|\tilde{\mathbf{x}}|$ is nearly equal to $1/2m$ both for the initial and the final electron* and the observable part becomes to the second order of \not{p}/m ,

$$\left\{ 1 - \frac{1}{2} (\mathbf{p} - \mathbf{q})^2 |\tilde{\mathbf{x}}|^2 \right\}^2 \approx 1 - \frac{\not{p}^2}{2m^2} (1 - \cos \theta), \quad (3.13)$$

in agreement with (3.10). It is clear, therefore, that this reduction of the scattering cross-section is due to the interference effect of the Zitterbewegung of initial and final states.

It is to be remarked that the radiative correction¹⁰⁾ for the elastic scattering of a slow electron has again the same angle-dependent form as (3.10), because, as Welton has clarified, the reactive part of the correction can be interpreted as caused by the position fluctuation (Welton's Zitterbewegung!). The detailed discussion about the scattering problems will be given in our later works.

* We have to employ not the average but the maximum displacement in order to secure numerical agreement.

iv) *Magnetic Moment.* Suppose, for simplicity, that an electron with vanishing momentum lies in a homogeneous magnetic field B in z -direction. One can employ the external potential

$$\dot{A}_x = -\frac{1}{2} By, \quad \dot{A}_y = \frac{1}{2} Bx \quad (3.14)$$

and examine the expectation value of interaction energy $e(\alpha, \dot{A})$. Since the electron has no macroscopic velocity p/E , α consists only of the anticommutable part, so that the energy expectation value is given by the product of the anticommutable parts of α and \dot{A} . Substituting (3.3) and (3.4),

$$\begin{aligned} \tilde{\alpha}_x \cdot \tilde{\dot{A}}_x &= -\frac{1}{2} B \tilde{\alpha}_x \tilde{y} = -\frac{1}{2} B \tilde{\alpha}_x \tilde{\alpha}_y \left(-\frac{H^{-1}}{2i} \right) \\ &= -\frac{B}{2} \frac{1}{2m} i \tilde{\alpha}_x \tilde{\alpha}_x, \end{aligned}$$

$$\tilde{\alpha}_y \cdot \tilde{\dot{A}}_y = \frac{1}{2} B \tilde{\alpha}_y \tilde{x} = \frac{B}{2} \cdot \frac{1}{2m} i \tilde{\alpha}_y \cdot \tilde{\alpha}_x,$$

so that

$$e(\alpha, \dot{A}) = \frac{B}{2} \frac{1}{2m} (-i) (\overline{a_x a_y - a_y a_x}) = B \cdot \frac{e}{2m} \overline{\sigma_z}, \quad (3.15)$$

where

$$\sigma_z = -i a_x a_y = i a_y a_x \quad (3.16)$$

has a commutable part of magnitude 1 for a rest electron.

Thus we have verified the "normal" magnetic moment is $e/2m$. This way of derivation strongly suggests that one had better represent the normal magnetic moment as a ring-current of diameter $1/m$ caused by the Zitterbewegung, while for the additional magnetic moment due to the Pauli-type interaction $\ell \gamma_\mu \gamma_\nu F_{\mu\nu}$ with

$$\gamma_j = i a_j = \beta, \quad \gamma_4 = \beta, \quad F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}, \quad x_4 = it,$$

a magnetic needle is perhaps a more appropriate model, although these are equivalent to each other.

v) In §2 we have decomposed the interaction energy into four terms (2.11). We see that the third term is the commutable part of α multiplied by both parts of \dot{A} , so that the other terms come from $\tilde{\alpha}$. The last term has an appreciable commutable part, which is the product of $\tilde{\alpha}$ and $\tilde{\dot{A}}$, while the remaining two are practically anticommutable ones, as will be verified more precisely below.

§4. Improvement of Welton's model.

i) Welton asserts that, while the infinite energy of the vacuum electro-

magnetic field due to its fluctuation has no physical meaning, an electron in vacuum performs a sort of Brownian motion caused by this zero-point oscillation, which yields certain observable effects. He puts namely

$$m \frac{d^2 x_j}{dt^2} = e \sum_k E_j(\mathbf{k}) e^{ikt}, \quad (4.1)$$

where $E_j(\mathbf{k})$ denotes the Fourier amplitude of the electric field produced by the vacuum fluctuation. The above x corresponds to \bar{x} in our notation, and we can rewrite (4.1) into

$$\frac{d\bar{x}_j}{dt} = \frac{e}{m} \sum_k \frac{E_j(\mathbf{k})}{ik} e^{ikt}, \quad (4.2)$$

where we have employed a first order differential equation instead of the second-order one, because the former is more adequate for mediating between classical and quantum-mechanical equations of motion.

Here we claim further that the anticommutable part of the electron coordinate \tilde{x}_j is also coupled with the vacuum field fluctuation. Now when no external force acts on the electron \tilde{x}_j performs, for a rest electron say, already an oscillation with frequency $2m$ and amplitude $1/2m$, so that the "free" equation for \tilde{x}_j is

$$\frac{d}{dt} \tilde{x}_j + \tilde{x}_j (2im) = 0 \quad (4.3)$$

If, moreover, the same external fluctuating force is acted on it as in (4.2), the equation of motion becomes*

$$\frac{d}{dt} \tilde{x}_j + \tilde{x}_j (2im) = -\frac{e}{m} \sum_k \frac{E_j(\mathbf{k}) e^{ikt}}{ik}, \quad (4.4)$$

which together with (4.2) makes the basis of our following arguments.

ii) In the above reasoning (4.4) has been derived in a simple analogy to (4.2), and of course needs to be justified by a more strict consideration. So we interpose here a brief outline of such a verification.**

According to (3.3), (3.1), and (3.2), one has

$$\begin{aligned} \tilde{x}_j &= \frac{1}{2iH} \tilde{a}_j = \frac{1}{2iH^2} H(a_j - \tilde{a}_j) \\ &= \frac{(\boldsymbol{\sigma} \times \mathbf{p})_j + \rho_j \sigma_j m}{2(\rho^2 + m^2)} \end{aligned} \quad (4.5)$$

where ρ 's and σ 's are related to α 's and β in the same way as employed in Dirac's book.¹¹⁾ (4.5) gives indeed, when no external force is exerted

* The minus sign is due to the fact that $\bar{x} + \tilde{x} = x$ commutes with $e(a_j^2)$.

** A more detailed discussion has been carried out by S. Tani. His work will appear in this journal soon.

$$\frac{d}{dt} \tilde{x} = i [H, \tilde{x}] = -\tilde{x} (2iH), \quad (4.6)$$

and when the perturbation energy

$$H' = -e (\mathbf{a}, \mathbf{A}) \quad (4.7)$$

is added,

$$\frac{d}{dt} \tilde{\mathbf{x}} + \tilde{\mathbf{x}} (2iH) = i [H', \tilde{\mathbf{x}}] \quad (4.8)$$

$$= + \frac{em}{p+m^2} \beta \mathbf{A} + (\text{terms involving } \mathbf{p} \text{ or derivatives of potentials}).$$

For an electron with vanishing momentum one puts $\mathbf{p}=0$, and for a rough evaluation one is also allowed to neglect terms involving derivatives of potentials, because in the following computation one is always confined to the region $k \lesssim m$, k being the frequency of the vacuum fluctuation. Further we put $\beta=1$ and $\mathbf{A} = -\sum_k \frac{\mathbf{E}(\mathbf{k})}{ik} e^{ikt}$ and (4.8) is reduced to (4.4), what was to be proved.

iii) Now the solution of (4.2) and (4.4) with the boundary condition that when $\epsilon \rightarrow 0$, $\bar{x}_j=0$, $\tilde{x}_j = \frac{1}{2m} e^{i2mt}$, can be obtained quite easily:

$$\bar{x}_j = \sum_k \bar{x}_j(\mathbf{k}) e^{ikt}, \quad \bar{x}_j(\mathbf{k}) = -\frac{e E_j(\mathbf{k})}{m \cdot k^2} \quad (4.9)$$

$$\tilde{x}_j = \frac{1}{2m} e^{i2mt} + \sum_k \tilde{x}_j(\mathbf{k}) e^{ikt}, \quad \tilde{x}_j(\mathbf{k}) = +\frac{e E_j(\mathbf{k})}{mk(2m+k)} \quad (4.10)$$

Therefore the time average of the square of the position fluctuation turns out

$$\begin{aligned} \langle (\Delta x_j)^2 \rangle_{Av} &= \sum_k | \bar{x}_j(\mathbf{k}) + \tilde{x}_j(\mathbf{k}) |^2 \\ &= \sum_k \{ | \bar{x}_j(\mathbf{k}) |^2 + | \tilde{x}_j(\mathbf{k}) |^2 + 2 \bar{x}_j^*(-\mathbf{k}) \tilde{x}_j(\mathbf{k}) \}. \end{aligned} \quad (4.11)$$

The first two terms on the right-hand side of (4.11) represent the commutable part of the correction while the last term gives a modification of anticommutable part, so that we obtain as the corrected value of \tilde{x}_j ,

$$\begin{aligned} \tilde{x}_j &= \frac{1}{2m} \left\{ 1 + \frac{1}{(1/2m)^2} 2 \sum \bar{x}_j^*(-\mathbf{k}) \tilde{x}_j(\mathbf{k}) \right\}^{\frac{1}{2}} \\ &= \frac{1}{2m} \left\{ 1 + 4e^2 \sum \bar{x}_j^*(-\mathbf{k}) \tilde{x}_j(\mathbf{k}) \right\} \end{aligned} \quad (4.12)$$

Inserting (4.9) and (4.10) into (4.12)

$$\tilde{x}_j = \frac{1}{2m} \left\{ 1 + 4e^2 \sum \frac{E_j(\mathbf{k})^2}{k^3(m+k)} \right\}. \quad (4.13)$$

Here we use, in accordance with Welton, for the vacuum fluctuation electric

field

$$\sum_k E_j(k)^2 \rightarrow \frac{1}{3} \cdot \frac{2}{\pi} \int k^3 dk, \quad (4.14)$$

and arrive at the result:

$$\tilde{x}_j = \frac{1}{2m} (1 + \delta_2) \quad (4.15)$$

with

$$\delta_2 = \frac{8e^2}{3\pi} \int \frac{dk}{2m+k} \quad (4.16)$$

When we cut it off at $k \approx m$

$$\delta_2 \approx \frac{3 \cdot 2}{3} \frac{e^2}{\pi} \approx \frac{e^2}{\pi}. \quad (4.17)$$

The readers may remember that the magnetic moment of the electron was given by $e|\tilde{x}|$, so that δ_2 is nothing but one radiative correction for the magnetic moment.

In this way we have really derived a positive correction, which, when added to the negative correction $\delta_1 = -\frac{e^2}{2\pi}$ deduced by Welton, yields the correct value

$$\delta_{\text{total}} = \delta_1 + \delta_2 = \frac{e^2}{2\pi}. \quad (4.18)$$

Of course the agreement of the numerical factor cannot be taken seriously, because an error of order 1 is unavoidable in the semi-classical treatment, as will be seen in §5. However we hope to have clarified that a positive correction can indeed be obtained owing to the effect considered here.

iv) Roughly speaking we can express the content of the above computation by means of an intuitive image. In a homogeneous magnetic field the normal magnetic moment of an electron is represented by a ring current, as was mentioned before. (Fig. 2). The arrow of dotted line shows the equivalent dipole. When the electron is coupled with the vacuum fluctuation the plane of



Fig. 2.

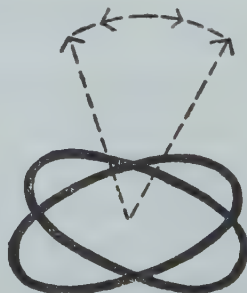


Fig. 3.



Fig. 4.

ring current will roll and pitch so that the equivalent dipole will swing its head (the effect considered by Welton) reducing the effective magnetic moment, but at the same time the ring itself will vibrate in its own plane and when properly coupled with the fluctuation of the center of the ring (\bar{x}), it results in the magnification of the diameter of the ring current, increasing the effective magnetic moment. (Fig. 3) The second effect surpasses the first and we are left with the positive correction, which can be represented by the original ring-current plus a small additional dipole as in Fig. 4.

§ 5. Correspondence to the precise calculation.

When we take into account the radiative reactions according to the Tomonaga-Schwinger formalism and write down the expectation value of a certain operator Q with regard to a rest electron, we have, besides those which can be attributed to infinite mass and charge, finite terms, for instance, of the following types

$$\frac{\phi^+(0) \phi^+(\mathbf{k}) Q \phi^+(\mathbf{k}) \phi^+(0) \cdot A(-\mathbf{k}) A^*(-\mathbf{k})}{(m - E_k - k)^2} \quad (5.1)$$

$$\frac{\phi^+(\mathbf{k}) \bar{\phi}(0) Q \phi^+(\mathbf{k}) \phi^+(0) \cdot A(-\mathbf{k}) A^*(\mathbf{k})}{(m - E_k - k)(-m - E_k - k)} \quad (5.2)$$

$$\frac{\bar{\phi}(\mathbf{k}) \phi^+(0) Q \phi^+(0) \bar{\phi}^+(\mathbf{k}) \cdot A(-\mathbf{k}) A^*(\mathbf{k})}{(-m - E_k - k)^2} \quad (5.3)$$

where A 's and ϕ 's are the Fourier amplitudes of electromagnetic potentials and electron field spinors, A and ϕ including annihilation operators and A^* and ϕ^\dagger , $\bar{\phi}^\dagger$ creation operators.

The interchange of A and A^* into the right order gives rise to the δ -function which may be well called vacuum fluctuation or zero-point oscillation and we can replace it according to the notations employed in § 4 by $E(\mathbf{k})^2/k^2$. Further, since we are aware of the fact that the integral with regard to k surely converges, resulting in a finite correction, we can safely confine ourselves to the case $k < m$ and neglect $(k/m)^2$, and the above expressions are, apart from a factor of the order 1 due to the electron field amplitudes, reduced to

$$\frac{E^2(\mathbf{k})}{k^4}, -\frac{E^2(-\mathbf{k})}{k^3(2m+k)}, \frac{E^2(\mathbf{k})}{k^2(2m+k)^2}, \times \text{some average value of } Q,$$

respectively. In these one will at once recognize the characteristic forms of $\bar{x}(\mathbf{k})^2$, $\bar{x}(\mathbf{k}) \bar{x}^*(-\mathbf{k})$ and $\bar{x}(\mathbf{k})^2$ derived in § 4. Thus we can infer that $m\bar{x}(k)$ and $m\bar{x}(k)$ introduced as Fourier components of position fluctuation corresponds to the translation operators in momentum space by an amount k with and without pair creation respectively.

The above statement is quite provisional, more detailed analysis will be made in later works. Nevertheless we can remark that our semi-classical derivation corresponds to the following approximations:

- i) *The recoil of the electron is neglected.*
- ii) *The numerical factor of order 1 is neglected. It will suffice, however, for a semi-quantitative estimation provided that one employs a suitable classical image.*
- iii) *The non-relativistic formula is out of at $k \sim m$, since a more rigorous treatment is sure to yield a convergent result.*

Welton's model corresponds, in addition to the above three, to a fourth assumption:

- iv) *Virtual pair creation and annihilation can be neglected.*

This was permissible in the case of the Lamb-shift, but proved fatal in the case of the anomalous magnetic moment.

As for the positron-theoretical subtraction it is utilized only as far as it guarantees the convergence of the result. Apart from this point the characteristic difference between one-body treatment and many-body treatment (subtraction of the vacuum) does not appear in this stage of approximation. If, however, $\tilde{x}(\mathbf{k})^2$ should play an essential role this difference would become appreciable. Indeed we have to subtract a certain part of $|\tilde{x}(\mathbf{k})|^2$ in a positron-theoretical treatment instead of adding it in the one-body treatment. Of course when the virtual quantum has a mass comparable or greater than the electron, this subtraction procedure becomes quite important, what is the case with the cohesive meson hypothesis.¹²⁾

Acknowledgement. The author wishes to express his sincere thanks to Prof. S. Tomonaga for his kind interest and critical discussions throughout this work and also to other members of Tokyo group, especially to Mr. T. Kinoshita, Mr. Y. Miyamoto, Mr. Y. Nambu, and Mr. S. Tani for various helps and valuable discussions.

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On the Method of the Third Quantization.

Yôichirô NAMBU

Department of Physics, Tokyo University.

(Received May 16, 1949)

1. Introduction.

The mathematical formalism presented here¹⁾ owes its origin to the recent development of quantum electrodynamics. The theory of Tomonaga²⁾ and Schwinger,³⁾ which deals with the reaction of the interacting fields in a completely relativistic way, has led us to a deeper insight into the nature of the interaction of the elementary particles. This reaction of the fields manifests itself in such phenomena like the self-energy of the fields and the modification of various physical quantities, and may be considered as effects of the zero-point fluctuation of the quantized fields. Mathematically, it means that a quantity composed of q -number wave functions has a non-vanishing expectation value even in the lowest state. Thus, for instance, the product of two quantized waves ψ^*, ψ' describing the electron field, or A, A' describing the radiation field, will not have in general a zero expectation value even if there are no electrons or no photons (vacuum). In order to avoid this zero-point fluctuation, we usually decompose field quantities into creation and annihilation operators, and rearrange the products of these operators so that they become sums of two terms, the one giving no fluctuation, while the other corresponding to pure fluctuation effect. Then the latter may be subtracted unless it gives rise to any physically observable effect. This was the case for the zero-point energy of the radiation field and the total energy of the completely filled negative states in the hole theory of Dirac electron since they did not depend on the field variables. The self-energy of the electron arises from the one-body fluctuation in the essentially two body operators of the interaction energy of the form $\psi^* \psi \psi^{*'} \psi'$. In this case the fluctuation term turns out bilinear in ψ^* and ψ , so that it may be amalgamated to the originally existing mass term, thus making only a mass renormalization which is physically undetectable. In some cases, however, the fluctuation terms do really cause observable effects such as the anomalous interaction of the electron with external field.

The above mentioned separation of the fluctuation terms may be carried out in any quantity by the prescription that we rearrange the field variables in such a way that the creation operators (marked with asterisk) stand always to the left of the annihilation operators (without asterisk). For exchanging any two

variables into correct order we make use of the commutation relation between them. We are therefore allowed to regard every quantity to be a well-ordered function of the field variables which may be written as

$$F = \sum f(a^*, b^*, \dots) \cdot g(a, b, \dots),$$

where each of the set a^*, b^*, \dots and a, b, \dots is commutative or anti-commutative according to the statistics. The well-ordered character is preserved on addition of two such quantities, but unfortunately it is not on multiplication. Hence a rearrangement will be necessary after every operation such as simple multiplication of two quantities, transformation of representation, and making commutation relation. To give a clear perspective of this rearrangement process we shall introduce in the subsequent sections some new algebraical concepts.

2. Operators operating on well-ordered quantities.

(a) Case of Fermi statistics.

Let the quantized wave function of a fermion field be denoted by ψ and ψ^* , each bearing the meaning of destruction and creation operator respectively. Between these quantities exist some commutation relations:

$$\begin{aligned} \{\psi_r(X), \psi_s(X')\} &= \{\psi_r^*(X), \psi_s^*(X')\} = 0, \\ \{\psi_r(X), \psi_s^*(X')\} &= \text{a function of } (r, X; s, X'). \end{aligned} \quad (2.1)$$

Here X and X' are in general the four-dimensional coordinates, and r and s the spin coordinates of the field. The bracket $\{ \}$ means the "plus" type commutator: $\{A, B\} = AB + BA$. We shall write for short the commutation relation $\{\psi_r(X), \psi_s(X')\}$ simply as $\{rX, sX'\}$ (or more briefly, (X, X') or $\{r, s\}$), which is, of course, a c -number quantity.

Now let us take a "well-ordered" operator consisting of several ψ 's and ψ^* 's: $\psi_1^* \psi_2^* \dots \psi_n^* \psi_1 \psi_2 \dots \psi_n$, the suffixes standing for both space and spin variables. Since the ψ 's and ψ^* 's anticommute among themselves, the above expression is indeterminate only in the order of the ψ 's and ψ^* 's, together with a \pm sign. Hence it is possible, as in the case of wave functions in the configuration space, to regard it as a "state" characterized by the coordinates of the $n+n'$ "particles" $\psi_1^* \psi_2^* \dots \psi_n$, that is,

$$\psi_1^* \psi_2^* \dots \psi_n^* \psi_1 \psi_2 \dots \psi_n \equiv (12\dots n; 1'2'\dots n'). \quad (2.2)$$

It changes sign against interchange of two "particles", and vanishes when two particles occupy the same state. The ψ^* and ψ are to be regarded as different kinds of particles, though not completely independent. With these assumptions, the operation of multiplying a ψ^* or ψ either from the left or from the right means an operation of producing a new state of which the number of particles is changed by one. Of course we should make this interpretation only after re-

covering the correct order of the quantities which may have been destroyed on multiplication. Thus

$$\begin{aligned}
 \vec{\phi}_r^* \cdot (12\dots n; 1, 2, \dots n; 1'2'\dots n') &= (r12\dots n; 1'2'\dots n') \\
 \vec{\phi}_r \cdot (12\dots n; 1'2'\dots n') &= \sum_{i=1}^n (-1)^{i-1} \{r, i\} (12\dots i-1, i+1\dots, 1'2'\dots n') \\
 &\quad + (-1)^n (1, 2, \dots nr; 1'2'\dots n') \\
 \overleftarrow{\phi}_r \cdot (12\dots n; 1'2'\dots n') &= (12\dots n; 1'2'\dots n' r) \\
 \overleftarrow{\phi}_r^* \cdot (12\dots n; 1'2'\dots n') &= \sum_{i=n'}^n (-1)^{n'-i} \{i, r\} (12\dots n; 1'2'\dots i'-1, i'+1\dots n') \\
 &\quad + (-1)^{n'} (12\dots nr; 1'2'\dots n'). \tag{2.3}
 \end{aligned}$$

The meaning of the arrow \rightarrow or \leftarrow will be obvious. The suffixes which characterize the states are here used at the same time as simple ordinal numbers. According to the above interpretation we introduce now such quantities that, operated on the state $(12\dots n; 1'2'\dots n')$, are capable of creating or annihilating a particle ϕ^* or ϕ . If we denote them by $\vec{\phi}^*$ and $\vec{\phi}$, $+$ and $-$ meaning creation and annihilation respectively, then the relation (2.3) can be written by means of these operators as follows:

$$\begin{aligned}
 \vec{\phi}_r^* &= \vec{\phi}_r^+, \\
 \vec{\phi}_r &= \vec{\phi}_r^+ + \sum_i \{r, i\} \vec{\phi}_i^*, \\
 \overleftarrow{\phi}_r &= \overleftarrow{\phi}_r^-, \\
 \overleftarrow{\phi}_r^* &= \overleftarrow{\phi}_r^* + \sum_i \tilde{\phi}_i \{i, r\}. \tag{2.4}
 \end{aligned}$$

We have included in these operators the sign function $(-1)^{i-1}$ or $(-1)^{n'-i'}$, so that left-hand operation and right-hand operation must be distinguished with respect to the sign function. This distinction is made by the symbol \sim . Of course we can also express them by operators which explicitly indicate this point as was done in the second quantization process. Thus

$$\begin{aligned}
 \vec{\phi}_r^* &= \sigma_1^* \sigma_2^* \dots \sigma_{r-1}^* \tau_r^+, \quad \overleftarrow{\phi}_r^* = \sigma_1^* \sigma_2^* \dots \sigma_{r-1}^* \tau_r^-, \\
 \vec{\phi}_r &= \sigma_1^* \sigma_2^* \dots \sigma_N^* \sigma_1 \sigma_2 \dots \sigma_{r-1} \tau_r^+, \quad \overleftarrow{\phi}_r = \sigma_1^* \sigma_2^* \dots \sigma_N^* \sigma_1 \sigma_2 \dots \sigma_{r-1} \tau_r^-, \\
 \overleftarrow{\phi}_r^* &= \tau_r^* \sigma_{r-1}^* \dots \sigma_N^* \sigma_1 \sigma_2 \dots \sigma_N', \quad \overleftarrow{\phi}_r^* = \tau_r^* \sigma_{r+1}^* \dots \sigma_N^* \sigma_1 \sigma_2 \dots \sigma_N', \\
 \overleftarrow{\phi}_r &= \tau_r \sigma_{r+1} \dots \sigma_{N-1} \sigma_N', \quad \overleftarrow{\phi}_r = \tau_r \sigma_{r+1} \dots \sigma_{N-1} \sigma_N', \tag{2.5}
 \end{aligned}$$

where

$$\sigma^*, \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \tau^+, \tau = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad \tau^*, \tau = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \tag{2.6}$$

and N and N' are the total number of the states for ϕ^* and ϕ respectively (we are here treating only the case of discrete states). Although the operators $\overset{\pm}{\phi}^*, \overset{\pm}{\phi}$ and $\overset{\mp}{\phi}^*, \overset{\mp}{\phi}$ are quite different from each other, there are simple relations between the products of two operators:

$$\begin{aligned}\overset{\pm}{\phi}_r^* \overset{\pm}{\phi}_s^* &= -\overset{\mp}{\phi}_r^* \overset{\mp}{\phi}_s^*, \quad \overset{\pm}{\phi}_r \overset{\pm}{\phi}_s = -\overset{\mp}{\phi}_r \overset{\mp}{\phi}_s, \quad \overset{\pm}{\phi}_r \overset{\pm}{\phi}_s^* = -\overset{\mp}{\phi}_r \overset{\mp}{\phi}_s^*, \\ \overset{\pm}{\phi}_r^* \overset{\mp}{\phi}_s^* &= \overset{\mp}{\phi}_r^* \overset{\mp}{\phi}_s^*, \quad \overset{\pm}{\phi}_r \overset{\mp}{\phi}_s = \overset{\mp}{\phi}_r \overset{\mp}{\phi}_s, \quad \overset{\pm}{\phi}_r \overset{\mp}{\phi}_s^* = \overset{\mp}{\phi}_r \overset{\mp}{\phi}_s^*,\end{aligned}\quad (2.7)$$

where the alternative signs should be changed simultaneously.

Commutation relations between the new operators are easily deduced from (2.5). We get

$$\begin{aligned}\{\bar{\phi}_r^*, \phi_s^*\} &= \{\bar{\phi}_r^+, \phi_s^+\} = \delta_{rs}, \\ \{\bar{\phi}_r^*, \bar{\phi}_s^*\} &= \{\phi_r^+, \phi_s^+\} = \{\bar{\phi}_r^-, \bar{\phi}_s^-\} = \{\phi_r^-, \phi_s^-\} = \dots = 0.\end{aligned}\quad (2.8)$$

Similar relations hold for the right-hand operators $\overset{\mp}{\phi}^*$ and $\overset{\mp}{\phi}$.

If we introduce further new quantities by

$$\begin{aligned}\bar{\Psi}_r^* &\equiv \phi_r^*, \quad \bar{\Psi}_r \equiv \phi_r^+, \\ \bar{\Psi}_r^* &\equiv \sum_i \{r, i\} \bar{\phi}_i, \quad \bar{\Psi}_r \equiv \sum_i \bar{\phi}_i \{i, r\},\end{aligned}\quad (2.9)$$

the commutation relations for them become just the same as those for the original quantized field quantities:

$$\{\bar{\Psi}_r^*, \bar{\Psi}_s^*\} = \{\bar{\Psi}_s^+, \bar{\Psi}_r^+\} = \{r, s\}, \quad (2.10)$$

all other anticommutators being zero. However, the number of states is doubled since there are quantities with as well as without the asterisk which are regarded as distinct, and the symbols $+$ and $-$ now play the rôle of the asterisk in the original field.

We shall reserve detailed investigations about the above defined operators for later sections, and turn to the case of Bose statistics.

(b) Case of Bose statistics.

Let the quantized wave function of a boson field be denoted by φ^* and φ , each responsible for creation (emission) and destruction (absorption) of a boson respectively. The process of going over to new operators is quite analogous to the case of the fermion. A state $(12\dots n; 1'2'\dots n')$ shall mean the quantity $\varphi_1^* \varphi_2^* \dots \varphi_n^* \varphi_1' \varphi_2' \dots \varphi_n'$ where φ^* 's and φ 's are respectively commutable among themselves, and

$$[\varphi_r(X), \varphi_s^*(X')] \equiv [rX, sX'] = a \text{ } c\text{-number function}, \quad (2.11)$$

$[A, B]$ being the ordinary commutator $AB - BA$. The "wave function" $(12\dots n;$

$1'2'\dots n')$ is symmetrical with respect to interchange of "particles", so that more than one "particles" can occupy one and the same state, contrary to the fermion case. The multiplication formula is now given by

$$\begin{aligned}\vec{\varphi}_r^* \cdot (12\dots n; 1'2'\dots n') &= (r12\dots n; 1'2'\dots n'), \\ \vec{\varphi}_r \cdot (12\dots n; 1'2'\dots n') &= \sum_{i=1}^n [r, i] (12\dots i-1, i+1\dots n; 1'2'\dots n') + (12\dots n; r'1'2'\dots n'), \\ \overleftarrow{\varphi}_r \cdot (12\dots n; 1'2'\dots n') &= (12\dots n; 1'2'\dots n'r), \\ \overleftarrow{\varphi}_r^* \cdot (12\dots n; 1'2'\dots n') &= \sum_{i'=n'}^{1'} [i', r] (12\dots n; 1'2'\dots i'-1, i'+1\dots n') + (12\dots nr; 1'2'\dots n').\end{aligned}\quad (2.12)$$

Introducing such operators that generate or destroy a wave field φ^* or φ , we can write (2.12) as

$$\begin{aligned}\vec{\varphi}_r^* &= \varphi_r^*, & \vec{\varphi}_r &= \varphi_r + \sum_i [r, i] \bar{\varphi}_i^*, \\ \overleftarrow{\varphi}_r &= \varphi_r, & \overleftarrow{\varphi}_r^* &= \varphi_r^* + \sum_i \bar{\varphi}_i [i, r].\end{aligned}\quad (2.13)$$

In the present case, distinction of left- and right-operators is unnecessary since we need not to do with the sign function. The commutation relations are

$$[\bar{\varphi}_r^*, \varphi_s^+] = [\bar{\varphi}_r, \varphi_s^+] = \delta_{rs}, \quad (2.14)$$

other commutators being zero. Introduction of new operators, analogous to (2.9), defined by

$$\begin{aligned}\bar{\Psi}_r^* &\equiv \varphi_r^*, & \bar{\Psi}_r &\equiv \varphi_r, \\ \bar{\Psi}_r^* &\equiv \sum_i [r, i] \bar{\varphi}_i^*, & \bar{\Psi}_r &\equiv \sum_i \bar{\varphi}_i [i, r]\end{aligned}\quad (2.15)$$

recovers the original commutation relations:

$$[\bar{\Psi}_r^*, \bar{\Psi}_s^+] = [\bar{\Psi}_r, \bar{\Psi}_s^+] = [r, s], \quad (2.16)$$

other commutators = 0.

Unfortunately, however, this analogy to the second quantization is not complete. The new operators $\bar{\varphi}^*$ and φ^* (or $\bar{\varphi}$ and φ) are not Hermitian conjugate to each other. (Of course, to terms like "state" and "Hermitian operators" should be given a mathematically precise meaning, which will be attempted later.) This we can see as follows. Let two operators q and p (in the sense of the conventional quantum theory) be related by the commutation relation $[q, p] = 1$, and take a well-ordered function $p^n q^m = (n, m)$. Then the multiplication law reads

$$\vec{p}(n, m) = (n+1, m),$$

$$\begin{aligned}
 \vec{q}(n, m) &= n(n-1, m) + (n, m+1), \\
 \overleftarrow{q}(n, m) &= (n, m+1), \\
 \overleftarrow{p}(n, m) &= m(n, m-1) + (n+1, m).
 \end{aligned}
 \tag{2.17}$$

Hence the "matrix elements" of \vec{p} and \overleftarrow{q} in the representation where n and m are diagonal are given by

$$\vec{p}, \overleftarrow{q} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \end{pmatrix}, \quad \overleftarrow{p}, \overleftarrow{q} = \begin{pmatrix} 0 & 1 & & \\ & 0 & 2 & \\ & & 0 & 3 \\ & & & \ddots & \ddots \end{pmatrix}. \tag{2.18}$$

They are clearly not Hermitian conjugates. In order to amend these defects, we modify the definition of the state and put

$$(12\dots n; 1'2'\dots n') \equiv \frac{1}{\sqrt{a! \beta! \gamma! \dots}} \varphi_1^* \varphi_2^* \dots \varphi_n^* \varphi_1' \varphi_2' \dots \varphi_n', \tag{2.19}$$

where a, β, γ, \dots are the number of particles simultaneously occupying the same state. Then the matrix elements get changed to

$$\vec{p}, \overleftarrow{q} = \begin{pmatrix} 0 & & & \\ \sqrt{1} & 0 & & \\ & \sqrt{2} & 0 & \\ & & \ddots & \ddots \end{pmatrix}, \quad \overleftarrow{p}, \overleftarrow{q} = \begin{pmatrix} 0 & \sqrt{1} & & \\ & 0 & \sqrt{2} & \\ & & \ddots & \ddots \end{pmatrix}. \tag{2.20}$$

That such a definition is reasonable will be further demonstrated in the next section.

We see thus the new operators defined in (a) and (b) exactly duplicate all essential features of the second quantization operators, not speaking of their physical implications. On this reason we should like to call the present procedure *the third quantization* and the new operators the third quantization operators, or for short *the q -3 numbers*. An ordinary operator expressed by means of the quantized field quantities (*q -2 numbers*) may be regarded here as a vector or a wave function in the representation of *the second configuration space*, onto which the *q -3 numbers* operate in turn, transforming it into another vector. Because of this conservation of the quantization character, a sequence of *q - n numbers* is now conceivable of which the *q -1 numbers* will correspond to the ordinary *c -numbers*.⁵⁾ There are, however, many points that need more careful investigation such as the definition of vectors and linear operators for them in the called *q -3 Hilbert space*. They will be treated, though quite incomplete, in some detail in the next section.

3. Vectors and operators in the $q-3$ Hilbert space.

Let ψ^*, ψ and φ^*, φ be the aforementioned $q-2$ numbers obeying certain commutation relations. We call them elementary vectors in the $q-3$ Hilbert space, which describe the properties of the corresponding elementary particles in the free particle (inertial) reference system. A general vector A is the well-ordered function of these elementary vectors:

$$A = \sum a \psi_1^* \psi_2^* \dots \varphi_1^{*\alpha} \varphi_2^{*\beta} \dots \psi_{1''} \psi_{2''} \dots \varphi_{1'''}^{\alpha'} \varphi_{2'''}^{\beta'} \dots, \quad (3.1)$$

where α, β, \dots take the values $1, 2, \dots$. In the representation of the second configuration space it is written as

$$A = \sum a (12\dots; \overbrace{1'1'\dots1'}^{\alpha}, \overbrace{2'2'\dots2'}^{\beta}, \dots; 1''2''\dots; \overbrace{1'''1'''\dots1'''}^{\alpha'}, \overbrace{2'''2'''\dots2'''}^{\beta'}, \dots), \quad (3.2)$$

while in the alternative representation, that is, the proper $q-3$ representation, it reads as

$$A = \sum a (\dots 1_1 \dots 1_2 \dots; \alpha_1 \dots \beta_2 \dots; \dots 1_{1''} \dots 1_{2''} \dots; \dots \alpha'_{1'''} \dots \beta'_{2'''} \dots). \quad (3.3)$$

The norm and the scalar product of vectors shall be by definition

$$\|(12\dots; 1'1'\dots; \dots)\| = \alpha! \beta! \dots,$$

$$\text{and} \quad ((12\dots), (1'2'\dots)) = 0 \quad (3.4)$$

unless all of the state variables coincide; and for composite quantities like (3.2),

$$\begin{aligned} \|A\| &= \sum_i |a_i|^2 \|(i)\|, \\ (A, B) &= \sum_i a_i^* b_i \|(i)\|. \end{aligned} \quad (3.5)$$

This definition is compatible with unitary transformation of the elementary vectors ψ^*, ψ and φ^*, φ . If, for example

$$\psi_i' = \sum_k a_{ik} \psi_k, \quad \varphi_i' = \sum_k b_{ik} \varphi_k, \quad (3.6)$$

with (a_{ik}) and (b_{ik}) being unitary, then

$$\begin{aligned} \|\psi_i'\| &= \sum_k |a_{ik}|^2 \|\psi_k\| = \sum_k |a_{ik}|^2 = 1, \\ \|\varphi_i'\| &= \sum_k |b_{ik}|^2 \|\varphi_k\| = \sum_k |b_{ik}|^2 = 1. \end{aligned} \quad (3.7)$$

Moreover,

$$\begin{aligned} \|\psi_1' \psi_2'\| &= \left\| \sum_k a_{1k} \psi_k \sum_j a_{2j} \psi_j \right\| = \left\| \sum_{k>j} (a_{1k} a_{2j} - a_{1j} a_{2k}) \psi_k \psi_j \right\| \\ &= \sum_{k>j} |a_{1k} a_{2j} - a_{1j} a_{2k}|^2 = \sum_{k>j} \left| \frac{a_{1k} a_{1j}}{a_{2k} a_{2j}} \right|^2 = \sum_k |a_{1k}|^2 \sum_j |a_{2j}|^2 = 1, \end{aligned} \quad (3.8)$$

and

$$\|\varphi_1' \varphi_2'\| = \left\| \sum_k b_{1k} \varphi_k \sum_j b_{2j} \varphi_j \right\| = \sum_{k>j} |b_{1k} b_{2j} + b_{1j} b_{2k}|^2 + 2! \sum_k |b_{1k} b_{2k}|^2$$

$$\begin{aligned}
&= \sum_{k>j} \{ |\delta_{1k}|^2 |\delta_{2j}|^2 + |\delta_{1j}|^2 |\delta_{2k}|^2 + \delta_{1k} \delta_{2k}^* \delta_{1j} \delta_{2j}^* + \delta_{1k}^* \delta_{2k} \delta_{1j}^* \delta_{2j} \} \\
&+ 2 \sum_k |\delta_{1k}|^2 |\delta_{2k}|^2 = \sum_{k,j} |\delta_{1k}|^2 |\delta_{2j}|^2 + \sum_{k,j} \delta_{1k} \delta_{2k}^* \delta_{1j} \delta_{2j}^* = 1 + \delta_{12}.
\end{aligned} \quad (3.9)$$

In general, we can prove that

$$\begin{aligned}
&\|\phi'_1 \phi'_2 \dots \phi'_n\| = 1, \\
&\|\varphi'^\alpha_1 \varphi'^\beta_2 \dots \varphi'^\omega_n\| = \alpha! \beta! \dots \omega!.
\end{aligned} \quad (3.10)$$

This is just the same relation as in the case of ordinary wave function in the configuration space. (3.10) equally holds if we replace the ϕ 's and the φ 's by ϕ^* 's and φ^* 's respectively. These two kinds of vectors are, in our present convention, regarded as representing different kinds of particles. Mixing of them is not here considered as it leads to more complicated situations.

Once the vectors and their metric in configuration space are defined consistently, it is now an easy task to introduce in this space linear operations, and also translate these relations into the named third quantization form. We shall not carry it out here, but only refer to the works by Fock⁴⁾ and others on the second quantization. There are, however, some interesting features in our case which appear to merit special attention. As was mentioned in Section 2, a left- (\vec{A}) or right- (\overleftarrow{A}) multiplication can be considered to be a linear operation in the second configuration space. From them we can construct new operations, for example,

$$\begin{aligned}
[A, X] &\equiv AX - XA = (\vec{A} - \overleftarrow{A}) \cdot X, \\
\{A, X\} &\equiv AX + XA = (\vec{A} + \overleftarrow{A}) \cdot X,
\end{aligned} \quad (3.11)$$

where A is regarded as an operator, while X is the vector on which they operate. It is to be noted that left-hand and right-hand operators always commute:

$$AXB = \vec{A}\vec{B} \cdot X = \overleftarrow{B}\overleftarrow{A} \cdot X. \quad (3.12)$$

Now we take the equation for the unknown X :

$$[A, X] = \lambda X, \quad (3.13)$$

λ being a c -number. This is the analog of the eigenvalue problem. X may be called an eigenvector or an eigenmatrix. If we write this equation in such a representation where A is a diagonal matrix, we see that

$$(E_n - E_m) X_{nm} = \lambda X_{nm}, \quad E_n = \text{eigenvalue of the matrix } A, \quad (3.14)$$

hence $\lambda = E_n - E_m$ if $X_{nm} \neq 0$; that is, λ is a difference of two eigenvalues of A . Product of two eigenvectors yields again an eigenvector with the sum of their eigenvalues for its own eigenvalue:

$$[A, X_1 X_2] = [A, X_1] X_2 + X_1 [A, X_2] = (\lambda_1 + \lambda_2) X_1 X_2. \quad (3.15)$$

The meaning of the eigenvector X may be seen from the following. Let ψ_m be an eigenfunction for A (in the ordinary sense) :

$$A\psi_m = E_m \psi_m. \quad (3.16)$$

Then $AX\psi_m = X(A + \lambda)\psi_m = X(E_m + \lambda)\psi_m = (E_m + \lambda)X\psi_m. \quad (3.17)$

Hence $X\psi_m$ is a new eigenfunction with an eigenvalue $E_m + \lambda$, provided that X does not annihilate ψ_m . In other words, X transforms eigenvectors into other eigenvectors. Such a relation is not novel to us, but has frequently been used in deriving eigenvalues and selection rules for various observables. The Hermitian conjugate of the equation (3.13), that is,

$$[X^*, A] = \lambda X^*, \text{ or } [A, X^*] = -\lambda X^*, \quad (3.18)$$

where A is supposed to be Hermitian, shows that X^* is an eigenvector with eigenvalues $-\lambda$. Thus we have always positive and negative eigenvalues coupled with each other. An eigenvector with $\lambda > 0$ may be called a creation operator, and one with $\lambda < 0$ an annihilation operator. But, of course, this interpretation depends on the physical meaning of A . Evidently

$$[A, X^*X] = [A, XX^*] = 0. \quad (3.19)$$

Such eigenvectors with zero eigenvalues constitute an algebra—the commutator algebra of A . The elementary vectors are special kinds of eigenvectors X and X^* for some A (energy, momentum, and charge operators) which obey simple commutation relations. A remark that may be added here is that in the equation (3.14), the eigenvalue λ is given by the difference of two energies or other physical observables, which is the only quantity that is really observed. This is understandable because X is an operator of dynamical nature that concerns the transition from one state to another. Such a viewpoint seems to promise some advantages over the static eigenvalue problems for stationary states.

The integral form of the commutator equation (3.13) is

$$e^{At} X e^{-At} = e^{\lambda t} X, \text{ or } UXU^{-1} = \lambda X, \quad (3.20)$$

t being a parameter. For the product of two eigenvectors (3.20) becomes

$$UX_1 X_2 U^{-1} = \lambda_1 \lambda_2 X. \quad (3.21)$$

If U is unitary, λ is a complex number with $|\lambda| = 1$, and X remains invariant under such transformations except for a phase factor.

Hitherto we have assumed the operators to be Hermitian or unitary only in the ordinary sense, but a question naturally arises as to whether such properties also exist in the $q=3$ Hilbert space. In connection with this, it is interesting to not that a different, or rather more concrete, definition of the metric is possible. In fact, let

$$\{\psi_r, \psi_s\} = \delta_{rs}, \quad (3.22)$$

and put

$$\|\phi_r\| \equiv M\phi_r^*\phi_r \equiv \text{Tr}\phi_r^*\phi_r = \frac{1}{2} \text{Tr}\{\phi_r^*, \phi_r\} = 1. \quad (3.23)$$

Then

$$\begin{aligned} \|\phi_r\phi_s\| &= M\phi_s^*\phi_r^*\phi_r\phi_s = M\phi_s^*\phi_s M\phi_r^*\phi_r = 1, \quad (r \neq s) \\ \|\phi_r^*\phi_r\| &= M\phi_r^*\phi_r\phi_r^*\phi_r = M\phi_r^*\phi_r = 1, \quad ((\phi_r^*\phi_r)^2 = \phi_r^*\phi_r), \\ \|\phi_r^*\phi_s\| &= M\phi_s^*\phi_r\phi_r^*\phi_s = M\phi_s^*\phi_s M\phi_r\phi_r^* = 1, \quad (r \neq s). \end{aligned} \quad (3.24)$$

In general,

$$\begin{aligned} &\|\phi_1^*\phi_2^*\dots\phi_n^*\phi_1'\phi_2'\dots\phi_n'\| \\ &= M\phi_n^*\phi_{n-1}^*\dots\phi_1^*\phi_n\phi_{n-1}\dots\phi_1\phi_1^*\phi_2^*\dots\phi_n^*\phi_1'\phi_2'\dots\phi_n'. \end{aligned} \quad (3.25)$$

If $\phi_1' = \phi_1$ we first displace ϕ_n^* and ϕ_1 to the left and right side of ϕ_1 and ϕ_1^* respectively (no sign change), then we can take the trace independently of other factors:

$$\text{Tr}\phi_1^*\phi_1\phi_1^*\phi_1' = \text{Tr}\phi_1^*\phi_1 = 1. \quad (3.26)$$

This procedure goes on until all the factors are exhausted. The orthogonality holds for the scalar products of different states as shown by

$$\begin{aligned} (\phi_r, \phi_s) &\equiv M\phi_r\phi_s = \frac{1}{2} \text{Tr}\{\phi_r^*, \phi_s\} = \delta_{rs}, \\ (\phi_r^*, \phi_s) &\equiv M\phi_r\phi_s = 0. \end{aligned} \quad (3.27)$$

In general we can easily show the orthogonality between any different states.

The case of a boson field is somewhat embarrassing because the trace diverges as it contains an infinite number of quantum states. So we modify the definition of the matrix elements of φ^* and φ by a convergence factor:

$$(n+1 | \varphi^* | n) = (n | \varphi | n+1) = \sqrt{n+1} x^{n/2}. \quad (3.28)$$

$$\text{Then,} \quad \text{Tr}\varphi^*\varphi = \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}, \quad (3.29)$$

and in the limit $x \rightarrow 0$, $\text{Tr}\varphi^*\varphi = 1$. This modification is also equivalent to defining the metric by

$$\begin{aligned} \|\varphi\| &= M\varphi^*F\varphi F = \text{Tr}(\varphi^*F\varphi F) / \text{Tr}F^2, \\ F &\equiv \exp(-\sum x\varphi^*\varphi/2\theta), \end{aligned} \quad (3.30)$$

together with

$$\begin{aligned} [\varphi_r, \varphi_s^*] &= \delta_{rs} e^{-1/\theta}, \\ (n+1 | \varphi^* | n) &= (n | \varphi | n+1) = \sqrt{n+1} e^{-1/2\theta} \end{aligned} \quad (3.31)$$

x corresponds to the factor $\exp(-1/\theta)$. Thus

$$\begin{aligned}\|\varphi\| &= \sum_{n=0}^{\infty} (n+1) x^n / \sum x_n = \frac{1}{1-x}, \\ \|\varphi^2\| &= \sum (n+2)(n+1) x^n / \sum x^n = \frac{2!}{(1-x)^2}, \\ \|\varphi^* \varphi\| &= \sum (n+1)^2 x^n / \sum x^n = \sum \{ (n+2)(n+1) - (n+1) \} x^n / \sum x^n \\ &= \frac{2!}{(1-x)^2} - \frac{1}{(1-x)}.\end{aligned}\quad (3.32)$$

They agree with the original definition by putting $x=0$. This means that we take only the first non-vanishing diagonal element. As is seen from above, such a definition is invariant with respect to the unitary transformation of the basis. Of course, it is not essential to take the limiting case $x \rightarrow 0$, or $\theta \rightarrow 0$. Any value of θ (some analog of temperature!) may be admitted. In conformity with (3.30) we may also put for a fermion field

$$\begin{aligned}\|\psi\| &\equiv M(\psi^* F \psi F) = \frac{1}{1+x}, \\ F &= \exp(-\sum x \psi^* \psi / 2\theta), \\ \{\psi_r, \psi_s^*\} &= \delta_{rs} e^{-1/\theta}.\end{aligned}\quad (3.33)$$

In general, we can introduce some suitable convergence factor F , in terms of which the metric is defined for any quantity A :

$$\begin{aligned}\|A\| &= \text{Tr}(A^* F A F) / \text{Tr} F^2, \\ (A, B) &= \text{Tr}(A^* F B F) / \text{Tr} F^2,\end{aligned}\quad (3.34)$$

where A^* is the adjoint of A .

Now let us consider linear operators in the above mentioned metric space, such as

$$\begin{aligned}\vec{A} \cdot X &\equiv AX = Y, \\ i(\vec{A} - \overleftarrow{A}) \cdot X &\equiv i[A]X = Y.\end{aligned}\quad (3.35)$$

We shall use the notations $[A]$ and $\{A\}$ for the commutation and anticommutation instead of $\vec{A} - \overleftarrow{A}$ and $\vec{A} + \overleftarrow{A}$. Then for any two vectors X and Y ,

$$\begin{aligned}(Y, \vec{A}X) &= M(Y^* F \vec{A} X F), \\ (Y, i[A]X) &= M(iY^* F A X F - iY^* F X A F).\end{aligned}\quad (3.36)$$

Hence if A commutes with the metric operator F , we have

$$\begin{aligned}(Y, AX) &= M(Y^* F A X F) = M(Y^* A F X F) = ((A^* Y), X), \\ (Y, i[A]X) &= M(iY^* F A X F - iY^* F X A F) \\ &= M(iY^* A F X F - iY^* F X F A) = (-i[Y, A^*], X) = (i[A^*]Y, X).\end{aligned}\quad (3.37)$$

Thus we can speak of a Hermitian operator H defined by

$$(Y, HX) = (HY, X),$$

which can really be satisfied in the above mentioned case. Clearly such a condition is fulfilled if F is a function of A in (3.36) and (3.37). More generally, assuming that A is the Hamiltonian of a dynamical system, we may take for F any function of the constants of motion. The particular choice

$$F = \exp(-A/2\theta) \quad (3.38)$$

is probably the most interesting since it is just the Boltzmann factor, although it is not yet certain whether such an analogy is merely superficial or not.

The problem of the orthogonal system of eigenvectors and the spectral decomposition of Hermitian operators is very important. But we shall here content ourselves only with the analogy to the ordinary spectral theory and not go too far into mathematics.

Thus far we have treated the degrees of freedom to be discrete. But the transition to continuous degrees which is encountered in the actual case does not cause much trouble, at least so long as we consider only those quantities like (2.9) and (2.15) of Section 2, and not those like (2.8) and (2.14) which have delta-function type commutation relations. On the other hand it seems quite dangerous to regard the totality of the elementary vectors ϕ, ϕ^* and φ, φ^* in space-time as independent states while the actual degrees of freedom make only a three-dimensional manifold. This point will be clarified later.

4. The wave equation in the q-3 formalism.

We shall here consider the theory of Tomonaga and Schwinger, and attempt to write it down in terms of our present formalism. There are two methods of approach, the one starting from the pure Schrödinger representation which was originally done by Tomonaga, and the other from the pure Heisenberg representation as appears adopted by Schwinger³⁾. However, Schwinger's deduction is somewhat obscure. We shall therefore first treat the Schrödinger picture and then try to formulate the Heisenberg picture in a more reasonable fashion.

Let the Tomonaga equation of motion for the Schrödinger functional $\Psi[C]$ and its conjugate $\Psi^*[C]$ be given by

$$i \frac{\partial}{\partial C} \Psi[C] = H \Psi[C], \quad -i \frac{\partial}{\partial C} \Psi^*[C] = H \Psi^*[C], \quad (4.1)$$

where H is the interaction Hamiltonian density. In the case of the interacting electron and electromagnetic field,

$$H = -j_\mu A_\mu, \quad j_\mu = e i \psi^\dagger \gamma_\mu \psi. \quad (4.2)$$

The transformation function $U[CC_0]$ which transforms a Schrödinger functional

$\Psi[C_0]$ at the surface C_0 to $\Psi[C]$ at the surface C obeys the equation of motion

$$\begin{aligned} i \frac{\partial}{\partial C} U[CC_0] &= H(x) U[CC_0], \quad x \in C, \\ -i \frac{\partial}{\partial C_0} U[CC_0] &= U[CC_0] H(x_0), \quad x_0 \in C_0, \end{aligned} \quad (4.3)$$

with

$$U^*[CC_0] = U[C_0 C] = U^{-1}[CC_0]. \quad (4.4)$$

$U[CC_0]$ is a unitary operator composed of the quantized field quantities. Hence we can look upon it as a vector in the second configuration space. Then (4.3) becomes

$$\begin{aligned} i \frac{\partial}{\partial C} U[CC_0] &= \vec{H}(x) U[CC_0], \\ -i \frac{\partial}{\partial C_0} U[CC_0] &= \hat{H}(x_0) U[CC_0], \end{aligned} \quad (4.5)$$

where \vec{H} and \hat{H} are understood to be expressed in terms of the q -3 operators. Integral equations equivalent to (4.5) are given by

$$\begin{aligned} U[CC_0] &= 1 - i \int_{C_0}^C \vec{H}(x') U[C' C_0] dx', \\ U[CC_0] &= 1 - i \int_{C_0}^C \hat{H}(x') U[CC'] dx', \end{aligned} \quad (4.6)$$

which lead to

$$\begin{aligned} U[CC_0] &= (1 - i \int_{C_0}^C \vec{H}' dx' - \int_{C_0}^C \vec{H}' dx' \int_{C_0}^{C'} \vec{H}'' dx'' - \dots) \cdot 1 \\ &= 1 - i \int_{C_0}^C H' dx' - \int_{C_0}^C H' dx' \int_{C_0}^{C'} H'' dx'' - \dots, \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} U[CC_0] &= (1 - i \int_{C_0}^C \hat{H}' dx' - \int_{C_0}^C \hat{H}' dx' \int_{C'}^C \hat{H}'' dx'' - \dots) \cdot 1 \\ &= 1 - i \int_{C_0}^C H' dx' - \int_{C'}^C H'' dx'' \int_{C_0}^{C'} H' dx' - \dots. \end{aligned} \quad (4.8)$$

By interchanging the order of integration in the latter, it is easily verified that the both expressions coincide.

The transformation from the present interaction representation to the Heisenberg representation for any quantity $\mathcal{Q}[C]$ expressed in terms of the elementary vectors on a surface C is performed by

$$\underline{\mathcal{Q}}[CC_0] = U[C_0 C] \mathcal{Q}[C] U[CC_0] = \vec{U}[C_0 C] \hat{U}[CC_0] \cdot \mathcal{Q}[C], \quad (4.9)$$

or, using (4.7) and (4.8),

$$\begin{aligned}\underline{Q}[CC_0] &= (1 + i \int_{c_0}^c \vec{H}' dx' - \int_{c_0}^c \vec{H}' dx' \int_{c'}^c \vec{H}'' dx'' - \dots) \times \\ &\quad \times (1 - i \int_{c_0}^c \overleftarrow{H}' dx' - \int_{c_0}^c \overleftarrow{H}' dx' \int_{c'}^c \vec{H}'' dx'' - \dots) \underline{Q}[C] \\ &= (1 - i \int_{c_0}^c \overleftarrow{H}' dx' - \int_{c_0}^c \overleftarrow{H}' dx' \int_{c'}^c \vec{H}'' dx'' - \dots) (1 + i \int_{c_0}^c \vec{H}' dx' - \int_{c_0}^c \vec{H}' dx' \int_{c'}^c \vec{H}'' dx'' - \dots) \times \\ &\quad \times \underline{Q}[C].\end{aligned}\quad (4.10)$$

Alternatively (4.9) obeys the equation of motion

$$\begin{aligned}i \frac{\delta}{\delta C_0} \underline{Q}[CC_0] &= [H(x), \underline{Q}[CC_0]] = (\vec{H}(x_0) - \overleftarrow{H}(x_0)) \underline{Q}[CC_0] \\ &= [H(x_0)] \cdot \underline{Q}[CC_0].\end{aligned}\quad (4.11)$$

(4.11) can be replaced by the integral equation

$$\underline{Q}[CC_0] = \underline{Q}[C] + i \int_{c_0}^c [H(x')] \cdot \underline{Q}[CC'] dx' \quad (4.12)$$

which leads to the development

$$\underline{Q}[CC_0] = (1 + i \int_{c_0}^c [H(x')] dx' - \int_{c_0}^c [H(x')] dx' \int_{c'}^c [H(x'')] dx'' - \dots) \underline{Q}[C]. \quad (4.13)$$

This must agree with the expression (4.10). In fact

$$\begin{aligned}& (1 + i \int_{c_0}^c \vec{H}' dx' - \int_{c_0}^c \vec{H}' dx' \int_{c'}^c \vec{H}'' dx'' - \dots) (1 - i \int_{c_0}^c \overleftarrow{H}' dx' - \int_{c_0}^c \overleftarrow{H}' dx' \int_{c'}^c \vec{H}'' dx'' - \dots) \\ &= 1 + i \int_{c_0}^c (\vec{H}' - \overleftarrow{H}') dx' + \left(\int_{c_0}^c \vec{H}' dx' \int_{c_0}^c \overleftarrow{H}' dx' - \int_{c_0}^c \vec{H}' dx' \int_{c'}^c \vec{H}'' dx'' - \int_{c_0}^c \overleftarrow{H}' dx' \int_{c'}^c \vec{H}'' dx'' \right) - \dots \\ &= 1 + i \int_{c_0}^c [H'] dx' + \\ &+ \left(\int_{c_0}^c \vec{H}' dx' \int_{c_0}^c \overleftarrow{H}'' dx'' + \int_{c_0}^c \overleftarrow{H}' dx' \int_{c'}^c \vec{H}'' dx'' - \int_{c_0}^c \vec{H}' dx' \int_{c'}^c \vec{H}'' dx'' - \int_{c_0}^c \overleftarrow{H}' dx' \int_{c'}^c \vec{H}'' dx'' \right) - \dots \\ &= 1 + i \int_{c_0}^c [H'] dx' + \\ &+ \left(\int_{c'}^c \vec{H}' dx' \int_{c_0}^c \overleftarrow{H}'' dx'' + \int_{c_0}^c \vec{H}' dx' \int_{c'}^c \overleftarrow{H}'' dx'' - \int_{c_0}^c \vec{H}'' dx'' \int_{c_0}^c \vec{H}' dx' - \int_{c_0}^c \overleftarrow{H}' dx' \int_{c'}^c \overleftarrow{H}'' dx'' \right) - \dots \\ &= 1 + i \int_{c_0}^c [H'] dx' - \int_{c_0}^c [H'] dx' \int_{c'}^c [H''] dx'' - \dots,\end{aligned}\quad (4.14)$$

using the commutability of left- and right-operators.

On this occasion it may be instructive to derive the equation of motion (4.11) in the interaction representation starting from the pure Heisenberg picture. The Tomonaga equation was derived from the pure Schrödinger picture:

$$i \frac{\partial}{\partial t} \Psi_0 = (H_0 + H_i) \Psi_0, \quad (4.15)$$

where H_0 is the inertial Hamiltonian (integrated over space), by the transformation

$$\Psi_0 = S\Psi, \quad H_i \rightarrow S^{-1}H_iS, \quad S = \exp(-iH_0t). \quad (4.16)$$

On the other hand, the equation of motion in the pure Heisenberg picture for any quantity Q_0 is

$$-i \frac{\partial}{\partial t} Q_0 = [H_0 + H_i, Q] = ([H_0] + [H_i]) \cdot Q_0. \quad (4.17)$$

As the total Hamiltonian $H_0 + H_i$ itself is a constant of motion, we may insert for H_0 and H_i in (4.17) their initial value at time $t=0$. Now we perform the transformation

$$Q_0 = SQ, \quad S = \exp(i[H_0]t) \quad (4.18)$$

which changes the equation of motion (4.17) into

$$-i \frac{\partial}{\partial t} Q = [H'_i] Q \quad (4.19)$$

with

$$[H'_i] = S^{-1}[H_i]S.$$

The transformation (4.18) and (4.19) means that

$$\begin{aligned} Q &= S^{-1}Q_0 = \exp(-i[H_0]t) \cdot Q_0 = \left(\sum_{n=0}^{\infty} \frac{1}{n!} (-i[H_0]t)^n \right) \cdot Q_0 \\ &= \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} [H_0 [H_0 \dots [H_0, Q_0] \dots]], \end{aligned}$$

$$\text{or} \quad = \exp(-i\vec{H}_0 t) \exp(i\vec{H}_0 t) Q_0 = e^{iH_0 t} Q_0 e^{iH_0 t}; \quad (4.20)$$

$$\begin{aligned} \text{and} \quad [H'_i] &= \exp(-i[H_0]t) [H_i] \exp(i[H_0]t) \\ &= e^{-iH_0 t} H_i e^{iH_0 t} - e^{-iH_0 t} H_i e^{iH_0 t}. \end{aligned} \quad (4.21)$$

Thus these transformed quantities varies with time according to their inertial motion. The equation (4.19) directly leads to its super-many-time generalization

$$-i \frac{\partial}{\partial C} Q[C] = [H'_i(X)] \cdot Q[C] \quad (4.22)$$

in which $Q[C]$ means that it is regarded as expressed in terms of the elementary vectors on a space-like surface C . This is the equation of motion in the

interaction representation for a dynamical variable Q in contrast to the ordinary equation which describes the motion of the wave functional Ψ . When the Hamiltonian $[H_i]$ is expressed in terms of the q -3 operators, $Q[C]$ just plays the rôle of a wave functional on which they operate. In other words, the Heisenberg equation is turned into a Schrödinger equation. The integral form of (4.22) is

$$Q[C] = Q[C_0] + i \int_{C_0}^C [H'] dx' Q[C'] \quad (4.23)$$

$$= (1 + i \int_{C_0}^C [H'] dx' - \int_{C_0}^C [H'] dx' \int_{C_0}^{C'} [H''] dx'' - \dots) \cdot Q[C_0]. \quad (4.24)$$

The difference between the equation (4.11) to (4.13) on one hand and (4.22) to (4.24) on the other comes from the fact that, in the former case, we consider the change of $\Psi[C]$ due to the interaction while the operators are assumed to be varying as free; in the latter the situation is contrary, and the operators suffer changes through interaction while the wave functional vector is moving as free. In order to obtain agreement, we have only to interchange the rôle of C and C_0 in either expressions.

If the operator $[H_i]$ in (4.22) turns out to be Hermitian in the sense of the third quantization (as an ordinary operator, H_i is of course Hermitian), then we can speak of the conservation of the norm of the vector $Q[C]$. According to the second definition of the norm in Section 3, this will be the case provided that H_i commutes with the metric operator F . But it seems fairly difficult to investigate this problem from such a viewpoint, so that we shall touch it later by using more explicit expressions for the q -3 operators.

Another important problem is that of the supplementary condition. Here the Heisenberg picture alone does not suffice since the supplementary condition is inevitably connected with the wave functional. To dispense with the explicit use of the wave functional, it may be proposed to restrict the Hamiltonian operator itself to the suboperator which operates on the allowed domain of the wave functionals. Thus we replace the Hamiltonian H by $PHP = PH = HP$, where P is the projection operator for the subspace in which the supplementary condition $A\Psi = 0$ holds. However, the construction of such a projector would meet with various difficulties so that it does not appear adequate, although the conventional formalism is also accompanied by inconveniences. (*To be continued.*)

References.

- 1) Outline of the present article is reported in *Prog. Theor. Phys.* **4** (1949), 96.
- 2) Tomonaga, *Phys. Rev.* **74** (1948), 224; Tati and Tomonaga, *Prog. Theor. Phys.* **3** (1948), 391.
- 3) Schwinger, *Phys. Rev.* **73** (1948), 416; **74** (1948), 1439.
- 4) Fock, *Zeits. f. Phys.* **75** (1932), 622.
- 5) The q -1 number is the one-body wave function (fermion) or the classical field (boson); the q -2 number is the ordinary quantized wave.

On the γ -Decay of Neutral Meson.

Hiroshi FUKUDA and Yoneji MIYAMOTO.

Physics Institute, Tokyo University.

(Received May 16, 1949)

Introduction.

Recently Tomonaga¹⁾ and Schwinger²⁾ have independently developed a covariant formulation of quantum electrodynamics (super-many-time theory), and have successfully applied it to the explanation of the Lamb shift in the hydrogen atom and the anomalous magnetic moment of the electron. They have shown that, although the present theory of fields, in general, gives infinite answers to such field reaction problems, since it cannot be formulated in a Lorentz- and gauge-covariant way without introduction of the singular delta function of Jordan and Pauli, it is nevertheless possible to avoid the divergences by amalgamating them into the mass and charge, and that the remaining finite term can well account for the experimental facts.^{2,3)} But there remains the question, whether or not such finite term, as is separated from infinity, can be free from any ambiguity arising from the pathological character of delta function of Jordan and Pauli.

One typical example of the appearance of such an ambiguity is the photon self energy. As first pointed out by Schwinger,²⁾ the photon self energy should be zero from the gauge covariant point of view, while, completely against Schwinger's prediction, the recent calculation by Wentzel⁴⁾ leads to a finite but non-gauge covariant result for the photon self energy. It is evident that this inconsistent result comes from the mathematical difficulty of obtaining a definite expression for the singularity of the light cone of Jordan-Pauli's delta function.

A very similar situation is also encountered in the γ decay of neutral meson. By using the method of evaluation which has been applied by Schwinger²⁾ to the calculation of the anomalous magnetic moment of the electron, we have obtained the convergent but non-gauge covariant result for the γ decay of neutral meson. Further our result is inconsistent with the recent discussions of Dyson, Sawada

1) S. Tomonaga: *Prog. Theor. Phys.* **1**, (1946) 27.

Koba, Tati and Tomonaga: *Prog. Theor. Phys.* **2**, (1947) 101 and 198.

2) J. Schwinger: *Phys. Rev.* **74**, (1948) 1439; **75**, (1949) 651 and his unpublished manuscripts.

3) Z. Koba and S. Tomonaga: *Prog. Theor. Phys.* **2**, (1948) 218,

T. Tati and S. Tomonaga: *Prog. Theor. Phys.* **3**, (1948) 391.

H. Fukuda, Y. Miyamoto and S. Tomonaga: *ibid.* in press.

4) G. Wentzel: *Phys. Rev.* **74**, (1948) 1070.

and Nambu⁵⁾ on the equivalence between the pseudoscalar and pseudovector couplings of the pseudoscalar meson field. Thus, in the present state of the field theory, we cannot give an unambiguous life-time for neutral meson. It is very desirable to find some general prescription which enables one to get the definite answer to field reaction problems.

§ 1. General Formulation.

In this paper we shall be chiefly interested in the two quanta disintegration of neutral meson which was discussed by Tanikawa⁶⁾ and Finkelstein⁷⁾ in perturbation theory. To use the word of perturbation theory this process can be described by the following several steps:

$$\mu \rightarrow P^+ + P^- \rightarrow P^+ + P^- + \gamma_1 \rightarrow \gamma_1 + \gamma_2 \text{ etc.}$$

P^+ , P^- , μ and γ represent respectively a proton, an antiproton, a meson and a photon. The matrix elements for these processes are generally divergent, since there is no restriction on the momentum of the virtually created proton and antiproton pairs. However, our recalculation by perturbation theory yields the convergent life-time for neutral meson in contrast to those of the above mentioned authors. This seems to us to be due to some errors in the former calculations.

Covariant formulation of wave fields is required in order to treat the process associated with infinity without loss of Lorentz- and gauge-covariance. Thus Tomonaga and Schwinger theory will be employed throughout this paper.

In the super-many-time theory, the generalized Schrödinger equation, describing the system of a neutral meson, nucleon and electromagnetic field, is given by

$$\left\{ -ie[\varphi^+ \tau_p \gamma_i \varphi] A_i + H - i\hbar c \frac{\delta}{\delta C} \right\} \Psi = 0, \quad (1.1)$$

where the first and second terms represent the interaction energy between proton and electromagnetic field, and the interaction energy between neutral meson and nucleon respectively. In (1.1) φ means the spinor describing the nucleon, A_i the potential for electromagnetic field. Further τ_p is the projection operator in the proton state, and $[\varphi^+ \tau_p \gamma_i \varphi]$ means the abbreviation for

$$\frac{1}{2} (\varphi^+ \tau_p \gamma_i \varphi - \varphi \tau_p \tilde{\gamma}_i \varphi^+).$$

The interaction term H is given as follows;

5) F. J. Dyson: Phys. Rev. **73**, (1948) 929.

Y. Nambu and K. Sawada: Lectures at Tokyo meeting on October in (1948).

6) Y. Tanikawa: Proc. Phys. Math. Soc. **24**, (1940) 610.

7) R. J. Finkelstein: Phys. Rev. **72**, (1947) 414. He has overlooked the contributions from the product of the first terms in numerator and the second terms in denominator in the expansion of the wavenumber of photon.

for the scalar meson field

$$H = f[\varphi^+ \varphi] V \quad (1.2)$$

for the pseudoscalar meson with pseudovector and pseudoscalar couplings,

$$H = \frac{if}{\mu} [\varphi^+ \gamma_5 \gamma_i \varphi] \partial_i V + \frac{2\pi}{\mu^2} f^2 ([\varphi^+ \gamma_5 \gamma_N \varphi])^2 \quad (1.3)$$

and

$$H = if[\varphi^+ \gamma_5 \varphi] V \quad (1.4)$$

and for pseudovector meson with pseudovector coupling, $\partial_i V$ in (1.3) is to be replaced by V_i , where $V(V_i)$ is the potential for meson field. We denote by μ , the reciprocal compton wavelength of the meson. The coupling constant is denoted by f . Here the interaction energies with vector or tensor coupling are not written down, since, according to Furry's theorem, they do not contribute to the two quanta disintegration of the neutral meson (see (A) in this section.)

The matrixelement responsible for the γ decay of the neutral meson can be obtained by the following canonical transformation.

$$\Psi = U \Psi_1, \quad \text{with } U \text{ determined by}$$

$$\left\{ -ie[\varphi^+ \tau_p \gamma_i \varphi] A_i - i\hbar c \frac{\delta}{\delta C} \right\} U = 0 \quad (1.5)$$

Now the original equation (1.1) is transformed by (1.5) into

$$\left\{ U^{-1} H U - i\hbar c \frac{\delta}{\delta C} \right\} \Psi_1 = 0 \quad (1.6)$$

On inserting the expression (1.2), (1.3) and (1.4) in $U^{-1} H U$, we obtain:

$$\text{for (1.2)} \quad f V U^{-1} [\varphi^+ \varphi] U, \quad (1.2')$$

$$\text{for (1.3)} \quad \frac{if}{\mu} \partial_i V U^{-1} [\varphi^+ \gamma_5 \gamma_i \varphi] U, \quad (1.3')$$

$$\text{and for (1.4)} \quad if V U^{-1} [\varphi^+ \gamma_5 \varphi] U, \quad (1.4')$$

where the term in f^2 plays no essential role in our problem, so it may be dropped off. The matrixelement for the γ decay of the neutral meson, in which there appear no proton and antiproton in the initial and final states, can be deduced from (1.2'), (1.3') and (1.4') by evaluating the vacuum expectation value of the bilinear expression $U^{-1} [\varphi^+ L \varphi] U$. Further, it is allowed to treat potential A_i as if it were a c-number, if we are not interested in the radiative correction to the γ instability of neutral meson. Consequently, according to the well known formula for $U^{-1} T U$ (T being some general operator)

$$U^{-1} T U = \sum_{n=0}^{\infty} \left(\frac{e}{\hbar c} \right)^n \int dC_1 \int dC_2 \cdots \int dC_n [B(X^{(n)}) \times \\ [B(X^{(n-1)}) \cdots [B(X^{(1)}), T] \cdots]] \quad (1.7)$$

(where $B=[\varphi^+ \tau_p \gamma_i \varphi] A_i$),

$$U^{-1} \varphi U = \sum_{n=0}^{\infty} \left(\frac{i\epsilon}{\hbar c} \right)^n \int dC_1 \int dC_2 \cdots \int dC_n^{n-1} S(X - X^{(1)}) (\gamma A^{(1)}) \cdots \times \\ S(X^{(i)} - X^{(i+1)}) (\gamma A^{(i+1)}) \cdots S(X^{(n-1)} - X^{(n)}) (\gamma A^{(n)}) \varphi^{(n)} \quad (1.8)$$

$$U^{-1} \varphi^+ U = \sum_{n=0}^{\infty} \left(\frac{-i\epsilon}{\hbar c} \right)^n \int dC_1 \int dC_2 \cdots \int dC_n^{n-1} \varphi^{+(n)} (\gamma A^{(n)}) S(X^{(n)} - X^{(n-1)}) \cdots \times \\ S(X^{(i)} - X^{(i-1)}) (\gamma A^{(i-1)}) \cdots (\gamma A^{(1)}) S(X^{(1)} - X), \quad (1.9)$$

where we have used the commutation relation

$$[\varphi_a^+, \varphi_b]_+ = \frac{1}{i} S_{\beta\alpha}(X - X') = \frac{1}{i} (\gamma \partial - x)_{\beta\alpha} \mathcal{A}(X - X'), \quad (1.10)$$

$\mathcal{A}(x)$ being the Jordan-Pauli's delta function for proton field. (x being the reciprocal compton wave length of the proton.) The vacuum expectation value for the expression $[\varphi_a^+, \varphi_b]$ can be constructed;

$$\langle [\varphi_a^+, \varphi_b] \rangle_0 = + \frac{1}{2} S_{\beta\alpha}^{(1)}(X - X') \quad (1.11)$$

By virtue of (1.8), (1.9) and (1.11), the vacuum expectation value of $U^{-1} [\varphi^+ L \varphi] U$ becomes

$$\langle U^{-1} [\varphi^+ L \varphi] U \rangle_0 = \sum_{n=0}^{\infty} ([\varphi^+ L \varphi])_n, \quad (1.12)$$

$$\text{where } ([\varphi^+ L \varphi])_n = + \frac{1}{2} \left(\frac{\epsilon}{\hbar c} \right)^n \sum_{(s+t'=n)} (i)^{s-t'} \left(\int dC_1 \cdots \int dC_s^{s-1} \right) \left(\int dC_1' \cdots \int dC_{t'}^{t'-1} \right) \times \\ Sp \{ S(X - X^{(1)}) (\gamma A^{(1)}) \cdots S(X^{(s-1)} - X^{(s)}) (\gamma A^{(s)}) S^{(1)}(X^{(s)} - X^{(t')}) (\gamma A^{(t')}) \times \\ S(X^{(t')} - X^{(t'-1)}) \cdots (\gamma A^{(1)'}) S(X^{(1')} - X) L \}$$

Following Schwinger's procedure, we introduce the relation;

$$\int dC_j^{j-1} = \frac{1}{2} \int_{-\infty}^{+\infty} (1 + \epsilon(X^{(j-1)} - X^{(j)})) dX^{(j)},$$

where $\epsilon(x) = \pm 1$, according as $x_0 \leq 0$, noting that $\int_{-\infty}^{+\infty} dX^{(j)}$ can be discarded, since we are not interested in the real process in the intermediate state. Accordingly, when $\int dC_j^{j-1}$ is replaced by $\frac{1}{2} \int_{-\infty}^{+\infty} \epsilon(X^{(j-1)} - X^{(j)}) dX^{(j)}$, we obtain

$$([\varphi^+ L \varphi])_n = + \frac{1}{2} \left(\frac{-i\epsilon}{\hbar c} \right)^n \sum_{(s=0, \dots, n)} \int_{-\infty}^{+\infty} dX^{(1)} \cdots dX^{(n)} Sp \{ \bar{S}(X - X^{(1)}) (\gamma A^{(1)}) \cdots \times \\ (\gamma A^{(s)}) S^{(1)}(X^{(s)} - X^{(s+1)}) (\gamma A^{(s+1)}) \bar{S}(X^{(s+1)} - X^{(s+2)}) \cdots (\gamma A^{(n)}) \bar{S}(X^{(n)} - X) L \} \quad (1.13)$$

where $\bar{S}(X) = -\frac{1}{2} \epsilon(X) S(X)$. Evidently $([\varphi^+ L \varphi])_n$ will cause the n -quanta disintegration of neutral meson. Before carrying through the evaluation of (1.13) we will now derive some general properties of the expression (1.13). (A) Here we shall show the Furry's theorem⁸⁾ in our formulation. For the proof of this theorem, the following property of spur will be necessary;

$$Sp \{ (\gamma a_1)(\gamma a_2) \cdots (\gamma a_{n-1})(\gamma a_n) \} = Sp \{ (\gamma a_n)(\gamma a_{n-1}) \cdots (\gamma a_2)(\gamma a_1) \} \quad (1.14)$$

Applying lemma (1.14) to (1.13), and changing the integration variable $X^{(i)}$ into $X^{(n+1-i)}$, the integrand in (1.13) with $i=n-i$ becomes

$$Sp \{ \bar{S}^*(X - X^{(i)})(\gamma A^{(i)}) \cdots (\gamma A^{(n)}) S^{*(1)}(X^{(i)} - X^{(i+1)})(\gamma A^{(i+1)}) \cdots \times \\ \bar{S}^*(X^{(n)} - X) L^* \}, \quad (1.15)$$

where

$$\begin{aligned} \bar{S}^*(X)(S^{(1)*}(X)) &= -(\gamma \partial + x) \bar{A}(X)(A^{(1)}(X)) \\ L^* &= \epsilon L, \quad \epsilon = +1 \quad \text{for } L=1, \gamma_i, \gamma_{1234} \\ &\quad \epsilon = -1 \quad \text{for } L=\gamma_{ijk}, \gamma_{ij}. \end{aligned} \quad (1.16)$$

Noting the fact that spur of odd number of γ 's vanishes, it can be shown that for L of product of even (odd) number of γ 's,

$$(1.15) = \pm(\mp)\epsilon \cdot (1.13), \quad \text{according as } n=\text{even or odd}. \quad (1.17)$$

Combining (1.17) with (1.16),

(1.15) = $\pm \epsilon_0 \cdot (1.13)$ according as $n=\text{even or odd}$, and $\epsilon_0 = +1$ for $L=1, \gamma_{ijk}, \gamma_{1234}$, and $\epsilon_0 = -1$ for $L=\gamma_i, \gamma_{ij}$. Thus, on addition of (1.15) and (1.13), we find the theorem; neutral meson with scalar, pseudoscalar or pseudovector coupling cannot disintegrate into odd number of photons, while neutral meson with vector or tensor couplings cannot disintegrate into even number of photons.

Further, the expression (1.13) has another two important properties; (B) gauge invariance and (C) $\partial_i ([\varphi^+ \gamma_5 \gamma_i \varphi])_n = -2x([\varphi^+ \gamma_5 \varphi])_n$ and $\partial_i ([\varphi^+ \gamma_i \varphi])_n = 0$. These facts are evident without any proof, if we remember that $U^{-1}[\varphi^+ L \varphi]U$ is regarded as the Heisenberg representation of $[\varphi^+ L \varphi]$ in the presence of unquantized potential. But the direct proof of (B) and (C) shall be given for the terms of order e^2 in (1.13), which are

$$\begin{aligned} ([\varphi^+ L \varphi])_2 &= -\frac{1}{2} \left(\frac{e}{\hbar c} \right)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dX' dX'' Sp \{ S^{(1)}(X - X'')(\gamma A'') \bar{S}(X'' - X')(\gamma A') \times \\ &\quad \bar{S}(X' - X) L + \bar{S}(X - X'')(\gamma A'') \bar{S}(X'' - X')(\gamma A') S^{(1)}(X' - X) L \\ &\quad + \bar{S}(X - X'')(\gamma A'') S^{(1)}(X'' - X')(\gamma A') \bar{S}(X' - X) L \}. \end{aligned} \quad (1.18)$$

8) W. H. Furry: Phys. Rev. 51, (1937) 125.

The proof runs as follows;

(B) The gauge invariance, i.e., the expression (1.18) is invariant under the transformation

$$A_i \rightarrow A_i + \partial_i A. \quad (\square A = 0). \quad (1.19)$$

Inserting (1.19) into (1.18), the terms bilinear in A are

$$\begin{aligned} & -\frac{1}{2} \left(\frac{e}{\hbar c} \right)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dX' dX'' A' \partial_j'' A'' \partial_i' A' S \rho \{ S^{(0)}(X-X'') \gamma_j S(X''-X') \gamma_i \bar{S}(X'-X) L \\ & \quad + \bar{S}(X-X'') \gamma_i \bar{S}(X''-X') \gamma_j S^{(0)}(X'-X) L \\ & \quad + \bar{S}(X-X'') \gamma_j S^{(0)}(X''-X') \gamma_i \bar{S}(X'-X) L \end{aligned} \quad (1.20)$$

Integrating by parts, (1.20) becomes

$$\begin{aligned} & + \frac{1}{2} \left(\frac{e}{\hbar c} \right)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dX' dX'' A' \partial_j'' A'' S \rho \{ S^{(0)}(X'-X'') \gamma_j (\partial_i(X''-X') \bar{S}(X'-X) \\ & \quad - \bar{S}(X''-X') \partial(X'-X)) L + \bar{S}(X-X'') \gamma_j (\partial(X''-X') \times \\ & \quad S^{(0)}(X'-X) - S^{(0)}(X''-X') \partial(X'-X)) L \} \\ & = \frac{1}{2} \left(\frac{e}{\hbar c} \right)^2 \int_{-\infty}^{+\infty} dX' (A'-A) \partial_j' A' S \rho \{ S^{(0)}(X-X') \gamma_j \bar{S}(X'-X) L \\ & \quad + \bar{S}(X-X') \gamma_j S^{(0)}(X'-X) L \}. \end{aligned} \quad (1.20')^*$$

By integrating by parts again,

$$\begin{aligned} & -\frac{1}{2} \left(\frac{e}{\hbar c} \right)^2 \int_{-\infty}^{+\infty} \frac{1}{2} (A'-A)^2 S \rho \{ -S^{(0)}(X-X') L \partial(X'-X) \\ & \quad + \partial(X-X') S^{(0)}(X'-X) L \} = 0, \end{aligned} \quad (1.21)$$

where we have utilized the relations $(\gamma \partial + \kappa) \bar{S}(X-X') = -\partial(X-X')$ and $\bar{S}(X'-X)(\gamma \partial - \kappa) = +\partial(X'-X)$. The terms linear in A cancel each other in the similar way.

(C) The identity (C). We can verify identity $\partial_i([\varphi^+ \gamma_5 \gamma_i \varphi])_2 = -2\kappa([\varphi^+ \gamma_5 \varphi])_2$. Performing the differentiation ∂_i on the left-hand side we obtain

$$\begin{aligned} \partial_i([\varphi^+ \gamma_5 \gamma_i \varphi])_2 &= -2\kappa([\varphi^+ \gamma_5 \varphi])_2 - \frac{1}{2} \left(\frac{e}{\hbar c} \right)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dX' dX'' S \rho \{ (\gamma A'') \bar{S}(X''-X') \times \\ & \quad (\gamma A') (-\partial(X'-X) \gamma_5 S^{(0)}(X-X'') - S^{(0)}(X'-X) \gamma_5 \partial(X-X'')) \\ & \quad + (\gamma A'') S^{(0)}(X''-X') (\gamma A') (-\partial(X'-X) \gamma_5 \bar{S}(X-X'') \\ & \quad - \bar{S}(X'-X) \gamma_5 \partial(X-X'')) \}. \end{aligned} \quad (1.22)$$

The second term on the right of (1.22) becomes

* See note added in proof (1).

$$\frac{1}{2} \left(\frac{e}{\hbar c} \right)^2 \int_{-\infty}^{+\infty} dX' S p \{ (\gamma A') \bar{S}(X' - X) ((\gamma A) \gamma_5 + \gamma_5 (\gamma A)) S^{(1)}(X - X') \\ + (\gamma_5 (\gamma A) + (\gamma A) \gamma_5) \bar{S}(X - X') (\gamma A') S^{(1)}(X' - X) \}$$

which vanishes by the property of γ_5 that anticommutes with (γA) . We have thus established the required proof of the identity.

§ 2. Evaluation of the Matrixelement.

In this section we shall give only the outline of the evaluation of (1.18), since it can be carried out in the same method as was used by Schwinger in computing the anomalous magnetic moment of the electron.

$$([\varphi^+ L \varphi])_2 = -\frac{1}{2} \left(\frac{e}{\hbar c} \right)^2 \int_{-\infty}^{+\infty} dX' dX'' A_j' A_i' K_{ji}(X' - X, X - X''), \quad (2.1)$$

where

$$K_{ji}(\xi, \eta) = S p \{ S^{(1)}(\eta) \gamma_j \bar{S}(-\xi - \eta) \gamma_i \bar{S}(\xi) L + \bar{S}(\eta) \gamma_j S^{(1)}(-\xi - \eta) \gamma_i \bar{S}(\xi) L \\ + \bar{S}(\eta) \gamma_j \bar{S}(-\xi - \eta) \gamma_i S^{(1)}(\xi) L - \frac{1}{4} S^{(1)}(\eta) \gamma_j S^{(1)}(-\xi - \eta) \gamma_i S^{(1)}(\xi) L \} \quad (2.2)$$

For easier calculation, we supplement the fourth term in (2.2), which is shown to be zero. Inserting in (2.2)

$$\bar{S}(X) = -\frac{i}{2(2\pi)^4} \int_{-\infty}^{+\infty} d^4 k \frac{a}{|a|} da (i\gamma k - x) e^{i(kX) + ia(k^2 + x^2)}$$

$$\text{and} \quad \bar{S}^{(1)}(X) = \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} d^4 k db (i\gamma k - x) e^{i(kX) + ib(k^2 + x^2)}, \quad (2.3)$$

we obtain

$$K_{ji}(\xi, \eta) = \frac{-1}{4(2\pi)^{12}} \int_{-\infty}^{+\infty} da db dc d^4 k d^4 k' d^4 k'' \left(\left(\frac{a}{|a|} + \frac{b}{|b|} \right) \frac{c}{|c|} + \frac{a}{|a|} \frac{b}{|b|} + 1 \right) \\ \exp \{ i(k\xi) + i(k'\eta) + i(k'', \xi + \eta) + ia(k^2 + x^2) + ib(k'^2 + x^2) + ic(k''^2 + x^2) \} \\ \times S p \{ \gamma_i (i(\gamma k) - x) L (i(\gamma k') - x) \gamma_j (-i(\gamma k'') - x) \} \quad (2.4)$$

By making the substitution

$$k + k'' \rightarrow k, \quad k' + k'' \rightarrow k', \quad k'' \rightarrow k'' + \frac{ak + bk'}{a + b + c}, \quad (2.5)$$

noting that, as for new variable, $k^2 = k'^2 = 0$, and using the identity

$$\left(\frac{a}{|a|} + \frac{b}{|b|} \right) \left(\frac{a+b}{|a+b|} \right) = \frac{a}{|a|} \cdot \frac{b}{|b|} + 1$$

(2.4) can be simplified as follows

$$K_{ji}(\xi, \eta) = \frac{-1}{4(2\pi)^{12}} \int_{-\infty}^{+\infty} da db dc d^4 k d^4 k' d^4 k'' \left(\frac{a}{|a|} + \frac{b}{|b|} \right) \left(\frac{c}{|c|} + \frac{a+b}{|a+b|} \right) \times \\ \exp \left\{ i(k\xi) + i(k'\eta) + i(a+b+c)(k'^2 + x^2) - i \frac{2ab(kk')}{a+b+c} \right\} \times \\ Sp \{ \gamma_i (i(\gamma, k - k'' - P) - x) L(i(\gamma, k' - k'' - P) - x) \gamma_j (-i(\gamma, k'' + P) - x) \}, \quad (2.6)$$

where $P = \frac{ak + bk'}{a + b + c}$. Integrating over k'' , and introducing the new variables u , v , and w by means of

$$a = \frac{1}{2x^2} u w (1+v), \quad b = \frac{1}{2x^2} u w (1-v), \quad \text{and} \quad C = \frac{1}{x^2} u (u-1) w \quad (2.7)$$

we obtain

$$K_{ji}(\xi, \eta) = \frac{-i\pi^2}{4(2\pi)^{12}} \int_{-\infty}^{+\infty} d^4 k d^4 k' e^{i(k\xi) + i(k'\eta)} \int_{-1}^{+1} dv \int_1^\infty du \int_{-\infty}^{+\infty} d\tau e^{i\tau w} \left\{ u^2 - \frac{(kk')}{2x^2} (1-v^2) \right\} \times \\ \left[\frac{i}{x^2} \frac{2}{u} \frac{w}{|w|} Sp \{ \gamma_i (i(\gamma, k - P) - x) L(i(\gamma, k' - P) - x) \gamma_j (-i\gamma P - x) \} \right. \\ \left. - i \frac{1}{u^3} \frac{1}{|w|} Sp \{ \gamma_i \gamma_\mu L \gamma_\mu \gamma_j (-i\gamma P - x) - 2\gamma_i (i(\gamma, k - P) - x) L \gamma_j \right. \\ \left. - 2\gamma_i L (i(\gamma, k' - P) - x) \gamma_j \} \right], \quad (2.8)$$

where

$$P = \frac{1+v}{2u} k + \frac{1-v}{2u} k'$$

Integrating over w , according to

$$\int_{-\infty}^{+\infty} \frac{w}{|w|} d\tau e^{i\tau w} = \frac{2i}{a}, \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{d\tau w}{|w|} e^{i\tau w} = -2 \log |a| + 2 \int_0^\infty \frac{\cos \tau w}{w} d\tau \quad (2.9)$$

(2.8) becomes

$$K_{ji}(\xi, \eta) = \frac{\pi^2}{(2\pi)^{12} x^2} \int_{-\infty}^{+\infty} d^4 k d^4 k' e^{i(k\xi) + i(k'\eta)} \int_{-1}^{+1} dv \int_1^\infty \frac{du}{u} \left[\frac{1}{u^2 - \frac{(kk')}{2x^2} (1-v^2)} \times \right. \\ Sp \{ \gamma_i (i(\gamma, k - P) - x) L(i(\gamma, k' - P) - x) \gamma_j (-i(\gamma P) - x) \} + \frac{x^2}{u^2} \left\{ \int_0^\infty \frac{\cos \tau w}{\tau w} d\tau - \right. \\ \left. \log \left(u^2 - \frac{(kk')}{2x^2} (1-v^2) \right) \right\} \left. Sp \left\{ -\frac{1}{2} \gamma_i \gamma_\mu L \gamma_\mu \gamma_j (-i\gamma P - x) + \right. \right. \\ \left. \left. \gamma_i (i(\gamma, k - P) - x) L \gamma_j + \gamma_i L (i(\gamma, k' - P) - x) \gamma_j \right\} \right]. \quad (2.10)$$

The divergent term $\int_0^\infty \frac{\cos \tau v}{\tau v} d\tau v$ appears in the second term of (2.10). However, the coefficient of this term can be shown to vanish. For $L=1$ and γ_{1234} , the proof is trivial, since spur, following the divergent term, vanishes. For $L=\gamma_{ijk}$, evaluation of spur yields the factor $\frac{1}{u^3} \left(1 - \frac{3}{2u}\right)$ which vanishes on integration over u . It is to be emphasized that we have the convergent result without any recourse to mass or charge renormalization.

From the computation of spur in (2.10), the final expression for $([\varphi^+ L \varphi])_2$ is; for $L=1$, $([\varphi^+ \varphi])_2 = -\frac{x}{8\pi^2} \left(\frac{e}{\hbar c}\right)^2 \int_{-1}^{+1} d\tau \int_1^\infty \frac{du}{u} \left[-\frac{1}{u^2} A^2 + \frac{1/x^2}{u^2 - \square(1-v^2)/4x^2} \times \left\{ \left(1 - \frac{1-v^2}{u^2}\right) \frac{1}{2} F_{ij}^2 + \left(1 - \frac{2}{u} + \frac{1-v^2}{u^2}\right) (\partial_i A_i)^2 \right\} \right]$, (2.11)

where $A_i \partial_i \partial_j A_j$ that appears at first sight, vanishes on integration;

$$\text{for } L=\gamma_{ijk}, ([\varphi^+ \gamma_{ijk} \varphi])_2 = \frac{1}{8\pi^2} \left(\frac{e}{\hbar c}\right)^2 \int_{-1}^{+1} d\tau \int_1^\infty \frac{du}{u} \left[\frac{1}{u^2} \sum_{cycl. (i,j,k)} A_i F_{jk} + \frac{1/x^2}{u^2 - \square(1-v^2)/4x^2} \left\{ \frac{1-v^2}{u^2} \left(\sum_{cycl. (i,j,k)} F_{ii} \partial_i F_{jk}^{(*)} \right) - \left(\frac{2}{u} - \frac{1+v^2}{u^2} \right) \times \left(\sum_{cycl. (i,j,k)} \partial_i (\partial_i A_i) F_{jk} \right) \right\} \right], \quad (2.12)$$

where $\gamma_{ijk} = \gamma_5 \gamma_i \gamma_j$ and (ijk) assumes (324), (134), (214), (123) as s runs from 1 to 4;

and for $L=\gamma_{1234}=\gamma_5$

$$([\varphi^+ \gamma_5 \varphi])_2 = \frac{1}{4\pi^2} \left(\frac{e}{\hbar c}\right)^2 \frac{1}{x} \int_{-1}^{+1} d\tau \int_1^\infty \frac{du}{u} \left\{ \frac{1}{u^2} + \frac{\square(1-v^2)/4x^2}{u^2 \left(u^2 - \frac{\square}{4x^2} (1-v^2) \right)} \right\} \times (F_{12} F_{34} + F_{23} F_{14} + F_{31} F_{24}). \quad (2.13)$$

Here we have introduced the field intensity $F_{ij} = \partial_i A_j - \partial_j A_i$. We see that non-gauge covariant terms of the type A^2 and $(A_i F_{jk})$ still survive in the expression $([\varphi^+ \varphi])_2$ and $([\varphi^+ \gamma_{ijk} \varphi])_2$, which is the serious contradiction to the assertion of § 1 (B) that $([\varphi^+ L \varphi])_2$ should be gauge covariant. This circumstance is similar to that of the photon self energy. We think that these inconsistent results arise from the pathological character of Jordan-Pauli's delta function. If any prescription for the correct treatment of this singular function could be found in the future, our non-gauge covariant terms would vanish, with the photon self-energy.

The further test on our results is provided by examining whether our results satisfy the identity $\partial_i ([\varphi^+ \gamma_5 \gamma_i \varphi])_2 = -2x ([\varphi^+ \gamma_5 \varphi])_2$. From (2.12), we obtain

$$(*) \sum_{cycl. (i,j,k)} F_{ii} \partial_i F_{jk} = -\partial_l (F_{23} F_{14} + F_{31} F_{24} + F_{12} F_{34}), \text{ where } l \text{ is defined by } \gamma_5 \gamma_l = \gamma_{ijk}.$$

$$\begin{aligned}\partial_i([\varphi^+\gamma_5\gamma_i\varphi])_2 &= \sum_{(i,j,k)}^{(*)} \partial_i([\varphi^+\gamma_{ijk}\varphi])_2 \\ &= \frac{-1}{2\pi^2} \left(\frac{e}{\hbar c} \right)^2 \int_{-1}^{+1} d\eta \int_1^\infty \frac{d\eta}{\eta} \left\{ \frac{1}{2\eta^3} + \frac{\square/4x^2(1-\eta^2)}{\eta^2(\eta^2-\square)(1-\eta^2)/4x^2} \right\} \times \\ &\quad (F_{23}F_{14} + F_{31}F_{24} + F_{42}F_{13}),\end{aligned}\quad (2.14)$$

where we have used the identities

$$\begin{aligned}\sum_{(i,j,k)} \partial_i \left\{ \sum_{cyclic(i,j,k)} A_i F_{jk} \right\} &= -2(F_{14}F_{23} + F_{24}F_{31} + F_{31}F_{12}), \\ \sum_{(i,j,k)} \partial_i \left\{ \sum_{cyclic(i,j,k)} \partial_i(\partial_i A_i) F_{jk} \right\} &= (1)\end{aligned}$$

and

$$\begin{aligned}\sum_{(i,j,k)} \partial_i \left\{ \sum_{cyclic(i,j,k)} F_{ij} \partial_i F_{jk} \right\} &= \sum_{i=1}^4 \partial_i \left\{ (-\partial_i)(F_{23}F_{14} + F_{31}F_{24} + F_{12}F_{34}) \right\} \\ &= -\square(F_{23}F_{14} + F_{31}F_{24} + F_{12}F_{34}).\end{aligned}$$

Comparing (2.14) with (2.13) we see that identity does not hold; we obtain a very puzzling results that the identity does not hold between the first term in (2.13) and the corresponding term in (2.14), though it holds for the terms of higher order in expansion of \square/x^2 . Probably this is regarded as due to the same cause which leads to the nongauge-covariant result. In the future theory, in which gauge covariance and $\partial_i([\varphi^+\gamma_5\gamma_i\varphi])_2 = -2x([\varphi^+\gamma_5\varphi])_2$ hold simultaneously, the first term in (2.13) might disappear, since such a term as $(F_{12}F_{34} + F_{23}F_{14} + F_{31}F_{24})$ cannot be deduced from $\sum_{(i,j,k)} \partial_i([\varphi^+\gamma_{ijk}\varphi])_2$ in the absence of $\sum A_i F_{jk}$ in $([\varphi^+\gamma_{ijk}\varphi])_2$. Therefore, in the present state of theory, we fail to give the unique life-time for neutral meson. So we shall give in the following two kinds of life-time, according as whether the first terms in (2.11), (2.12) and (2.13) are retained or not. The matrix-element for two-quanta disintegration of neutral meson reads as follows; for scalar meson

$$\frac{x}{8\pi^2} \left(\frac{e}{\hbar c} \right)^2 f l^2 \left(A^2 - \frac{1}{3x^2} F_{ij}^2 \right), \quad (2.15)$$

for pseudoscalar meson with pseudovector and pseudoscalar couplings

$$\frac{1}{8\pi^2} \left(\frac{e}{\hbar c} \right)^2 \frac{f}{\mu} \sum_{(i,j,k)} \partial_i l^2 \left(\frac{2}{3} \sum_{cyclic(i,j,k)} A_i F_{jk} + \frac{1}{3x^2} \sum_{cyclic(i,j,k)} F_{ij} \partial_i F_{jk} \right), \quad (2.16)^{(**)}$$

and

$$\frac{1}{4\pi^2} \left(\frac{e}{\hbar c} \right)^2 \frac{f}{x} l^2 \left(1 + \frac{1}{12} \frac{\square}{x^2} \right) (F_{23}F_{14} + F_{31}F_{24} + F_{12}F_{34}). \quad (2.17)$$

It is interesting to remark that the first terms in (2.15) and (2.16) do not agree with the result by perturbation theoretic calculation, while the second terms

(*) $\sum_{(i,j,k)}$ means the sum over (1,324), (2,134), (3,214), and (4,123).

(**) See note added in proof (2).

in (2.15) and (2.16), and the first term in (2.17) also follow from perturbation theory. The perturbation theoretic results corresponding to the first terms in (2.15) and (2.16), which are derived by the special choice of the integration variables, are $\frac{x}{6\pi^2} \left(\frac{e}{\hbar c} \right)^2 f I' (A^2 - A_4^2)$ and

$$\frac{1}{3\pi^2} \left(\frac{e}{\hbar c} \right)^2 \frac{f}{\mu} \sum_{(i,j,k)} \partial_i I' \sum_{cyclic (i,j,k)} \{ A_i (\partial_{j1} \partial_4 A_k - \partial_{k1} \partial_4 A_j) - A_i F_{jk} \partial_{i4} \}$$

Table: The life time of the neutral meson.

Meson	Coupling	I First term retained	II First term discarded	Expression for the life time	
				I	II
Scalar	Scalar	1.1×10^{-18} (1.7×10^{-18})	7.2×10^{-14} (2.1×10^{-14})	$\frac{1}{8} \left(\frac{\mu}{x} \right)^2 \tau_0$	$\frac{9}{8} \left(\frac{x}{\mu} \right)^2 \tau_0$
Pseudo-scalar	Pseudo-scalar	0.9×10^{-16} (0.6×10^{-16})	1×10^{-10} (1.3×10^{-11})	$\frac{1}{8} \tau_0$	$18 \left(\frac{x}{\mu} \right)^4 \tau_0$
Pseudo-scalar	Pseudo-vector	8.4×10^{-16} (5.6×10^{-16})	1×10^{-10} (1.3×10^{-11})	$\frac{9}{8} \tau_0$	$18 \left(\frac{x}{\mu} \right)^4 \tau_0$
Pseudo-vector	Pseudo-vector	∞	∞		

where $f_s^2/\hbar c = f_{ps}^2/\hbar c = 0.05$, and $f_{ps}^2/\hbar c = \left(\frac{2x}{\mu} \right)^2 \times 0.05$; meson mass = 200 (300) m.

$$\tau_0 = 32\pi^2 \left(\frac{\hbar c}{e^2} \right)^2 \left(\frac{\hbar c}{f^2} \right) \frac{1}{\mu}.$$

The three quanta disintegration of neutral meson, which has not been touched here, can be evaluated in the similar way. The detailed account will be published in the following paper. The rather long life of neutral meson, given in table suggests that the γ -decay of neutral mesons can not be the agency of extensive air shower,⁹ and may greatly influence the interpretation of mixed showers. Further $\tau \rightarrow 3\pi$ decay which has recently been discovered by Powell¹⁰ et al, will probably be of the same mechanism as γ -decay of neutral meson. The life of τ -meson may be less than 10^{-11} sec because the length of its track in photographic emulsion is at least 3000μ . Such a value of life can only be derived from our procedure.

Further considerations on this problem will be treated in the following paper.

In conclusion the authors wish to express their cordial thanks to Prof. S. Tomonaga for valuable discussions and encouragements in this work, and also to Dr. T. Miyazima and Mr. S. Hayakawa for their helpful advices.

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10) Powell et al: Nature **168**, (1949) 47, 82. H. Yukawa, his private communication.

Note added in proof: (1) (1.20') can be shown to be zero by Furry's theorem. Thus further integration by parts is unnecessary.

(2) After completion of this work, this problem was discussed by J. Steinberger (Phys. Rev. in press) and S. Ozaki (Prog. Theor. Phys. in press). We should like to express our sincere thanks to Dr. Steinberger and Dr. Ozaki for sending us their manuscripts prior publication.

On the Dirac's Indefinite Metric Space.

Naomi SHÔNO and Nobuo ODA

Institute of Theoretical Physics, Kyûsai University.

(Received May 20, 1949)

We have investigated the mathematical foundation of the indefinite metric which was discussed by P.A.M. Dirac¹⁾ and by W. Pauli²⁾, then seen that their method is the only one which the total energy has a diagonal form concerning the numbers of positive and negative energy oscillators.

(I) Hilbert Space with Indefinite Metric.

- (a) Our space of quantum state is that of indefinite metric.
 (b) Definition of the metric

We define the innerproduct for any two vectors f and g by

$$\langle f, g \rangle = (f, \eta g), \quad (1)$$

where the symbol $(,)$ in the right-hand side means the usual innerproduct of the Hilbert space \mathfrak{H} . And η is a bounded linear operator on \mathfrak{H} . Thus follow

$$\begin{aligned} \langle f_1 + f_2, g \rangle &= \langle f_1, g \rangle + \langle f_2, g \rangle, \\ \langle f, g_1 + g_2 \rangle &= \langle f, g_1 \rangle + \langle f, g_2 \rangle, \\ \langle af, g \rangle &= a \langle f, g \rangle \end{aligned} \quad (2)$$

and

$$\langle f, \beta g \rangle = \bar{\beta} \langle f, g \rangle,$$

where a, β are scalars and $\bar{\beta}$ is the conjugate complex to β , so that $\langle f, g \rangle$ also has the character of the innerproduct of \mathfrak{H} . From the requirement that the normalization $\langle f, f \rangle$ is real, we have

$$\eta^+ = \eta, \quad (3)$$

here the symbol $+$ means hermitic conjugate.

- (c) Decomposition of the space

We assume that the operator η is represented by

$$\eta = I - P, \quad (4)$$

where P is a projection operator which has characters $P^+ = P$ and $P^2 = P$, and

I a unit operator. Then it follows

$$\eta^2 = I \quad \text{and} \quad \eta^{-1} = \eta. \quad (5)$$

If the operator P projects an element of whole space R to its subspace $R_- (<f, f> < 0)$, the operator projecting to the subspace $R_+ (<f, f> > 0)$ is given by $(I - P)$. That is, a element of R_- is given by $f_- = Pf$, and a element of R_+ by $f_+ = (I - P)f$, so that any vector f may be decomposed into the direct sum

$$f = f_+ + f_- \quad (6)$$

We show the structure of this space schematically by the Fig 1 and Fig. 2.

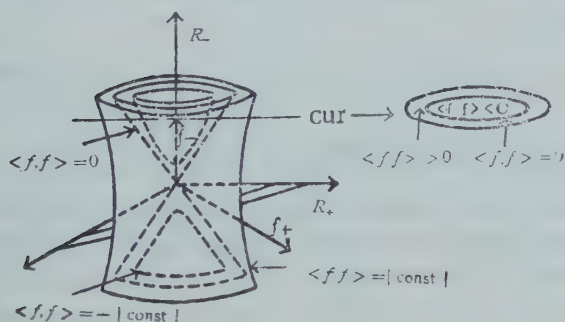


Fig. 1. Pictured three dimensionally.

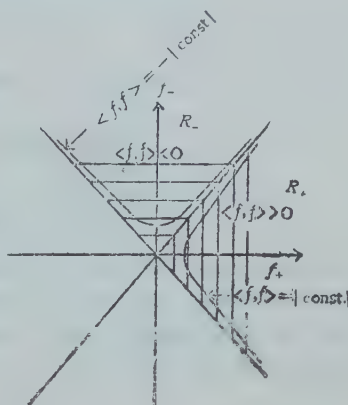


Fig. 2. Pictured two dimensionally. The line $<f, f> = 0$ is as if the lightpath of the special relativity.

(d) Operator

The operator corresponding to an observable must be "self-adjoint"

$$A^* = \eta^{-1} A^+ \eta = \eta A^+ \eta = A. \quad (7)$$

This is the generalization of the hermitic conjugate operator in the positive definite Hilbert space, and then the self-adjoint Hamiltonian H conserves the normalization $<\psi, \psi>$ in accordance with the equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi. \quad (8)$$

The expectation value of an observable A is defined by

$$<A> = <f, Af> = (f, \eta Af). \quad (9)$$

Any operator U (self-adjoint or not) can be decomposed as follows

$$U = U_{++} + U_{+-} + U_{-+} + U_{--}, \quad (10)$$

where

$$\begin{aligned}
U_{++} &= (I-P)U(I-P), \\
U_{--} &= PUP, \\
U_{+-} &= (I-P)UP, \\
U_{-+} &= PU(I-P),
\end{aligned} \tag{11}$$

then we have

$$\langle U_{++}f, U_{++}f \rangle = \|(I-P)U(I-P)f_+\|^2 > 0 \tag{12}$$

and

$$\langle U_{+-}f, U_{+-}f \rangle = \|(I-P)UPf_-\|^2 > 0, \tag{13}$$

so that U_{++} and U_{+-} are the operators which always transform a vector into the subspace R_+ ; U_{++} operated on f eliminates its f_- part, and U_{+-} operated on f eliminates its f_+ part. Further

$$\langle U_{--}f, U_{--}f \rangle = -\|PUPf_-\|^2 < 0 \tag{14}$$

and

$$\langle U_{-+}f, U_{-+}f \rangle = -\|PU(I-P)f_+\|^2 < 0, \tag{15}$$

which show that U_{--} and U_{-+} are the operators always transforming a vector into the subspace R_- ; U_{--} operated on f eliminates its f_+ part, and U_{-+} operated on f eliminates its f_- part.

Now putting

$$U_{++} + U_{--} = U_+ \tag{16}$$

and

$$U_{+-} + U_{-+} = U_-, \tag{17}$$

we obtain

$$(U_+)^* = \eta(U_+)^i \eta = (U^*)_+ = U_+^* = U_+^\dagger \tag{18}$$

and

$$(U_-)^* = \eta(U_-)^i \eta = (U^*)_- = U_-^* = -U_-^\dagger. \tag{19}$$

If the operator U is self-adjoint, we have

$$U^*_+ = U_+^\dagger = U_+ \tag{18'}$$

and

$$U_-^* = -U_-^\dagger = U_-, \tag{19'}$$

namely U_+ is a hermitic operator and U_- anti-hermitic operator. From (18) and (19), it follows respectively for any operator

$$\eta U_+^* = U_+^* \eta \quad \text{and} \quad \eta U_-^* = -U_-^* \eta. \tag{20}$$

Especially for the self-adjoint operator U , we have

$$\eta U_+ = U_+ \eta \quad \text{and} \quad \eta U_- = -U_- \eta. \tag{20'}$$

(II) Harmonic Oscillator.

We consider a harmonic oscillator in this indefinite metric space and show the relation to Dirac's method. We can describe a oscillator system in suitable unites by two variables p, q satisfying the commutation relation

$$i[p, q] = 1, \quad (21)$$

and by the Hamiltonian

$$H = \frac{1}{2} (p^2 + q^2). \quad (22)$$

The p and q are self-adjoint operators, and we can decompose these operators respectively

$$p = p_+ + p_- \quad \text{and} \quad q = q_+ + q_-, \quad (23)$$

here p_+ and q_+ are hermitic operators having the character (18') or (20'), and p_- , q_- anti-hermitic operators having the character (19') or (20').

Considering a harmonic oscillator consistently with the decomposition of the operator in this indefinite metric Hilbert space, we must decompose the Hamiltonian (22) into the hermitian and the anti-hermitian by the decomposition of (23). But inserting the expression (23) into (22), as is easily known, the cross term $p_+ p_-$ or $q_+ q_-$ appears. We can not get a diagonal form of the Hamiltonian concerning the number of positive energy oscillator and negative owing to these cross terms.

So, considering a harmonic oscillator hereafter, we always take as Dirac's method, at the same time, two systems given by

$$H_+ = \frac{1}{2} (p_+^2 + q_+^2) \quad \text{and} \quad H_- = \frac{1}{2} (p_-^2 + q_-^2), \quad (24)$$

then assume that total energy H is given by

$$H = H_+ + H_-. \quad (25)$$

Transforming p, q to the new variables u, u^* by

$$p_+ = \frac{1}{\sqrt{2}} (u_+ + u_+^*), \quad q_+ = \frac{i}{\sqrt{2}} (u_+ - u_+^*) \quad (26)$$

and

$$p_- = \frac{1}{\sqrt{2}} (u_- + u_-^*), \quad q_- = -\frac{i}{\sqrt{2}} (u_- - u_-^*), \quad (27)$$

we have commutation relations as follows

$$i[p_+, q_+] = [u_+, u_+^*], \quad i[p_-, q_-] = -[u_-, u_-^*], \quad (28)$$

$$i[p_+, q_-] = \frac{1}{2} \{ [u_+, u_-] + [u_-^*, u_+] + [u_+^*, u_-] + [u_-^*, u_+^*] \} \quad (29)$$

and

$$i[p_-, q_+] = -\frac{1}{2} \{[u_+^*, u_-] + [u_-^*, u_+] - [u_+, u_-] - [u_-^*, u_+^*]\}. \quad (30)$$

Here the operator u_+ and u_- (not self-adjoint) has the character (18) and (19) respectively, and is decomposed in the sense of (16) and (17)

$$u_+ = u_{++} + u_{--} \quad \text{and} \quad u_- = u_{+-} + u_{-+}, \quad (31)$$

by the operators u_{++} , u_{+-} , u_{--} and u_{-+} having the characters (12), (13), (14) and (15).

Now we assume that the state of this system is described by the wave function $\phi(N_+, N_-)$, here N_+ , N_- has respectively values 0, 1, 2, 3, ..., and that the operators u_{++} , u_{--} , u_{-+} and u_{+-} operated on the $\phi(N_+, N_-)$ are given as follows

$$\begin{aligned} u_{++}\phi(N_+, 2N_-) &= \sqrt{N_+}\phi(N_+-1, 2N_-), \\ u_{--}\phi(N_+, 2N_-+1) &= \sqrt{N_+}\phi(N_+-1, 2N_-+1), \\ u_{-+}\phi(N_+, 2N_-) &= -\sqrt{2N_-}\phi(N_+, 2N_-+1) \end{aligned} \quad (32)$$

and

$$u_{+-}\phi(N_+, 2N_-+1) = -\sqrt{2N_-+1}\phi(N_+, 2N_-),$$

then the adjoint-operators mean emission. As is shown above, each operator is distinguished by the state being odd or even of N_- .

Consequently one finds easily for the operator u

$$\begin{aligned} u_+\phi(N_+, N_-) &= \sqrt{N_+}\phi(N_+-1, N_-), \\ u_+^*\phi(N_+, N_-) &= \sqrt{N_++1}\phi(N_++1, N_-), \\ u_-\phi(N_+, N_-) &= -\sqrt{N_-}\phi(N_+, N_-+1) \end{aligned} \quad (33)$$

and

$$u_-^*\phi(N_+, N_-) = -\sqrt{N_-+1}\phi(N_+, N_-+1),$$

for H_+ , H_-

$$H_+ = \frac{1}{2} (u_+^*u_+ + u_+u_+^*) = N_+ + \frac{1}{2} \quad (34)$$

and

$$H_- = \frac{1}{2} (u_-^*u_- + u_-u_-^*) = -\left(N_- + \frac{1}{2}\right), \quad (35)$$

for the commutation relation

$$i[p_+, q_+] = i[p_-, q_-] = 1 \quad (36)$$

and

$$i[p_+, q_-] = i[p_-, q_+] = 0. \quad (37)$$

Thus N_- is the number of negative energy oscillator, so the state having even numbers of negative energy oscillator is the vector of subspace R_+ and the state having odd numbers the vector of subspace R_- . The positive energy system and the negative, do not influence each other, and construct one system consistently. Starting from the Hamiltonian (22), we can get the commutation relations (36) and (37) by setting (32) or (33) for the operator u , but can not get a diagonal form for the total energy. Further we have the matrix representation

$$(N_+, N_- | \eta | N'_+, N'_-) = (-1)^{N_-} \delta_{N_+ N'_+} \delta_{N_- N'_-}, \quad (38)$$

from the condition for η

$$\eta u_+^* = u_+^* \eta \quad \text{and} \quad \eta u_-^* = -u_-^* \eta.$$

These results are identical with Dirac's and Pauli's one.

We are much indebted to Prof. T. Mutô and Prof. M. Nogami for their critical discussions on this work.

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Letters to the Editor

The Half-Life of Radioactive Decay of Free Neutron

Naomi Shôno and Hitoshi Hagihara

*Institute of Theoretical Physics,
Kyushu University.*

May 15, 1949

Recently an experiment¹⁾ was done in America, which is the measurement of the half-life of radioactive decay of free neutron. Assuming that the beta-decay is Yukawa's type, this process can be calculated mediating a π -meson as follows

$$N \rightarrow P + \pi^- \quad \pi^- + \nu \rightarrow e^-,$$

or

$$\nu \rightarrow e^- + \pi^+ \quad \pi^+ + N \rightarrow P. \quad (1)$$

And we must add the direct interaction²⁾ between the nucleon and light particles on this indirect interaction.

The transition probability per unit time is given by

$$w = \frac{2\pi}{\hbar} \rho_F |H_\beta|^2 \cdot \delta(E_A - E_F). \quad (2)$$

A) The case that the mediating meson is charged scalar.

We proceed the calculation standing on the assumption that the wave length of light particles is longer as $1/\mu_s = \hbar/\mu_s c$, here μ_s is the mass of meson emitted in the intermediate state.

We have the interaction energy between the nucleon and light particles as follows:

$$H_\beta^s = -\frac{4\pi f \cdot f'}{x_s^2} \int \psi_p^* \beta^{(2)} \varphi_{pn} \psi_p^* \beta^{(1)} \Psi_{p_0} dv, \quad (3)$$

where f, f' is respectively the coupling constant of meson with the nucleon and light particles, of charge dimension, and $p, p_n, P,$

and P_0 indicate respectively the electron, neutrino, proton, and the neutron; especially φ_{pn} is the wave function of negative energy state.

Taking the wave function of free particle for these particles, one has for expression (3) as follows:

$$H_\beta^s = -\frac{4\pi}{V} \cdot \frac{f \cdot f'}{x_s^2} \cdot \{u_p^* \beta^{(2)} u_{pn}\} \{U_p^* \beta^{(1)} U_{p_0}\}. \quad (4)$$

We average $|H_\beta^s|^2$ with respect to the spin orientation by taking

$$u_p = \frac{H_p + E_p}{2E_p} u_p, \quad u_{pn} = \frac{-H_{pn} + |E_{pn}|}{2|E_{pn}|},$$

$$U_p = \frac{H_p + E_p}{2E_p} U_p, \quad U_{p_0} = \frac{H_{p_0} + E_{p_0}}{2E_{p_0}}, \quad (6)$$

then have

$$w = \frac{2\pi}{\hbar} \cdot \rho_F \cdot \left(\frac{2\pi}{V}\right)^2 \cdot \left(\frac{f \cdot f'}{x_s^2}\right)^2 \cdot \frac{1}{E_p |E_{pn}| E_p \cdot E_{p_0}}$$

$$\times \{E_p \cdot |E_{pn}| + c^2(\vec{p} \vec{p}_n) - (mc^2)(m_n c^2)\}$$

$$\times \{E_p \cdot E_{p_0} - c^2(\vec{P} \vec{P}_0) + (M_N c^2)(M_P c^2)\}. \quad (7)$$

Assuming the neutron is at rest, the state density of final state is given by

$$\rho_F = \left(\frac{\partial E_p}{\partial E_F}\right) \cdot \rho_p \cdot \rho_P \cdot dE_p$$

$$= \frac{c^2 P |E_{pn}|}{c^2 P |E_{pn}| + \{c^2 P + c^2(\vec{p} \vec{P})/P\} E_p} \times$$

$$\times \frac{c^2 p P E_p E_p}{(2\pi \hbar c)^6} \cdot d\Omega_p d\Omega_P \cdot dE_p \cdot V^2. \quad (8)$$

Consequently the transition probability per unit time, which an electron having the energy dE_p in the interval $(E_p, E_p + dE_p)$ is emitted, is given by

$$P(E_p)dE_p = \frac{2\pi\hbar(\hbar c)^6}{4} \cdot \left(\frac{f \cdot f'}{x_s^2}\right)^2 \cdot \frac{c^2 p P^3(E_p + M_\nu c^2)}{E_p} \cdot dE_p \times \\ \times \left\{1 + \frac{\{E_p(E_p + |E_{pn}| + c^2 p^2 - mc^2 \cdot m_n c^2) - c^2 P^2(E_p + |E_{pn}|)\}}{2c^2 p P E_p}\right\} \times \\ \times \log \frac{c^2 P^2(E_p + |E_{pn}|) + c^2 p P E_p}{c^2 P^2(E_p + |E_{pn}|) - c^2 p P E_p} \quad (9)$$

Neglecting the energy of neutrino, the second term drops by the fact: $\lim_{x \rightarrow 0} x \log \frac{1}{x} = 0$. Finally the total transition probability is given, by assuming $E_p = M_\nu c^2$, as follows:

$$\frac{1}{\tau_s} = \left(\frac{f^2}{\hbar c}\right) \cdot \left(\frac{f'^2}{\hbar c}\right) \cdot \frac{1}{\tau}, \quad (10)$$

where

$$\frac{1}{\tau} = \frac{8}{2\pi\hbar} \frac{M_{Nc^2} - M_\nu c^2}{(x_s \hbar c)^4} \int_{m_\nu c^2}^{\infty} (\varepsilon^2 - m_\nu^2 c^4) \cdot d\varepsilon.$$

B) The case that the mediating meson is pseudoscalar.

We subtract the δ -like function term in the origin from

$$(\sigma^{(1)} \text{grad}^{(1)}) (\vec{\sigma}^{(2)} \text{grad}^{(2)}) e^{-x_{ps} \cdot \vec{r}/r},$$

then the interaction energy is given by

$$H_{ps}^{p.s.} = -\frac{4\pi\vec{f} \cdot \vec{f}'}{3x_{ps}^2} \int \psi_p^* \vec{\sigma}^{(2)} \varphi_{pn} \cdot \Psi_p^* \vec{\sigma}^{(1)} \Psi_{p0} dv. \quad (11)$$

Calculating equally as the case (A) hereafter, we get the total transition probability as follows

$$\frac{1}{\tau_{ps}} = \left(\frac{\vec{f}}{\hbar c}\right) \cdot \left(\frac{\vec{f}'}{\hbar c}\right) \cdot \frac{1}{3\tau}. \quad (12)$$

This result dose not correspond to the Nelson's calculation³⁾ contrary to our expectation.

C) The case that the mediating meson is vector.

Giving the interaction energy between the neutron and light particles

$$H_{ps}^v = \frac{4\pi}{x_s^2} \int dv [(g_1 g'_1) \psi_p^* \varphi_{vn} \cdot \Psi_p^* \Psi_{p0} \\ + \frac{2}{3} (g_2 g'_2) \psi_p^* \beta^{(2)} \vec{\sigma}^{(2)} \varphi_{vn} \cdot \Psi_p^* \beta^{(1)} \vec{\sigma}^{(1)} \Psi_{p0}], \quad (13)$$

the proper time is given, in the case $\beta^{(2)} = 1$, by

$$\frac{1}{\tau_v} = \frac{1}{\tau} \left\{ \frac{4}{3} \left(\frac{g_2^2}{\hbar c}\right) \cdot \left(\frac{g_2'^2}{\hbar c}\right) \times \right. \\ \left. - \left(\frac{g_1^2}{\hbar c}\right) \cdot \left(\frac{g_1'^2}{\hbar c}\right) \right\}, \quad (14)$$

and, in the case remaining $\beta^{(2)}$, by

$$\frac{1}{\tau_v} = \frac{1}{\tau} \left\{ -\frac{4}{3} \left(\frac{g_2^2}{\hbar c}\right) \left(\frac{g_2'^2}{\hbar c}\right) \right. \\ \left. - \left(\frac{g_1^2}{\hbar c}\right) \cdot \left(\frac{g_1'^2}{\hbar c}\right) \right\}. \quad (15)$$

D) The case that the mediating meson is pseudovector.

Giving the interaction energy

$$H_{ps}^{p.v.} = -\frac{4\pi}{x_{pv}^2} \int dv \left[\frac{2}{3} (\bar{g}_1 \bar{g}'_1) \psi_p^* \vec{\sigma}^{(2)} \varphi_{pn} \cdot \right. \\ \Psi_p^* \vec{\sigma}^{(1)} \Psi_{p0} + \frac{1}{3} (\bar{g}_2 \bar{g}'_2) \psi_p^* \beta^{(2)} \vec{\sigma}^{(2)} \varphi_{pn} \cdot \\ \left. \Psi_p^* \beta^{(1)} \vec{\sigma}^{(1)} \Psi_{p0} \right], \quad (16)$$

the proper time is given reasonably in spite of containing $\beta^{(2)}$ in this energy as the case (C) by

$$\frac{1}{\tau_{pv}} = \frac{1}{\tau} \left\{ \frac{4}{3} \left(\frac{\bar{g}_1^2}{\hbar c}\right) \left(\frac{\bar{g}_1'^2}{\hbar c}\right) \right. \\ \left. - \frac{1}{3} \left(\frac{\bar{g}_2^2}{\hbar c}\right) \left(\frac{\bar{g}_2'^2}{\hbar c}\right) \right\}. \quad (17)$$

We calculated this problem in the approximation: $|p| \cong |P|$, so that it is incorrect to put: $\beta^{(2)} \cong 1$, and the vector theory is not reasonable.

Giving the mass of meson 313 electron masses⁽⁴⁾, we have

$$\frac{1}{\tau} = 2.9 \times 10^{13} \quad (18)$$

so that the proper time of free neutron is 29 minutes in the case of pseudoscalar by

taking, for putting right the calculation to the experiment,⁽¹⁾ the coupling constant as

$$\frac{f^2}{4\pi} = 10^{-1}, \quad \frac{\bar{f}^2}{4\pi} = 6 \times 10^{-13} \quad (19)$$

But then the proper time of spontaneous decay of π -meson to light particles is given by

$$\tau(\pi \rightarrow e^-, \nu) \cong 6.5 \times 10^{-12} \text{ seconds.} \quad (20)$$

This is wrong on the standpoint of π and μ mesons, because the decay of π -meson to μ -meson is difficult.

We are grateful to Dr. Nakamura and Dr. Taketani for their kind advice.

- 1) A. H. Snell, L. C. Miller; *Phys. Rev.* **74** (1948) 1217.
- 2) S. Sakata; *Proc. Phys. Math. Soc., of Jap.* **23** (1941) 291.
- 3) E. C. Nelson; *Phys. Rev.* **60** (1941) 830.
- 4) Gardner, Latfès; *Science* **103** (1948) 270.

On the Eigenvalue Problem of Hydrogen

G. Araki.

*Department of Industrial Chemistry,
Kyoto University*

May 24, 1949

The behaviour of the fundamental system of solutions of the radial equation in the vicinity of the origin of the Coulomb field due to a point charge Z placed at the origin was examined in a letter⁽¹⁾ submitted to this section. But the discussion was incomplete and never satisfactory for solving the eigenvalue problem of the Hamiltonian H of this system. Therefore the supplementary account will be given in the following.

The above mentioned fundamental system for a negative value of energy is given by R_1 and R_2 . (Notations and units are the same as those in A).⁽¹⁾ In the vicinity of the origin they behave as

$$R_1(r) \propto r^L, \quad R_2(r) \propto r^{-(L+1)} \quad (1)$$

where L is the azimuthal quantum number. The asymptotic solution of the radial equation is given by $r^{\lambda-1}e^{-\epsilon r}$ or $r^{-\lambda-1}e^{+\epsilon r}$ where $\epsilon = \sqrt{-2E}$, $\lambda = Z/\epsilon$ and E is energy. Therefore another fundamental system of solutions is given by R_3 and R_4 where they asymptotically behave as

$$R_3(r) \propto r^{\lambda-1}e^{-\epsilon r}, \quad R_4(r) \propto r^{-\lambda-1}e^{+\epsilon r} \quad (2)$$

The general solution of the radial equation is represented as follows:

$$R(r) = aR_1(r) + bR_2(r) + cR_3(r) + dR_4(r) \quad (3)$$

In order to examine the quadratic integrability of the radial function R we shall distinguish two cases: (a) λ is equal to a positive integer and larger than L ; (b) the previous condition is not satisfied.

Case (a). R_1 must contain R_3 because R_1 and R_2 are linearly independent and R_1 does not contain R_4 .⁽²⁾ Therefore if b is different from zero R necessarily contains R_4 , and R is not quadratically integrable because of the asymptotic behaviour given by (2). Consequently R_1 only is quadratically integrable.

Case (b). If L is positive and b is different from zero R is not quadratically integrable owing to (1). Therefore there is no quadratically integrable solution in case of a positive L because R_1 contains R_4 .⁽³⁾ If L is equal to zero R_3 is quadratically integrable because R_2 behaves as r^{-1} in the vicinity of the origin (R_3 is given by R with $c=1$ and $d=0$, and in this case R_2 necessarily contains R_4 because R_1 contains R_4), but R_3 can not all be eigenfunctions of H because the point spectrum of a Hermitian operator is at most an enumerable set.⁽³⁾ In fact, all solutions involving R_2 do not belong to the domain of H .⁽⁴⁾

Thus we find that in case (b) the quadratic integrability can not exclude a non-eigenfunction R_3 which contains R_2 though R_2 itself can be ruled out by the quadratic integrability in both cases (a) and (b) as was

pointed out in $A^{(1)}$. This important fact was overlooked in A, and an incorrect expression was given in a later letter.⁵⁾

1) G. Araki; *Prog. Theor. Phys.*, **3** (1948), 97.

This will be referred to as A.

2) H. Bethe; *Handb. der Phys.*, **24/1** (1933), 277.

E. C. Kemble, *The Fundamental Principle of Quantum Mechanics* (1937), p. 159.

3) M. H. Stone, *Linear Transformation in Hilbert Space* (1932), p. 142.

4) W. Pauli; *Handb. der Phys.* (Geiger-Scheel), **24/1** (1933), 124.

5) G. Araki; *Prog. Theor. Phys.*, **3** (1948), 446.

An Improvement on the Integrations appearing in Perturbation Theory.

H. Umezawa and R. Kawabe.

*Institute of Theoretical Physics,
Nagoya University.*

June 2, 1949

The customary perturbation calculation is not only ambiguous as to its relativistic covariance, but often actually gives results evidently destroying the relativistic covariance. This can be seen for instance, from the fact, indicated by Pais¹⁾ and others, that a result with the transformation property of mass cannot be obtained from perturbation calculations for the self-energy of a moving electron due to an electromagnetic field. But, since the system of quantum field theory is of a relativistically invariant structure, it must be concluded that the cause of the failure of relativistic covariance lay in the process of the usual perturbation calculations. We intend to propose a method to remedy this defect of perturbation calculation.

We shall confine the following arguments to calculations in the momentum space. We consider the case when two particles 1 and 2 are created in the intermediate state, and denote their momentum-energy four-vectors by $P^{(1)} = (\mathbf{P}^{(1)}, E_P^{(1)})$ and $P^{(2)} = (\mathbf{P}^{(2)}, E_P^{(2)})$

respectively. Then, an integral of the following form occurs in the perturbation calculation:

$$\iint F(P^{(1)}, P^{(2)}) d\mathbf{P}^{(1)} d\mathbf{P}^{(2)} \quad (1)$$

where $F(P^{(1)}, P^{(2)})$ is a certain function of $P^{(1)}$ and $P^{(2)}$. Owing to the momentum conservation law, (1) can be rewritten as

$$\int G(\mathbf{P}^{(0)}) d_P \mathbf{P}^{(1)} \quad (2)$$

Hitherto, the method has been employed of taking a sphere as the integration domain for $\mathbf{P}^{(1)}$ in (2) and making its radius tend to infinity. This method destroys the relativistic covariance in the following two ways. (i) A sphere is not a relativistically invariant domain (ii) A momentum-conservation relation among particles which do not satisfy the energy conservation law is not a relativistically invariant relation.

(a) In order to remove the difficulty (ii), an integration domain must be prescribed for the integral (1).

(b) In order to remove the difficulty (i), the sphere must be replaced by a domain enclosed by a surface on which the momentum-space scalar quantity w takes a constant value. (c) Furthermore, we place the condition that the above domain becomes a momentum-space sphere when referred to a appropriate system of coordinates.

As the four dimensional scalar quantity w , we may take the four dimensional scalar product of the momentum energy four-vectors $(P^{(1)}, P^{(2)})$.

The actual method of calculation is as follow; we perform the transformation of variables $|\mathbf{P}^{(1)}| \rightarrow w$ in the integral (2), rewriting it as an integration for w . We may next set the bounds of the integration domain at $w=a, b$ (constants) under the condition (a). The values of a and b are determined by the condition (c). In general, when there are n particles 1, ..., n in the intermediate state, the following $(n-1)$ scalar quantities must be employed:

$$u^{(k)} \dots (P^{(1)}, P^{(2)}), \quad k=2, \dots, n$$

It can be ascertained, by actual calculation in various problems, that results having the correct relativistic covariance can be obtained by applying this method to perturbation theory. Detailed reports of various results thus obtained will be given elsewhere.

1) Pais; Verh. Ac. Amsterdam. Vol. 19, 1947.

General Formulae appearing in the Problem of the Vacuum Polarization.

II. Umezawa and R. Kawabe.

Institute of Theoretical Physics,

Nagoya University.

June 2, 1949.

We obtained general formulae for various quantities appearing in the problem of the vacuum polarization, independently of the nature of the charged particles. Perturbation calculations were performed in the interaction representation¹⁾.

Let ψ_A be the state vector for the state A in which the free charged particle (i) is. According to perturbation calculations

$$\psi_A = \psi_{0A} + e\psi_1 + e^2\psi_2 + \dots \quad (1)$$

The current $\langle \delta J \rangle$ induced by J is given by

$$\langle \delta J \rangle = \psi_A^* (X) J(X) \psi_A(X)$$

$$= \psi_{0A}^* (X) J(X) \psi_{0A}(X) = \langle J(X) \rangle_{\text{vac}}. \quad (2)$$

Here $\langle J(X) \rangle_{\text{vac}}$ is the current induced in vacuum. Taking only that part of δJ which is proportional to e^3 , and using the continuity equation to simplify the expression, we arrive, after some calculations, at the following form:

$$\begin{aligned} \delta J = & \int \int \left[\left\{ (f_{11}(l) + f_{12}(l)) \frac{2l}{\square} + (f_2(l) + f_3(l)) \right\} \right. \\ & \times (e_1 J^{(v)}(X')) e_1 + (f_1'(l) + f_3'(l)) (e J^{(v)}(X')) e \\ & \left. + (f_2'(l) + f_3'(l)) J_1^{(v)}(X') \right] e^{i q_1 X - X'} dX' dl \\ & \times (e_1 \pm l, e/l); \end{aligned} \quad (3)$$

$$\left. \begin{aligned} f_{11}(l) &= \sum_{\mathbf{P}} \frac{2(E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})}{\{E - (E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})^2\}} g_1(l), \\ f_2(l) &= \sum_{\mathbf{P}} \frac{-4l(E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})}{\{E - (E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})^2\}} g_1(l), \\ f_3(l) &= \sum_{\mathbf{P}} \frac{4l(E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})}{\{E - (E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})^2\}} \times \\ & \quad \frac{[\square]}{\{E - (E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})^2 + [\square]\}} g_1(l); \end{aligned} \right\} \quad (4)$$

$$\left. \begin{aligned} f_1'(l) &= \sum_{\mathbf{P}} \frac{2(E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})}{\{E - (E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})^2\}} g_2(l), \\ f_3'(l) &= \sum_{\mathbf{P}} \frac{2(E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})}{\{E - (E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})^2\}} \times \\ & \quad \frac{[\square]}{\{E - (E_{\mathbf{P}_v - \frac{l}{2}} + E_{\mathbf{P}_v + \frac{l}{2}})^2 + [\square]\}} g_2(l); \end{aligned} \right\} \quad (5)$$

$$f_{12}(l) = (i \sim e^2 H_2^{(v)} i \sim), \quad (6)$$

$$\begin{aligned} g_1(l) = & \frac{2\pi}{k} \left(\text{vac. } e_1 J_1^{(v)} | \mathbf{P}_v - \frac{l}{2}, -\mathbf{P}_v - \frac{l}{2} \right) \\ & \times \left(\mathbf{P}_v - \frac{l}{2}, -\mathbf{P}_v - \frac{l}{2} | e_1 J_1^{(v)} \text{vac.} \right), \end{aligned} \quad (7)$$

$$g_2(l) = (\text{vac. } | V^{(vv)} | \text{vac.}); \quad (8)$$

have, $J = J_1 + J_2$, $J_1 \propto e$, $J_2 \propto e^2$

$$V^{(vv)} = \frac{\rho^{(v)} \rho^{(v)}}{r}$$

\square : D'Alembertian.

For the charged particle (v) produced in the intermediate state, we write irrespective of its nature, $J_\mu^{(v)} = (J^{(v)}, \rho^{(v)})$.

As the result of calculations, we find that the relations

$$f_2(l) = f_2'(l), \quad f_3(l) = f_3'(l)$$

hold when (v) is a Fermi, scalar or vector

charged particle. The part δJ_2 of δJ which is proportional to $f_2(l)$ can be perfectly amalgamated into $J^{(i)}$, so we call it the "charge renormalization term" (c.r.). It can be seen from (6), that $f_2(l)$ have the sign " - " independent on the nature of the particle (v). The part δJ_3 of δJ proportional to $f_3(l)$ cannot be amalgamated into $J^{(i)}$ because it involves the \square . Consequently, this should be observable, so we call it the "observed current term" (o.c.).

The relations (6) allow us to draw the following conclusions. If n is the order of divergence of the self-energy of the photon due to $eH_1^{(n)}$, then those of δJ_2 and δJ_3 are $n=2$ and $n=4$ respectively. Since $n=2$ for Fermi and scalar charged particles, δJ_2 diverges logarithmically while δJ_3 remains finite. For vector charged particles, where $n=4$, δJ_2 diverges quadratically, and δJ_3 logarithmically. Since the (o.c.) δJ_3 cannot be amalgamated, it seems to be a fresh source of difficulty that this diverges.

Calculations of (6) for various particles will be reported in a separate paper.

1) S. Tomonaga: Prog. Theor. Phys. 1 (1946), 27.

J. Schwinger: Phys. Rev. 74 (1948), 1439.

Vacuum Polarization due to Various Charged Particles.

H. Umezawa and R. Kawabe.

*Institute of Theoretical Physics,
Nagoya University.*

June 2, 1949.

We previously obtained general formulae appearing in vacuum polarization independently of the type of charged particles¹⁾. In this note, we perform calculations, applying these formulae to charged spinor, scalar and vector fields.

In order to carry out the integrations in a relativistically invariant manner, we use the method described elsewhere.²⁾ That is, we

transform the variable p into a relativistically invariant quantity v , by

$$\left\{ \left(E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}} \right)^2 - l^2 \right\} \\ = -2(p, p') + 2\mu^2 = \frac{4\mu^2}{1-v^2} \quad (1)$$

where $E_{\mathbf{P}-\frac{\mathbf{l}}{2}}$, $E_{\mathbf{P}+\frac{\mathbf{l}}{2}}$ are the energies of the pair of charged particles created in the intermediate state. The scalar product of the energy-momentum four-vectors of these particles is written as (p, p') .

The current induced by the current $J^{(i)}$ is

$$\delta J(X) = \int \left[\left\{ (f_1(l) + f_1(l)) \frac{2l}{\square} \right. \right. \\ + (f_2(l) + f_3(l)) \left. \right\} (e, J^{(i)}(X')) e_1 \\ + (f_2'(l) + f_3'(l)) (e, J^{(i)}(X')) e \\ + (f_2'(l) + f_3'(l)) J_4^{(i)}(X') \left. \right] e^{i(l, X - X')} dX' dl. \quad (2)$$

We consider the cases when the charged particle created in the intermediate state is a spinor, scalar or vector particle respectively

$$\text{(Spinor)} \quad f_2^s(l) = f_3^s(l) \\ = -\frac{e^2}{3\pi} \log(2(p, p')/\mu^2) + 5e^2/9\pi, \quad (3)$$

$$\text{(Scalar)} \quad f_2^s(l) = f_3^s(l) \\ = -\frac{e^2}{12\pi} \log(2(p, p')/\mu^2) + e^2/18\pi, \quad (4)$$

$$\text{(Vector)} \quad f_2^v(l) = f_3^v(l) \\ = -\frac{e^2}{6\pi} \log((p, p')/\mu^2) \\ + \frac{e^2}{4\pi} \log(2(p, p')/\mu^2); \quad (5)$$

$f_3(l)$, $f_3'(l)$ are given by:

$$\text{(Spinor)} \quad f_3^s(l) = f_3'^s(l) \\ = -\frac{e^2 \square}{4\pi \mu^2} \int_0^1 v^2 \left(1 - \frac{v^2}{3} \right) \frac{dv}{\left\{ 1 - \frac{\square}{4\mu^2} (1-v^2) \right\}} \\ \approx -\frac{e^2 \square}{4\pi \mu^2} - \frac{e^2 \square^2}{140\pi \mu^4} \dots \quad (6)$$

$$\text{(Scalar)} \quad f_3^s(l) = f_3'^s(l) \\ = -\frac{e^2 \square}{24\pi \mu^2} \int_0^1 \frac{v^4 dv}{\left\{ 1 - \frac{\square}{4\mu^2} (1-v^2) \right\}}$$

$$= -\frac{e^2 \square}{120\pi\mu^2} - \frac{e^2 \square^2}{1630\pi\mu^4} \dots \quad (7)$$

$$\begin{aligned} \text{(Vector)} \quad f_3^v(l) &= f_3^{v'}(l) \\ &= -\frac{e^2 \square}{6\pi\mu^2} \int_0^1 \frac{v^4}{(1-v^2)} \left\{ 1 - \frac{\square}{4\mu^2} (1-v^2) \right\} dv \\ &\quad - \frac{e^2 \square}{8\pi\mu^2} \int_0^1 \frac{v^4 dv}{\left\{ 1 - \frac{\square}{4\mu^2} (1-v^2) \right\}} \quad (8) \end{aligned}$$

$$\begin{aligned} &= -\left\{ \frac{e^2}{12\pi\mu^2} \log \frac{2(p, p')}{\mu^2} - \frac{71e^2}{360\pi\mu^2} \right\} \square \\ &\quad - \frac{17e^2 \square^2}{1630\pi\mu^4} - \dots \quad (8') \end{aligned}$$

The self-energy of $w_l = f_{11}(l) + f_{12}(l)$ of the photon l is :

$$\begin{aligned} \text{(Spinor)} \quad w_l^s &= f_{11}^s(l) + f_{12}^s(l) \\ &= -\frac{2e^2}{3\pi l} \left[\frac{1}{2} (p, p') - \mu^2 \right] \quad (9) \end{aligned}$$

$$\begin{aligned} \text{(Scalar)} \quad w_l^s &= f_{11}^s(l) + f_{12}^s(l) \\ &= +\frac{e^2}{3\pi l} \left[\frac{1}{2} (p, p') - \mu^2 \right] \quad (10) \end{aligned}$$

$$\begin{aligned} \text{(Vector)} \quad w_l^v &= f_{11}^v(l) + f_{12}^v(l) \\ &= +\frac{e^2}{\pi l} \left[\frac{1}{2} (p, p') - \mu^2 \right] \quad (11) \end{aligned}$$

Since $f_2(l) = f_2'(l)$, $f_3(l) = f_3'(l)$, as seen in equations (3) to (8), (2) becomes

$$\begin{aligned} \delta J(X) &= \iint \left[w_l \frac{2l}{\square} (e_l J^{(X)}(X')) e_l + (f_2(l) \right. \\ &\quad \left. + f_3(l)) J^{(X)}(X') \right] e^{i(l, X - X')} dX' dl \quad (12) \end{aligned}$$

The first of these terms may be thought of as due to an additional term

$$\int \frac{w_l}{2(2\pi)} A_{\perp}^2(X') e^{i(l, X - X')} dX' dl \quad (13)$$

in the Hamiltonian function. This is the mass-term of the photon.

Observing the forms of (9), (11) the magnitude of (13) is seen to be proportional to the degree of freedom of the charged particle field, while the sign is “-” for a Fermi field and “+” for a Bose field. If we assume the existence of n , m and l Fermi scalar and vector fields respectively in the nature, it is in general, possible to annul the

self-energy of the photon by placing the following relations :

$$-2n + m + 3l = 0 \quad (i)$$

$$-2 \sum_{i=1}^n (\mu_i^s)^2 + \sum_{i=1}^m (\mu_i^s)^2 + 3 \sum_{i=1}^l (\mu_i^v)^2 = 0 \quad (ii)$$

(where $\mu_i^s (i=1 \dots n)$, $\mu_i^s (i=1 \dots m)$, $\mu_i^v (i=1 \dots l)$ are the respective masses).

However, the condition (ii) is not always necessary, since μ is involved in the same form in the finite terms of w^s , w^s , w^v . We need only take $(p, p') - 2\mu^2$ as the common variable for each particle. Since it is impossible to take a scalar variable not involving μ , it is at the present stage, totally unknown what function of (p, p') to take as the common variable.

As seen from (3), (4), (5), $f_2(l)$ can be perfectly amalgamated into $J^{(v)}$. This is not the case with $f_3(l)$ which involves the d'Alembertian \square . The fact that this diverges (8') seems to present a new difficulty to the Tomonaga-Schwinger theory.

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The Self-Energy of a Dirac Particle and its Relativistic Covariance.

R. Kawabe and H. Umezawa.

*Institute of Theoretical Physics,
Nagoya University.*

June 2, 1949

Hitherto, the electromagnetic self-energy of an electron in motion, when calculated directly by perturbation theory, did not as a whole show a relativistic covariance, so convenient prescriptions have been taken.¹⁾

But according to the prescription recently proposed by us, the self-energies of a Dirac particle due to an electromagnetic field and to a C -meson are both shown to possess just the transformation property of the mass term

That is, if we take

$$2(l, P-l) = (E_{P-l} + l)^2 - E_P^2 = \mu^2 w$$

where $(l, P-l)$ is the scalar product of the energy-momentum four-vectors of the two particles appearing in the intermediate state, the electromagnetic self-energy of a Dirac particle is

$$\Delta E^{(1)} = \frac{1}{2\pi^2 E_P} \left[\mu^2 \int \frac{1}{(E_{P-l} + l)^2 - E_P^2} \times \right. \\ \left. \frac{E_{P-l} + l}{E_{P-l}} \frac{dl}{l} + \frac{1}{2} \int \frac{E_{P-l} + l}{l E_{P-l}} dl \right. \\ \left. - \int \frac{dl}{E_{P-l}} \right] = \frac{e^2 \mu}{E_P} \frac{3\mu}{2\pi} \times \\ \left[\frac{1}{2} \log(1+w) - \frac{1}{6} \right].$$

$$- \frac{f^2 \mu^2}{2\pi E_P} \left[\frac{3}{4} \log w + \frac{1}{4} \left\{ 1 - \delta^2 + (\delta^4 - 6\delta^2) \log \delta \right. \right. \\ \left. \left. + \delta(\delta^2 - 4) \sqrt{\delta^2 - 4} \log \left(\frac{1}{2} (\delta - \sqrt{\delta^2 - 4}) \right) \right\} \right], \quad \delta > 2$$

$$\Delta E^{(2)} = \left| \begin{aligned} & - \frac{f^2 \mu^2}{2\pi E_P} \left[\frac{3}{4} \log w - \frac{1}{4} \{ 3 + 8 \log 2 \} \right], \quad \delta = 2 \\ & - \frac{f^2 \mu^2}{2\pi E_P} \left[\frac{3}{4} \log w + \frac{1}{4} \left\{ 1 - \delta^2 + (\delta^4 - 6\delta^2) \log \delta \right. \right. \\ & \left. \left. + \delta(\delta^2 - 4) \sqrt{4 - \delta^2} \cos^{-1} \frac{\delta}{2} \right\} \right], \quad \delta < 2. \end{aligned} \right.$$

If we place the condition⁴⁾ $f^2 = 2e^2$ and further put $\mu = M$ (the proton mass), $\Delta E = \Delta E^{(1)} + \Delta E^{(2)}$ gives the mass difference between proton and neutron.

$$\Delta E = - \frac{e^2 \mu^2}{4\pi E_P} \left[2 - \delta^2 + (\delta^4 - 6\delta^2) \log \delta + \delta(\delta^2 - 4) \sqrt{4 - \delta^2} \cos^{-1} \frac{\delta}{2} \right], \quad \delta < 2$$

This coincides with Pais⁵⁾ result when $P \rightarrow 0$, and consequently it is insufficient to account for the mass-difference, and it must be supplemented by considering the contribution of the meson field with the asymmetry of the nuclear forces between like particles and unlike particles taken into account. So that, the problem of the mirror nucleus⁶⁾ must be scrutinized from this new point of view.

where μ is the proper mass of the Dirac particle. This result coincides with that of Schwinger²⁾ inclusive of the finite terms. The third term of the first expression becomes, according to our method, an "invariant" term, and consequently Feynman's³⁾ arguments concerning this term and hence concerning the f -field (or the C -meson) are not valid. Also, the self-energy of a Dirac particle due to a C -meson can be shown to possess the transformation property of the mass-term if we take

$$2(l, P-l) + \kappa^2 = (E_{P-l} + \varepsilon_l)^2 - E_P^2 = w$$

where l and κ are the momentum and mass of the C -meson, respectively. Putting $\delta = \frac{\kappa}{\mu}$,

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- 2) J. Schwinger; *Phys. Rev.* **75**, (1949) 651.
- 3) R. P. Feynman; *Phys. Rev.* **74**, (1948) 1436.

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6) A. S. Wightman; *Phys. Rev.* **72**, (1947) 477.

A Preliminary Treatment of Nucleon Cascade.

H. Kita and H. Hasegawa

Physics Institute, Kyoto University

June 8, 1949

It is presumed that in high energy cosmic-

ray phenomena many high energy nucleons in addition to mesons are disrupted out from one air nucleus. Up to the present the hard component was often analysed under the assumption that primary protons lost a certain fraction of their energy by producing mesons. However, if the above mentioned process occurs and disrupted nucleons again give rise to nuclear disintegrations, then the analysis must be treated in a different way from the above.

As to the high energy nuclear disintegration we have no detailed information. Therefore we assume the following points phenomenologically; the cross section of collision of a nucleon with air nucleus is a per g/cm^2 . Spectra of protons and neutrons which are disrupted forward, including the incident nucleon, are $bE^{-t}E_0^u$, $dE^{-t}E_0^u$, where E_0 is the energy of the incident nucleon and E that of disrupted nucleons. In the case of air $b=d$. The effect of nucleons disrupted backward is neglected. Further we assume that the energy fraction k is constant, then the following relations are obtained.

$$u=t-1, \quad b+d=(2-t)k^{t-1}, \quad 2 > t \geq 1.$$

These with (II) reduce the number of parameters from 6 to 2.

Neglecting the energy loss except by nuclear processes, the following diffusion equations for $P(E, x)$, $N(E, x)$, the intensity of protons and neutrons with energy E at the depth $x g/cm^2$, are obtained.

$$\partial P(E, x)/\partial x = -aP(E, x)$$

$$+ ab \int_{E/k}^{\infty} \{P(E_0, x) + N(E_0, x)\} E^{-t} E_0^u dE_0.$$

$$\partial N(E, x)/\partial x = -aN(E, x)$$

$$+ ad \int_{E/k}^{\infty} \{P(E_0, x) + N(E_0, x)\} E^{-t} E_0^u dE_0.$$

A particular solution

$$P(E, x) + N(E, x)$$

$$= E^{-\tau} \exp[-a\{1 - (b+d)k^{\tau-t}/(\tau-t)\}x] \quad (I)$$

is easily obtained. We may take this as the solution, when the primary spectrum of the

proton is given by a power law $E^{-\tau}$.

For a single primary proton with energy E_0 we obtain

$$P(E, x; E_0) + N(E, x; E_0) = E^{-\tau} [2\pi y''(\bar{s})]^{-1/2} \\ \exp\{-a\bar{x} + y(\bar{s})\}, \quad y(s) = s \log(E_0/E) \\ + \alpha(b+d)xk^{\tau-t+1}/(s-t+1), \quad y'(s) = 0,$$

by using Mellin transformation and the saddle point method. Accordingly, for an arbitrary given primary spectrum we may gather these with respect to E_0 .

As is seen from (I), the absorption coefficient a' and the collision cross section of high energy nucleon α is connected by the relation

$$a' = a\{1 - (2-t)k^{1.8}/(2.8-t)\}, \quad (II)$$

when $\tau=2.8$ is adopted. If one takes $a'=1/125$ the value from the experiment and $\alpha=1/65$ corresponding to the geometrical cross section of air nucleus, the energy fraction k becomes too large. But if α is smaller than $1/65$, k can have a reasonable value. Therefore it seems necessary to treat α as a parameter instead of fixing it as the geometrical cross section. The determination of parameters is in course of investigation.

We wish to acknowledge the invaluable discussion and the continued encouragement of Mr. T. Inoue.

An Attempt to Pauli's Regulator.

Y. Katayama.

Department of Physics, Kyoto University.

June 15, 1949

Recent development in the quantum electrodynamics achieved by many authors has brought fruitful results in many problems, but there remain some ambiguities in the evaluation of infinite integrals. In these circumstances, the method of regulating these effects proposed by Feynman¹⁾, Stückelberg-Rivier²⁾ and Pauli³⁾ are very remarkable.

Here, we attempt to rewrite these methods in a more customary form and clarify the physical significance of these procedures. In this point of view, we modify Stückelberg-Rivier's method replacing the commutation relations of field variables by the sum of many delta functions with the different mass. For instance, in the electromagnetic and electronic fields, we have⁴⁾

$$\begin{aligned} [A_\mu(X), A_\nu(X')] \\ = i \sum_s C_s \left(\delta_{\mu\nu} - \frac{1}{\lambda_s^2} \frac{\partial^2}{\partial x_\mu \partial x_\nu} \right) D_s(X-X'), \\ \{ \psi_\alpha(X), \psi_\beta^\dagger(X') \} \\ = \frac{1}{i} \sum_s E_s \left(\gamma^\mu \frac{\partial}{\partial x_\mu} - x_s \right)_{\alpha\beta} A_s(X-X'), \quad (1) \end{aligned}$$

where $(\square^2 - \lambda_s^2) D_s(X) = 0$,

and $(\square^2 - x_s^2) A_s(X) = 0$.

Then the equations of field variables become

$$\begin{aligned} \square_s (\square^2 - \lambda_s^2) A_\mu(X) &= 0, \\ \square_s \left(\gamma^\mu \frac{\partial}{\partial x_\mu} + x_s \right) \psi(X) &= 0. \quad (2) \end{aligned}$$

We can decompose these into new variables which satisfy the usual equations of Proca and Dirac type with different mass respectively. That is,

$$\begin{aligned} A_\mu(X) &= \sum_s a_s U_\mu^s(X), \\ (\square^2 - \lambda_s^2) U_\mu^s(X) &= 0, \\ \psi_\alpha(X) &= \sum_s d_s \phi_\alpha^s(X), \\ \left(\gamma^\mu \frac{\partial}{\partial x_\mu} + x_s \right) \phi^s(X) &= 0. \quad (3) \end{aligned}$$

In virtue of (1), the commutation relations of these new variables take the following forms:

$$\begin{aligned} [U_\mu^s(X), U_\nu^t(X')] \\ = i \delta^{st} \frac{C_s}{a_s^2} \left(\delta_{\mu\nu} - \frac{1}{\lambda_s^2} \frac{\partial^2}{\partial x_\mu \partial x_\nu} \right) D_s(X-X'), \\ \{ \phi_\alpha^s(X), \phi_\beta^{\dagger t}(X') \} \\ = \frac{1}{i} \delta^{st} \frac{E_s}{d_s^2} \left(\gamma^\mu \frac{\partial}{\partial x_\mu} - x_s \right)_{\alpha\beta} A_s(X-X'). \quad (4) \end{aligned}$$

These results indicate the fact that $A_\mu(X)$

and $\psi(X)$ consist of many independent fields $U_\mu^s(X)$ and $\phi^s(X)$ and this is one form of the mixed theory. But we can not regard this method as introduction of new substances as C -meson or f -field theory. For this purpose, we compare this method with Pauli's regulator.

First we examine the self-energy of electron. In this case only the modification of electron field is necessary. Then the principal term of this problem is

$$\begin{aligned} \phi(X) \sim -\frac{e^2}{2} \sum_s C_s \int \gamma^\mu \left[\bar{D}_s(X-X') S^{(1)}(X-X') \right. \\ \left. + D_s^{(1)}(X-X') \bar{S}(X-X') \right] \gamma^\mu \psi(X') dX', \end{aligned}$$

which is equivalent to Pauli's result with

$$\rho(x) = \sum_s C_s \delta(x - \lambda_s^2). \quad (5)$$

The condition of regularization is

$$\sum_s C_s = 0. \quad (1)$$

Next, we have for the photon self energy,

$$\begin{aligned} \langle J_\mu(X) \rangle_0 &= \langle \sum_s j_\mu^s(X) \rangle_0 \\ &= -4e^2 \sum_s g^s \left(\frac{E_s}{d_s^2} \right)^2 \int K_{\mu\nu}^s(X-X') A_\nu(X') dX' \end{aligned}$$

and comparing with Pauli's result, we get

$$\rho(x) = \sum_s g^s \left(\frac{E_s}{d_s^2} \right)^2 \delta(x - x_s^2), \quad g^s = \pm 1 \quad (6)$$

Therefore, we have the conditions holding the gauge invariance throughout and regularizing the ambiguous calculations; that is

$$\sum_s g^s \left(\frac{E_s}{d_s^2} \right)^2 = 0, \quad \sum_s g^s \left(\frac{E_s}{d_s^2} \right)^2 x_s^2 = 0. \quad (11)$$

The condition (I) is necessarily satisfied by $C_0=1, C_1=-1, C_s=0$ ($s=2, 3, \dots$) which is equivalent to Feynman and Bopp-Podolsky theory. But there arises a well known difficulty related to the negative energy. And the conditions (II) indicate the co-existence of at least three spinor fields, but in this case there exists a difficulty concerning with the negative energy with vacancy due to $g^s=-1$ ⁵⁾ or imaginary coupling⁶⁾. Furthermore, if we regularize also the delta functions throughout,

the necessary number of particle is seven (three are nature of field and four are that of matter) and if we regularize higher order approximations, the number will increase. This is also a reason why we cannot regard these fields as realistic one.

In spite of these results, it is very useful to regard this method as an introduction of new substances for convenience. And then following Pauli, it is natural to bring the mass of particles except one to infinity at the end of calculations, holding the regularizing conditions throughout. In some simple cases, we can show that this is done and that other fields give no contribution to the final results.

I should like to express my heartily thanks to Prof. M. Kobayasi for his kind guidance and encouragement throughout this work.

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A Divergence-Free Field Theory.

K. Sawada.

Department of Physics, Kyoto University.

June 15, 1949

An analysis of commutation relations between field quantities, especially of Δ -functions leads us to generalize Feynman's relativistic cut-off¹⁾. His theory removes an inconsistency that, in usual calculation,

although the commutator of field variables vanishes when field mass becomes infinite since it is Δ -function and so the effect of quantum fluctuation should vanish in this limit, the calculation of self-energy shows a divergence depending logarithmically on the mass.

His commutation relation²⁾ can also be written as follows;

$$\begin{aligned} [C(X), C(X')]_- &= i[\Delta_{\mu 2}(X-X') - \Delta_{\lambda_0 2}(X-X')]_{\lambda_0 \rightarrow \infty} \\ &= -i \int_{\mu}^{\lambda_0 \rightarrow \infty} \frac{\partial}{\partial \lambda} \Delta_{\lambda 2}(X-X') d\lambda \\ &= i\Delta_{\mu 2}(X-X') \end{aligned} \quad (1)$$

for the scalar meson field, for example. Where $\Delta_{\mu 2}$ means Δ -function with meson mass μ , and field variable at the world point X is denoted by $C(X)$. The 2nd and 3rd expressions are in fact trivial since $\Delta_{\infty 2} = 0$; but using this expression holding λ_0 finite and tending after calculation $\lambda_0 \rightarrow \infty$ makes the reaction effect to vanish in the limit $\mu \rightarrow \infty$ in consistent way with the vanishment of commutator in this limit.

But the free field energy which was quantized remains in this case also. So we generalize that the field variables itself vanish in the limit $\mu \rightarrow \infty$; for this purpose we write;

$$C(X) = - \int_{\mu}^{\infty} \frac{\partial}{\partial \lambda} C^{(\lambda)}(X) d\lambda \quad (2)$$

Of course, the commutator (1) do not show any peculiar behaviour for $\mu \rightarrow \infty$, we should retain this since it comes from the Hamilton form of the system. For which, it is sufficient to take;

$$[C^{(\lambda)}(X), C^{(\lambda')}(X')]_- = i\Delta_{\lambda \lambda'}(X-X') \quad (3)$$

Then calculating self-energy of Dirac particle due to this field with interaction Hamilton density;

$$H = f\psi^\dagger \psi C \quad (4)$$

where C is defined by (2); we obtain for the mass correction in order f^2 ;

$$\frac{\delta m}{m} = \int_{\mu}^{\infty} \int_{\mu}^{\infty} \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda'} \left(-\frac{f^2}{32\pi} \right) \left[12 \int_0^{\infty} \frac{\cos(\lambda \lambda' z)}{z} dz + \right. \\ \left. + 2 \int_{-1}^1 (y+1)(y+5) \frac{y-1 + \frac{\lambda \lambda'}{m^2}}{(y-1)^2 + 2 \frac{\lambda \lambda'}{m^2} (y+1)} dy \right] d\lambda d\lambda' \quad (5)$$

The first term in bracket just behaves as $\log \frac{1}{\lambda \lambda' z_0} \Big|_{z_0 \rightarrow 0}$, but with the differentiation with λ and λ' , this term drops out, second term is finite over μ to ∞ for every λ and λ' , so we obtain:

$$\frac{\delta m}{m} = \frac{f^2}{8\pi} \left(2\delta^2 - 9 - 2(\delta^2 - 6\delta^2 - 6) \log \delta + \right. \\ \left. \begin{cases} + 2\delta(4 - \delta^2)^{3/2} \left(\tan^{-1} \frac{\delta}{\sqrt{4 - \delta^2}} - \tan^{-1} \frac{\delta^2 - 2}{\delta \sqrt{4 - \delta^2}} \right) & \delta < 2 \\ - \delta(\delta^2 - 4)^{3/2} \log \frac{\delta^2 + \delta \sqrt{\delta^2 - 4} - 2}{2} & \delta > 2 \end{cases} \right) \quad (\delta = \mu/m) \quad (6)$$

which is finite and, of course, for $\mu \rightarrow \infty$ this vanishes in consistent way with the vanishment of (1). For the photon interaction, some care is to be paid, since for the mass zero field, we cannot obtain with (2) and (3) the original commutator (1), and (6) diverges for $\mu \rightarrow 0$. So we take alternative way to impose the condition on ϕ ;

$$\phi(X) = - \int_m^{\infty} \frac{\partial}{\partial \lambda} \phi^{(\lambda)}(X) d\lambda; \\ [\phi^{(\lambda)}(X), \phi^{(\lambda')}(X')] + \\ = \frac{1}{i} \left(r^{\lambda} \frac{\partial}{\partial X_{\lambda}} - m \right) \Delta_{\lambda \lambda'} (X - X') \quad (7)$$

with this, calculating self-energy due to electro-magnetic field, one has for mass correction;

$$\frac{\delta m}{m} = \frac{5e^2}{8\pi} \quad (8)$$

Thus this theory gives finite mass correction for the field concerned. The scalar mesonic self-energy calculated in this alternative way do not diverge in the limit $\mu \rightarrow 0$ contrary to (6), and is given by;

$$\frac{\delta m}{m} = (E_0(6)) + \frac{f^2}{8\pi} (2 - 12 \log \delta) \quad (9)$$

It seems that the latter method will be preferable.

The details and the problem concerning

vacuum polarization and photon self-energy will appear in this journal.

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On the Photo-Disintegration of the Deuteron.

(Pseudoscalar Meson Theory.)

H. ENATSU and Y. TAKANO.

*Institute of Physics, Faculty of Science,
Kyoto University.*

June 22, 1949.

Recently van Hove¹⁾ and Araki²⁾ studied the singularities of nuclear potentials, and for example Araki obtained the expression for the nuclear potential by the pseudoscalar meson theory,

$$J = \frac{f^2 e^{-xr}}{(1+\phi)(1+2\phi)r} \left\{ \frac{1}{3} (\vec{\sigma}_1 \cdot \vec{\sigma}_2) + \left(-\frac{1}{3} + \frac{1}{x} + \frac{1}{x^2 r^2} \right) S \right\} T^r \quad (1)$$

$$\phi = |a| \left(\frac{M}{M_m} \right) \left(\frac{f^2}{\hbar c} \right) \frac{e^{-xr}}{xr},$$

$$S = \frac{3(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r})}{r^2} - (\vec{\sigma}_1 \cdot \vec{\sigma}_2)$$

M : Nucleon mass,
 M_m : Meson mass ($=286m$).

The feature of this potential is the $1/r$ singularity near the origin.

Therefore, the Schrödinger equation of the deuteron has the solution.

On the other hand, the experimental result of the photo-disintegration of the deuteron by Wilson, Collie and Halban³⁾ is

$$\sigma = 16 \times 10^{-28} \text{cm}^2 \quad (2)$$

(for $2.62 \text{ MeV } \gamma$ -ray.)

which is larger than the old ones. Taking account of these results we calculated the total cross-section of the photo-disintegration of the deuteron by the pseudoscalar meson theory. In our calculation the tensor force in the initial state was neglected, because the 3D state contribution is only 4% according to Rose and Goertzel¹⁾.

The initial state is determined by the variational method in which the potential (1) is further simplified, and the final states are the wave functions which correspond to the square well potentials for the brevity of numerical calculations. The cross-sections are as follows in the unit 10^{-28}cm^2 .

$h\nu$: incident γ -ray energy in MeV .
 σ_e : photoelectric cross-section for the transition $^3S \rightarrow ^3P$.
 σ_m : photomagnetic cross-section for the transition $^3S \rightarrow ^1S$.
 σ : total cross-section $\sigma_e + \sigma_m$.

(i) No nuclear forces in the final states.

$h\nu$	σ_e	σ_m	σ
2.6	2.59	0.07	2.66
6.2	12.19	0.17	12.36
17.1	5.13	0.18	5.31

(ii) Repulsive force (range $2.8 \times 10^{-13} \text{cm}$, depth 5 MeV .) in the final 3P state.

$h\nu$	σ_e	σ_m	σ
2.6	3.16	0.07	3.23
6.2	12.95	0.17	13.12
17.1	5.53	0.18	5.71

(iii) Repulsive force (range $2.8 \times 10^{-13} \text{cm}$, depth 10 MeV .) in the final 3P state.

$h\nu$	σ_e	σ_m	σ
2.6	2.78	0.07	2.85
6.2	12.95	0.17	13.12
17.1	6.43	0.18	6.61

The case (ii) is in good agreement with the experimental result (2). The detailed calculation will be published in a later issue.

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Three Dimensional Calculations of the Cosmic Ray Intensities I.

H. Hasegawa and H. Kita.

Physics Institute, Kyoto University.

June 23, 1949.

The conventional calculations of the cosmic ray intensities have been performed one dimensionally. But in the processes such as the multiple production of the mesons and the cascade multiplication of the nucleons, where many secondaries are produced with large angular spread, it is desired to calculate three dimensionally. We calculate on the meson production here.

An incident nucleon collides with one of the nucleons in the air nucleus, and produces several π -mesons, giving a fraction of its kinetic energy to mesons, and the remainder to nucleons, which causes the cascade multiplication of the nucleons. In the center of mass system of the two colliding nucleons, the π -meson spectrum produced after one collision is

$$\sigma(E, \epsilon, \theta) dE d\epsilon d\Omega / (4\pi).$$

At the point $x \text{ g cm}^{-2}$ from the top of the atmosphere, the number of the produced π -mesons of energy ϵ , moving vertically is given by

$$q(\epsilon_0, x) d\epsilon_0 dx = dx \int \int N_0 d\Omega_0 \int \sigma(E) P(E_0, x, \theta_0) dE, \quad (1)$$

where $\sigma(E)$ is the total cross section of the incident nucleon of energy E , N is the number of π -mesons emitted into the solid angle $d\Omega$ and the suffix o indicates the quantity in the earth system.

We assume that the number of the nucleons of a given direction depends only on the thickness of the material traversed and its energy as

$$P(E_0, x, \theta_0) = AE_0^\tau \exp(-\sigma' x \sec \theta_0),$$

σ' represents the apparent absorption coefficient of the nucleons. Then

$$q(\epsilon_0, x) d\epsilon_0 dx = d\epsilon_0 dx \times \int \frac{d\Omega}{4\pi} \cdot \frac{d\epsilon}{d\epsilon_0} \exp(-\sigma' x \sec \theta_0) \times \int \sigma(E, \epsilon, \theta) \sigma(E) P(2E) dE. \quad (2)$$

The same quantity derived one dimensionally is as follows:

$$q_1(\epsilon_0, x) d\epsilon_0 dx = d\epsilon_0 dx \int \frac{1}{2} \sigma(E, \epsilon) \sigma(E), P(2E) dE \exp(-\sigma' x).$$

As an example, we take

$$\sigma(E, \epsilon, \theta) dE d\epsilon d\Omega / (4\pi) = n\delta(2EH/n - \epsilon) dE d\epsilon d\Omega / (4\pi), \quad H \leq 1,$$

so that we get

$$q(\epsilon_0, x) d\epsilon_0 dx = \frac{n^2}{4H} \sigma A \left(\frac{n\epsilon}{H} \right)^\tau \frac{1}{2} \times \left(\exp(-\sigma' x) - (\sigma' x) \int_{\sigma' x}^{\infty} \exp(-z)/z \cdot dz \right) K_1(2')$$

where $(5/2)^\tau < K < 1$. The factor $1/2$ in $(2')$ comes from the existence of the mesons emitted upwards. The factor in the bracket represents the reduction of the produced mesons considered, as x increases, due to the exponential decrease of the ratio of the number of the nucleons inclined to that of the vertical with increasing x . The last factor K is interpreted by the fact that to

produce the mesons considered the incident nucleons inclined must have more energy than the incident vertically and the number of the nucleons decreases with increasing E_0 .

We calculated numerically the number of the hard component penetrating 8cm of lead in the upper atmosphere at the geomagnetic latitudes 50° , 40° and 0° respectively. The intensities show the remarkable reduction from the old values, for instance, this is about 30% at $\lambda=50^\circ$ and $x=200 \text{ g cm}^{-2}$. The agreement between the experiments and the old calculations may be rather apparent, and in order to obtain the true agreement we must assume the contribution from the new component — probably slow protons — besides the mesons and the high energy protons. These circumstances may be unaltered appreciably, when we use the other σ . The accurate calculation of the vertical and the directional intensities is in progress.

On the C-Meson Theory.

Y. Katayama and K. Sawada.

Dept. of Phys. Kyoto University.

June 30, 1949.

The C -meson theory proposed by Pais and Sakata et. al. has gained a brilliant success as far as the divergence of the self-energy of electron is concerned. A series of papers based on these ideas, however, especially those with regards to the problem of vacuum polarizations, are quite unsatisfactory. The greater part of their origin is due to the photon self-energy. In this case, according to Umezawa and others, although the photon self-energy itself does not vanish, there still remains the possibility to make it vanish if we take into account the co-existence of some charged particles, such as scalar and vector mesons.¹⁾ Be it ever the case, it is very doubtful that there are same numbers of corresponding fields which make the self-energy of C -meson vanish as in the case of

the photon self-energy and that their interaction constants satisfy the mutual relations which make the self-energy of charged particles finite.²⁾ In other words, if C -meson or other mutual relations were taken into account, it does not seem likely that we can get a closed model with which all self-energies become finite.

Before proceeding further, we must re-examine our methods. It seems to us that the problem of the photon self-energy stands in a different stage from that of electron and that we cannot manage this difficulty in exactly the same manner. The latter is an essential one inherent in the present theory while the former consists largely in the mathematical defects in carrying out calculations.

Such being the circumstances, it is natural to analyse the essential difficulties only by masking the apparent difficulties as far as possible in the present stage.

From the above mentioned point of view we have calculated the self-energy of C -meson due to the electron field according to Miyazima³⁾ and Pauli's⁴⁾ methods which make the photon self-energy vanish. If we define after Miyazima an indeterminate integral which appears in the course of calculations as

$$\int (dy) q_\lambda^2 e^{iayq^2} = i\pi^2 \frac{1}{a^2} \epsilon(a) \left[\frac{2i}{a} - 4i\epsilon(a)\delta(a) \right]$$

or, as Pauli has done,

$$R'(0) = 0$$

and dropping the indeterminate terms replace the remaining terms by $R(z) = e^{iz\mu^2}$, calculations can be uniquely performed. Then the self-energy as the mass correction of the C -meson reads :

$$\frac{\delta\mu_c}{\mu_c} = \frac{3}{2\pi} \left(\frac{f^2}{4\pi} \right) \left[\left(\frac{m^2}{\mu_c^2} - \frac{1}{6} \right) \times \right. \\ \left. \log \frac{1}{rw_0} + \frac{4}{3} \frac{m^2}{\mu_c^2} - \frac{5}{18} \right]$$

$$- \left\{ \begin{array}{l} \frac{1}{3} \left(4 \frac{m^2}{\mu_c^2} - 1 \right)^{3/2} \sin^{-1} \frac{\mu_c}{2m} \\ \frac{1}{6} \left(1 - 4 \frac{m^2}{\mu_c^2} \right)^{3/2} \times \\ \log \frac{\sqrt{\mu_c^2 - 4m^2} - \mu_c}{\sqrt{\mu_c^2 - 4m^2} + \mu_c} \end{array} \right\} \begin{array}{l} m > \frac{\mu_c}{2} \\ m < \frac{\mu_c}{2} \end{array}$$

Here m and μ_c are the mass of electron and C -meson, respectively.

We find only logarithmic divergence in contrast with the result of Umezawa et al. This is to be accepted from the fact that the self-energy of the photon vanishes by either of the methods here used, though calculations by usual perturbation theory give the quadratic divergence.

Now, as the explanation of the nucleon mass difference by the C -meson theory seems hopeless, we can take for both electrons and nucleus

$$2\sum \mu_c^2 = 6\sum m^2 + 6\sum M^2$$

in order to avoid the divergence of the self-energy of C -meson without destroying the consistency of the theory, where M is the mass of proton. By taking this value, however, we cannot but give up applications of C -meson theory to the self-energy of other charged particles which are not Fermions. In spite of this, if we confine ourselves only to the interaction of Fermions and electromagnetic field, we may again construct a closed model of the C -meson theory.⁵⁾ Of course, it is very desirable to construct a closed model including other charged particles, but there perhaps lie unexpected difficulties in its way.

- 1) H. Umezawa and R. Kawabe, *Prog. Theor. Phys.* **4** (1949) 369.
- 2) O. Hara, *Prog. Theor. Phys.* **3** (1948) 188.
- 3) T. Miyazima, Lecture at the annual meeting of Japan Physical Society (1949).
- 4) W. Pauli and F. Villars, unpublished.
- 5) H. Umezawa, J. Yukawa and E. Yamada, *Prog. Theor. Phys.* **4** (1949) 25.
- 6) R. Kawabe and H. Umezawa, *Prog. Theor. Phys.* **4** (1949) 370.

If we denote the finite self-energy by δm , the

mass difference to be observed can be written as $(1-2W_{i \rightarrow N})\delta m$. Here, $W_{i \rightarrow N}$ is the dissociation probability of nuclear meson. According to this, the correction becomes much more smaller and gives almost no contribution.

- 7) Another difficulty comes from the fact that, in spite of the condition $f_0^2=2c_0^2$ in order to make the self-energy of electron vanish, the renormalized charges $e=e_0+\delta e$ and $f=f_0+\delta f$ does not satisfy $f^2=2e^2$ owing to the lack of the corresponding relations between δe and δf . This difficulty is only a formal one, but still it is inconsistent.

Domain of Kinetic Energy and Perturbation Theory.

G. Araki.

Department of Industrial Chemistry,
Kyoto University.

June 30, 1949.

In a letter¹⁾ submitted to this section a mistake was made that the 1S-eigenfunction, ψ_1 , of hydrogen belongs to the domain of the square of its kinetic energy because of the identity $T^2\psi_1=(1/4-1/r)\psi_1$ where T is the non-relativistic energy (a Hermitian operator). This identity should have been replaced by $\Delta^2\psi_1=(1-4/r)\psi_1$ where Δ means the Laplacian operator. What should have been said is that ψ_1 belongs to the domain of a non-Hermitian extension, $\Delta^2/4$, of T^2 .

The reason is as follows. If ψ is an eigenfunction of hydrogen belonging to an energy eigenvalue E we have $T\psi=(1/r+E)\psi$. ψ behaves as r^L in the vicinity of the origin. Consequently $T\psi$ behaves as r^{-1} for $L=0$. As has been pointed out by Pauli²⁾, functions with such a singularity do not belong to the domain of T . Thus we find that all S-eigenfunctions of hydrogen do not belong to the domain of the square, T^2 , of its kinetic energy.

This can also be understood on another point of view. As will be seen from the following discussion, the Stieltjes integral of

$\lambda^2 d \|E(\lambda)\psi_1\|^2$ is convergent whereas that of $\lambda^4 d \|E(\lambda)\psi_1\|^2$ is divergent. This means that³⁾ ψ_1 belongs to the domain of T but does not to that of T^2 as has been pointed out by Kodaira and Kato⁴⁾.

The above mentioned fact suggests us that a caution is necessary for a perturbation theory involving higher powers of T . For example, we shall consider an electron in the Coulomb field where the kinetic energy of the electron is expressed relativistically. The Hamiltonian of this system is given by $H=W(T)-1/r$ where $W(T)=\sqrt{c^2+2Tc^2}-c^2$ is the relativistic energy of the electron (Atomic units are adopted throughout the present letter). A function of a self-adjoint operator must here be understood according to Neumann.⁵⁾ According to the customary way of the perturbation method, H is expanded in power series in $1/c$ as $H=H^{(0)}+H^{(2)}+H^{(4)}+\dots$ where

$$H^{(0)}=T-1/r, \quad H^{(2)}=-T^2/(2c^2),$$

$$H^{(4)}=T^4/(2c^4) \quad (1)$$

We shall consider $H^{(2)}$ and $H^{(4)}$ as perturbations. The unperturbed eigenvalues and eigenfunctions are then those of hydrogen in a non-relativistic case. If the non-degenerate 1S-state is the unperturbed one, the usual perturbation theory can not be applied to this case, because the 1S-eigenfunction, ψ , does not belong to the domain of $H^{(2)}$ and $H^{(4)}$. If we formally apply the perturbation theory we have a wrong result.

For example, if we wrongly make use of $\Delta^2\psi_1=(1-4/r)\psi_1$, in place of $4T^2\psi_1$ we have $E^{(2)}=3/(8c^2)$ whereas if we make use of $E^{(2)}=-\|T\psi_1\|^2/(2c^2)$ we have $E^{(2)}=-5/(8c^2)$, where $E^{(2)}$ is the second power term in $1/c$ of the energy eigenvalue. The expectation value of the square of a Hermitian operator must be positive. Therefore the former is incorrect whereas the latter is correct according to Neumann's standpoint.⁵⁾ The Fourier transform of ψ_1 is given by $(\sqrt{8/\pi})(1+p^2)^{-2}$ where $p^2=p_x^2+p_y^2+p_z^2$. It follows, from

this, that $d\|E(\lambda)\phi_1\|^2 = (32/\pi)p^2(1+p^2)^{-4}dp$ for $0 \leq \lambda$ or 0 for $\lambda < 0$ where $E(\lambda)$ is the resolution of identity corresponding to T and $p^2 = 2\lambda$. Therefore we have

$$\int_{-\infty}^{\infty} \frac{\lambda^3}{2e^4} d\|E(\lambda)\phi_1\|^2 = \frac{2}{\pi e^4} \int_0^{\infty} \frac{p^3 dp}{(1+p^2)^4} \quad (2)$$

and the perturbation theory diverges if we take into account the perturbation up to $H^{(4)}$. If we replace $-8T^3\phi_1$ with $\Delta^2\phi_1 = (1-6/r)\phi_1$, we have a wrong result, $(\phi_1, H^{(4)}\phi_1) = 5/(16e^4)$.

On the other hand, if we consider $W_1(T) = W(T) - T$ as a perturbation we have no difficulty because ϕ_1 belongs to the domain of $W_1(T)$ which can be seen from the convergence of the Stieltjes integral of $\{W_1(\lambda)\}^2 d\|E(\lambda)\phi_1\|^2$.

The above mentioned illustration provides us an example in which a formally applied perturbation method is divergent whereas the more exact solution is convergent.

- 1) G. Araki, *Prog. Theor. Phys.*, **3** (1948), 448.
- 2) W. Pauli, *Handb. der Phys.* (Geiger-Scheel), **24/1** (1933), 124.
- 3) M. H. Stone, *Linear Transformation in Hilbert Space* (1932), p. 180.
- 4) K. Kodaira and T. Kato, *Prog. Theor. Phys.*, **3** (1948), 439.
- 5) J. von Neumann, *Ann. of Math.*, **32** (1931), 191; *Math. Grundle. der Quantenmechanik* (1932), p. 74.

On the Gauge-difficulty in the γ -Decay of the Neutral Scalar Meson.

D. Itô and T. Miyazima.

Physics Institutes, Tokyo Bunrika Daigaku.

July 4, 1949.

Recently, Miyamoto and Fukuda¹⁾ obtained the convergent transition probability for the γ -decay of neutral mesons. But, as they pointed out, their results are not gauge-covariant, in spite of the gaugecovariance proof of the fundamental theory, and do not satisfy a certain identity, which they must be hold about them.

Of course, these difficulties are removed by W. Pauli's "Regulator Method",²⁾ but they can also be treated from another side, that is "realistic" point of view. In reality, the equivalent result with the regulator method is obtained, if we assume another proton pair interacting with the neutral meson field.

In this letter, we describe briefly the result of another attempt, in which we assume the charged scalar field in the intermediate states to compensate the non-gauge-covariant term. We assume that the interaction of this hypothetical scalar field $\phi(x)$ with the neutral scalar meson $U(x)$ and with the electromagnetic field $A_\mu(x)$ is described by the following interaction Hamiltonian density,³⁾

$$H_0 = g\phi^*\phi U, \quad (1)$$

$$H_1 = ie \left(\phi^* \frac{\partial \phi}{\partial x_\lambda} - \frac{\partial \phi^*}{\partial x_\lambda} \phi \right) A_\lambda, \quad (2)$$

$$H_2 = e^2 \phi^* \phi (A_\lambda^2 + (n_\lambda A_\lambda)^2). \quad (3)$$

and we calculate the transition probability for the γ -decay of the neutral meson $U(x)$ through the virtual pairs of both type, that is the ordinary Fermi proton-, and the hypothetical charged scalar Boson-pairs. ("Bose Proton" pair)

Then the non-gauge-covariant term $\sim A^2$ just compensates and there remains only the gauge covariant term by imposing the condition.

$$g = g', \quad \kappa = \kappa' \quad (4)$$

where, g, g' are the coupling constants of the neutral meson with the Fermi proton and with the Bose proton respectively, and κ, κ' are the masses of them.

Detailed calculation will be published in the later issue of this journal.

- 1) H. Fukuda and Y. Miyamoto, *Prog. Theor. Phys.* in press.
- 2) W. Pauli, letter to Prof. J. Schwinger, which is communicated from Prof. H. Yukawa to Prof. S. Tomonaga; the manuscript of Pauli and Villars sent from Prof. W. Pauli to Prof. S. Tomonaga.

- 3) S. Kanesawa and S. Tomonaga, *Prog. Theor. Phys.* **3**, (1948), 1.

On the Electronic Component in Extensive Air Showers.

Y. Fujimoto, S. Hayakawa and
Y. Yamaguchi.

*Department of Physics, Tokyo University and
Meteorological Research Institute.*

July 12, 1949.

A few years ago Tomonaga and one of us (S. H.)¹⁾ proposed a new mechanism of photon production by nucleon and tried to explain the origin of the electronic component in extensive air showers. Later developments of cosmic ray physics necessitate to alter some constants previously used and, further, to improve the calculation in such a way that it is performed analytically as far as possible, so that the constants can be substituted in the final results.

The number of electrons at depth l_0 by a primary proton with energy $\exp Y_0$ is given by

$$N(Y_0, l_0) = \sigma_0 \int_0^{l_0} \{Y_0 - a(l_0 - l)\} \int_0^Y dy C(y, l), \quad (1)$$

where depth and energy are measured in radiation unit of air, C means the cascade function at depth l with initiating photon energy $\exp y$, and a represents the energy absorption coefficient of the primary proton: $a = 1/A\tau = 0.160$; A is the mean free path of nucleons in air, 125 gcm^{-2} , and τ is the index of the power energy spectrum

$$F(Y_0) = Ae^{-\tau Y_0}, \quad \tau = 1.8, \quad A = 1.32 \times 10^{10}/m^2 hr,^{2)} \quad (2)$$

σ_0 represents the constant coefficient of the charge acceleration probability per radiation length: $\sigma_0 = 6.26 \times 10^{-3} k$,³⁾ k being a number of the order of unity. For the upper limit Y of the integration over y we should take the energy of the proton at the depth $l_0 - l$, or $Y = Y_0 - a(l_0 - l)$, but in order to carry

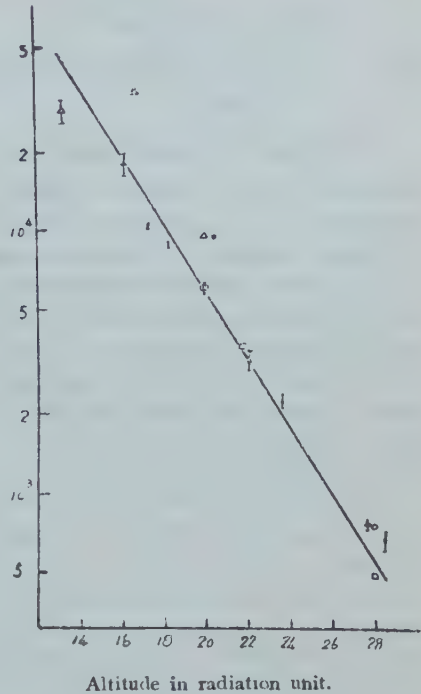


Fig. 1. Altitude variation of shower frequency, normalized at 20 radiation units.

- Cocconi and Tongiorgi, ref. 1).
Cocconi, Loverdo and Tongiorgi, *Phys. Rev.* **70**, (1946), 841.
Williams, *Phys. Rev.* **74**, (1948), 1689.
Hillberry, *Phys. Rev.* **60**, (1941), 1.
Kraybill and Overbo, *Phys. Rev.* **72**, (1947), 351.

out the integration (1), this upper limit is conveniently replaced by $Y_0/(1-a) - al_0/(1-a)$, which is allowed without introducing a serious error caused of the behaviour of C . Then we can integrate over l firstly. In this integration the upper limit l_0 may be put ∞ , since we are interested only in a lower altitude. Accounting for

$$\int_0^\infty dl C(y, l) = e^y, \quad \int_0^\infty dl C(y, l) l = (1.01y + 0.4)e^y,$$

(1) leads to

$$N(Y_0, l_0) - \sigma_0 \left\{ \frac{1}{1-a} (Y_0 - al_0) - 0.61a \right\} \times$$

$$\exp\left(\frac{1}{1-a} Y_0 - \frac{a}{1-a} l_0\right). \quad (3)$$

Then the density at the distance r from the shower axis is given by

$$A(Y_0, l_0, r) = N(Y_0, l_0) M(r, l_0), \quad (4)$$

multiplying (3) by Moliere function M . Solving (4), Y_0 is expressed as a function of l_0 and r , where l_0 in $Y_0 - a'l_0$ is replaced by a mean value, $l^* = 20.1$. Substituting Y_0 in $F(Y_0)$, the integrations over r and the angle of falling direction are carried out. Thus we get the shower frequency at the depth l_0 with density greater than Δ ,

$$H(\Delta, l) = 2\pi A \sigma_0 P^\tau (l^* - al_0)^{P\tau} l^{2P\tau-3} \times e^{-a' P\tau l_0 / \Delta^{P\tau}} \equiv K / \Delta^{P\tau} \quad (5)$$

where $a' = a/(1-a)$, $P = 1/(1-a)$, $P\tau = 1.51$ for $\tau = 1.8$.

Density spectrum tells that $\tau = 1.8$ may be appropriate. The absolute intensity K at sea level is found to be $K = 1.32 \times 10^{10}/hr$ if we assume $k = 1$. This value is twice as large as the experimental value⁴⁾. This seems to mean that the cross section for photon production may be suppressed in the highest energy, though considerable ambiguity is remained in the figures of A , A , k and τ . The altitude dependence, as shown in the figure, is somewhat steeper than experiments, since our approximation becomes worse at high altitude.

- 1) S. Hayakawa and S. Tomonaga, Prog. Theor. Phys. **2** (1947), 162, Jour. Sci. Res. Inst. **45** (1948), 67.
- 2) S. Hayakawa and J. Nishimura, Prog. Theor. Phys. to be published.
- 3) S. Hayakawa, Phys. Rev. **75** (1949), 1759.
- 4) G. Cocconi and V. Cocconi Tongiorgi, Phys. Rev. **75** (1949), 1058.

External Vacuum Polarization.

Z. Koba.

*Institute of Physics, Faculty of Science,
Tokyo University.*

July 9, 1949.

Recently Snyder¹⁾ has proposed a new interpretation of external vacuum polarization due to the electron wave field. The essential points of his argument seem to consist in the definition of positive and negative energy states by means of the eigensolutions for the Hamiltonian including the external potential, and their suitable combination so as to afford proper transformation characters to the current and charge density.

According to our opinion, however, this view would hardly help us out of difficulties of the present field theory with regard to the external vacuum polarization; because this problem arises only when a certain spacetime region I with Hamiltonian $H^{(I)}$ (e.g. without any external field) borders on another region II with different Hamiltonian $H^{(II)}$ (e.g. including a external field). Even if one defines vacuum properly in I, the state no longer remains vacuum in II, and *vice versa*, and hence the difficulties of external vacuum polarization. So Snyder's vacuum will remain pure vacuum as long as we pertain to the space of given external field, but, for example, when this external field is switched off, pairs might appear and thus the free-space would be "externally polarized". (Fig. 1).

I ((Pairs might appear))

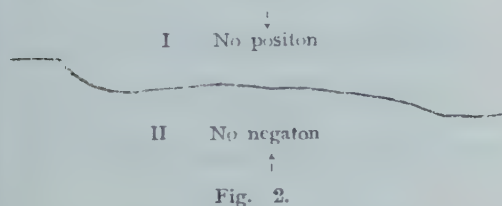
II Vacuum {no negaton
no positon}

↑ ↑

Fig. 1.

But it appears to us that, if we make use of the idea of Feynman's positon theory²⁾, there is one possibility of defining vacuum

so as to exclude pair formation and destruction even at a discontinuous space-time boundary (of space-like nature). When at such a boundary between I and II, there is no negaton wave (positive frequency component) on the past-side II and no positron wave (negative frequency component) on the future side I (Fig. 2), this state we should call "vacuum". (Of course we assume the conservation of the total charge of the system when passing the border.)



Indeed this "initial condition" in Feynman's sense for the narrow closed domain including the boundary will never yield any outgoing wave (i.e. negaton wave in I and positron wave in II), as long as interaction of electrons (internal vacuum polarization) is neglected.³⁾ And this fact seems to be one of the great advantages of Feynman's positron theory, though we are rather sceptical whether such a formal redefinition of vacuum would bring some profit in practical problems.

More detailed considerations will appear later.

- 1) H. S. Snyder: Phys. Rev. **75** (1949) 1623.
- 2) R. P. Feynman: Phys. Rev. in press. We express our cordial thanks to Professor Feynman for kindly having sent us a copy of his unpublished works.
- 3) Dr. Miyazima, to whom we are very much obliged, has pointed out that our "vacuum" would correspond to a suitable superposition of ordinary vacuum states.

Magnetic Moment of Nucleon.

K. Sawada.

Department of Physics, Kyoto University.

July 10, 1949.

Calculations of magnetic moment of nucleon were carried out in different ways by Case¹⁾ and Luttinger²⁾. In the calculation along Schwinger, there appears occasionally very curious singularity by the variable transformation involved. So we calculated in this theory with some care for all types of fields, finding coincidence with Luttinger's result for pseudo-scalar meson field with pseudo-vector coupling. We only list our results for additional magnetic moment;

i) Scalar meson theory. f_2 -coupling can be eliminated by a "Gauge" transformation. ($\delta = \mu/m$; μ : meson mass)

a) Neutral theory

$$\mu_N = 0, \quad \mu_P = \frac{f^2}{2\pi} \left(\frac{3}{2} - \delta^2 + \delta^2(3 - \delta^2) \log \frac{1}{\delta} - \frac{\delta(4 - 5\delta^2 + \delta^4)}{(4 - \delta^2)^{1/2}} \cos^{-1} \frac{\delta}{2} \right)$$

b) Charged theory

$$\mu_N = \frac{f^2}{2\pi} \left(-1 + (2 - \delta^2) \log \frac{1}{\delta} + \frac{\delta(4 - \delta^2)}{(4 - \delta^2)^{1/2}} \cos^{-1} \frac{\delta}{2} \right)$$

$$\mu_P = -\frac{f^2}{2\pi} \left(-\frac{5}{2} + \delta^2 + (2 - 4\delta^2 + \delta^4) \log \frac{1}{\delta} + \frac{\delta(8 - 6\delta^2 + \delta^4)}{(4 - \delta^2)^{1/2}} \cos^{-1} \frac{\delta}{2} \right)$$

c) Charge renormalization for Neutron:

$$\frac{f^2}{4\pi} \int_1^\infty \frac{2u-1}{u^3} du \cdot i c \psi^\dagger \gamma^\mu \hat{A}_\mu \tau_N \psi$$

ii) Pseudo-scalar meson theory. f_2 -coupling can be transformed into f_1 -type coupling with interaction constant $2 \frac{m}{\mu} f_2$ by a Gauge transformation. The results for f_2 -interaction can be obtained by the change in coupling constant mentioned.

a) Neutral theory

$$\mu_N=0, \mu_P=-\frac{f^2}{2\pi}\left(\frac{1}{2}+\delta^2-\delta^2(1-\delta^2)\log\frac{1}{\delta}\right. \\ \left.-\frac{\delta(3-\delta^2)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right)$$

b) Charged theory

$$\mu_N=-\frac{f^2}{2\pi}\left(1+\delta^2\log\frac{1}{\delta}\right. \\ \left.-\frac{\delta(2-\delta^2)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right)$$

$$\mu_P=\frac{f^2}{2\pi}\left(\frac{1}{2}-\delta^2+\delta^2(2-\delta^2)\log\frac{1}{\delta}\right. \\ \left.-\frac{\delta(2-4\delta^2+\delta^4)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right)$$

c) Neutron charge renormalization:

$$\frac{f^2}{4\pi}\int_1^\infty \frac{2u-1}{u^3} du \cdot ie\psi^\dagger \gamma^\mu \hat{A}_\mu \tau_N \psi$$

iii) Vector meson theory. Stueckelberg's formalism was adapted, in which B-field can be eliminated for this process by a gauge transformation analogous to scalar meson field (f_2 -coupling). Tensor coupling leads to logarithmically diverging results, but which can be eliminated by a regulator.³⁾ We only give for usual vector coupling:

a) Neutral theory

$$\mu_N=0, \mu_P=\frac{g^2}{\pi}\left(\frac{1}{2}-\delta^2+\delta^2(2-\delta^2)\log\frac{1}{\delta}\right. \\ \left.-\frac{\delta(2-4\delta^2+\delta^4)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right)$$

b) Charged theory

$$\mu_N=-\frac{g^2}{\pi}\left(-1+(1-\delta^2)\log\frac{1}{\delta}\right. \\ \left.+\frac{\delta(3-\delta^2)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right)$$

$$\mu_P=\frac{g^2}{\pi}\left(-\frac{1}{2}-\delta^2+(1+\delta^2-\delta^4)\log\frac{1}{\delta}\right. \\ \left.+\frac{\delta(1+3\delta^2-\delta^4)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right)$$

c) Charge renormalization for Neutron:

$$\frac{g^2}{2\pi}\int_1^\infty \frac{2u-1}{u^3} du \cdot ie\psi^\dagger \gamma^\mu \hat{A}_\mu \tau_N \psi$$

iv) Pseudo-vector meson theory. In this case also B-field can be transformed into pseudo-scalar coupling with interaction constant $2\frac{m}{\mu}g$ as for the pseudo-scalar f_2 -coupling. Only usual pseudo-vector coupling:

a) Neutral theory $\mu_N=0$

$$\mu_P=-\frac{g^2}{\pi}\left(\frac{7}{2}-\delta^2+\delta^2(5-\delta^2)\log\frac{1}{\delta}\right. \\ \left.-\frac{\delta(8-7\delta^2+\delta^4)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right) \\ +\left(\text{(ii) with } f^2=4\frac{g^2}{\delta^2}\right)$$

b) Charged theory

$$\mu_N=\frac{g^2}{\pi}\left(-7+2\delta^2+(3-10\delta^2+2\delta^4)\log\frac{1}{\delta}\right. \\ \left.+\frac{\delta(19-14\delta^2+2\delta^4)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right) \\ +\left(\text{(ii) with } f^2=4\frac{g^2}{\delta^2}\right)$$

$$\mu_P=-\frac{g^2}{\pi}\left(-\frac{7}{2}+\delta^2+(3-5\delta^2+\delta^4)\log\frac{1}{\delta}\right. \\ \left.+\frac{\delta(11-7\delta^2+\delta^4)}{(4-\delta^2)^{1/2}}\cos^{-1}\frac{\delta}{2}\right) \\ +\left(\text{(ii) with } f^2=4\frac{g^2}{\delta^2}\right)$$

c) Charge renormalization for Neutron:

$$\left(\frac{g^2}{2\pi}+\frac{1}{4\pi}\cdot 4\frac{g^2}{\delta^2}\right)\int_1^\infty \frac{2u-1}{u^3} du \cdot ie\psi^\dagger \gamma^\mu \hat{A}_\mu \tau_N \psi$$

From above results, firstly, the charge renormalization does not depend on the mass of meson and nucleon, for Neutron, apart from the "equivalent" pseudo-scalar coupling constant of the form $4\frac{g^2}{\delta^2}$ for pseudo-scalar f_2 and pseudo-vector B- field, so that this can be eliminated by a regulator with condition $R(0)=0$; thus obtaining consistent result for charge renormalization.

To compare above results with Luttinger's, we should take for pseudoscalar meon theory $f^2=8\frac{(f\mu)^2}{\delta^2}$ for charged theory and for neutral theory $4\frac{(f\mu)^2}{\delta^2}$. Then it will be seen that the charged theory and neutral theory both coincide with his results apart from a little change.* But for vector meson field, $g^2=2g_v^2$ for charged, $g^2=g_v^2$ for neutral, neutral theory only coincides with his result.

From the above results, though we cannot obtain exact coincidence with Luttinger's results, it may be expected that these two independent procedures produce the same

result, and that the constant use of regulator is necessary to obtain consistent result in Schwinger theory.

- 1) K. M. Case, Phys. Rev. **74** (1948) 1884.
- 2) J. M. Luttinger, Phys. Rev. **75** (1949) 309, 1277.
- 3) W. Pauli and F. Villars,

*After furnishing this calculation, we received Phys.

Rev. **76** No. 1 which contained Case's paper, from which we found that our results in (i) and (ii) just coincide with his results. Reference 2) seems to contain some misprints.

On the Electromagnetic Self-Energy of Meson.

S. Hanawa.

Department of Physics, Hokkaido University.

S. Kanesawa.

Physics Institute, Tokyo Bunrika Daigaku.

July 17, 1949.

In the case of electron interacting with radiation field, as was shown by Schwinger,¹⁾ one may ignore the auxiliary condition in the treatment of virtual photon processes, but the situation seems to be somewhat different in the case of meson interacting with radiation field. We have, in fact, found that the elimination of the auxiliary condition, which was left out of consideration by Baba and one of us in their calculation of the radiative reaction on meson,²⁾ gives rise to an additional self-energy term of the form

$$\delta H_{\text{self}} = CL \quad (1)$$

with

$$C = \frac{1}{2\pi} \frac{e^2}{\hbar c} \int_0^\infty \left(k - \sqrt{k^2 + \kappa^2} + \frac{\kappa^2}{\sqrt{\kappa^2 + k^2}} \right) dk, \quad (2)$$

where L is the Lagrangian density of free meson

$$L = \begin{cases} -\frac{1}{4\pi} \left(\frac{\partial \phi^*}{\partial x_\mu} \frac{\partial \phi}{\partial x_\mu} + \kappa^2 \phi^* \phi \right), & \text{(scalar) (3s)} \\ -\frac{1}{4\pi} \left(\frac{1}{2} \chi_{\mu\nu}^* \chi_{\mu\nu} + \kappa^2 \phi_\mu^* \phi_\mu \right). & \text{(vector) (3v)} \end{cases}$$

The electromagnetic mass, however, remains unaffected, since the expectation value of L in one-meson state vanishes.

- 1) J. Schwinger, Phys. Rev., **75** (1949), 651.
- 2) K. Baba and S. Kanesawa, Prog. Theor. Phys., **3** (1948), 443.

Application of Pauli's Regulator to the γ -Decay of Neutrettos.

H. Fukuda, Y. Miyamoto, T. Miyazima
and S. Tomonaga.

Physics Institutes, Tokyo University and
Tokyo Bunrika Daigaku.

July 17, 1949.

The effective Hamiltonian densities responsible to the two- γ decay of neutrettos were calculated by Fukuda and Miyamoto.¹⁾ The results were

$$fV \langle \bar{\psi} \psi \rangle = V \cdot \frac{f_m}{8\pi^2} e^2 \int_{-1}^1 dv \int_1^\infty \frac{du}{u} \times \left[\frac{1}{u^2} A_\mu^2 + \frac{1}{m^2} \left(1 - \frac{1-v^2}{u^2} \right) [u^2]^{-1} \frac{1}{2} F_{\mu\nu}^2 \right], \quad (1)$$

for the scalar neutretto V , and

$$g' \partial_\mu V' \cdot \langle \bar{\psi} \gamma_5 \gamma_\mu \psi \rangle = \partial_\mu V' \cdot \frac{g'}{8\pi^2} e^2 \int_{-1}^1 dv \int_1^\infty \frac{du}{u} \left[\frac{1}{u^5} \sum_{\text{cycle}} A_\nu F_{\sigma\tau} + \frac{1}{m^2} \frac{1-v^2}{u^2} [u^2]^{-1} \sum_\lambda \sum_{\text{cycle}} F_{\lambda\nu} \partial_\lambda F_{\sigma\tau} \right], \quad (2)$$

$$f' V' \langle \bar{\psi} \gamma_5 \psi \rangle = V' \cdot \frac{f'}{4\pi m} e^2 \int_{-1}^1 dv \int_1^\infty \frac{du}{u} \left[\frac{1}{u^5} + \frac{1-v^2}{4m^2 u^2} \square^2 [u^2]^{-1} \right] \times (E_{23} F_{14} + F_{31} F_{24} + F_{12} F_{34}) \quad (3)$$

for the pseudoscalar neutretto V' with the pseudovector and pseudoscalar couplings, g' and f' being respective coupling constants. In these equations A_μ and $F_{\mu\nu}$ represent the potential and field intensity of radiation, and $\gamma_5 \gamma_\mu = \gamma_\nu \gamma_\sigma \gamma_\tau$ and $(\nu\sigma\tau)$ assumes (324), (134), (214), (123) as μ runs from 1 to 4, and $[u^2]$ is written for $u^2 - \frac{1-v^2}{4m^2} \square^2$. That the non-gauge-invariant terms A_μ^2 and $\sum A_\nu F_{\sigma\tau}$ still survive in (1) and (2) is the serious contradiction to the assertion that they should

be gauge invariant. Besides (2) and (3) do not satisfy the Dyson theorem that the pseudovector and pseudoscalar couplings are equivalent if there is the following relation

$$\frac{1}{2m} f' = g', \quad (m = \text{proton mass}). \quad (4)$$

In fact, the difference

$$\begin{aligned} f' V' \langle \varphi \tau_5 \varphi \rangle - (-g' V' \partial_\mu \langle \varphi \tau_5 \tau_\mu \varphi \rangle) \\ = V' \cdot \frac{f' e^2}{12\pi^2 m} (F_{23} F_{14} + \dots) \end{aligned} \quad (5)$$

does not identically vanish.

(i) We think it appropriate to drop $\sum A F$ term in (2) from the following reasons: (a) presence of the term $\sum A F$ does not destroy, at first sight, the gauge invariance of the real processes, because $\sum A_\nu F_{\sigma\tau} \partial_\mu V'$ is equivalent with $2V' (F_{23} F_{14} + \dots)$ on account of conservation of momentum and energy. But this is not true in the case of pseudovector neutretto, because $\sum A_\nu F_{\sigma\tau} U_\mu$ (U_μ : a pseudovector) is no longer gauge invariant even for real processes. (b) As the coefficient of $\sum A F$ is independent from the mass m , this term is dropped by the regulator.²⁾ That is to say, if we suppose that there are auxiliary Fermi particles of mass m_i interacting with the neutrettos by the coupling constants g'_i which satisfy the condition

$$\sum_i g'_i = 0, \quad (6)$$

then $\sum A F$ disappears.

(ii) All terms in (3) are gauge invariant, but if we drop $\sum A F$ term in (2) the first term of (3) must also be dropped in order to preserve the equivalence between the pseudoscalar and pseudovector couplings. In order to drop the first term by using the regulator, we must impose the condition

$$\sum_i \frac{1}{m_i} f'_i = 0, \quad (7)$$

which is equivalent with (6) on account of the relation (4).

(iii) The coefficient of A^2 in (1) is only conditionally convergent, and there would be no reason to retain this non-gauge-invariant term. Pauli's condition is not sufficient to

drop this term, but a more strict condition

$$\sum_i m_i f'_i = 0 \quad (8)$$

must be required.

There are two alternatives interpreting the regulator method, either as a mixed field theory or as a formal procedure. If we take the first point of view and require the regulator to satisfy the condition

$$\sum_i f'_i = 0, \quad \sum_i m_i f'_i = 0, \quad \sum_i \frac{1}{m_i} f'_i = 0, \quad (9)$$

the life time of scalar neutretto will be changed seriously because then the first term of expansion in \square^2/m^2 of the second term of (1) will also be dropped. The anomalous magnetic moment of electron should also vanish, which would contradict with the experimental fact.

If we take the second point of view and require that the regulator must always be applied only to the functions of even power in m , we must first separate a factor of an odd power of m (m and m^{-1} in (1) and (3) respectively) and afterwards apply the regulator. This will yield a different result for (1). There remains ambiguity in separating an odd power of m , because it is also possible to separate m^3 in the case of (1).

We see that there remains still some ambiguity how to use the regulator, and this ambiguity would be solved only by some experiment which could detect the γ -decay of neutretto.

1) H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. in press.

2) W. Pauli and F. Villars, the manuscript sent to Tomonaga from Prof. Pauli.

On the Life of the Neutral Meson.

S. Hayakawa.

Meteorological Research Institute.

July 17, 1949.

The problem whether or not the neutral meson is stable and how long its life is if

it is unstable, is not only important for the interpretation of cosmic-ray phenomena, but also of theoretical interest because it gives ample informations about the nature of the interaction between mesons and nucleons. In this regard the present author has once remarked some experimental evidences which seems to indicate that the life of the neutral meson can not be so short as has been hitherto considered,¹⁾ and the author has also referred to the theoretical work of Fukuda and Miyamoto²⁾ who obtained rather long life of neutral mesons in contradicting earlier work on the same problem. Though the result of Fukuda and Miyamoto contained some ambiguity, this can be removed if one uses Pauli's regulator.^{3,4)} In this note the author will supplement the earlier consideration of the several experimental evidences.

The most likely processes taking part in the decay of neutral mesons are emission of two photons and creation of electron-positron pair. In both cases cascade showers will be formed, which accompanies high energy nucleon collisions, if the energy is high enough to produce mesons. This collision will be accompanied by a shower produced by the photon emitted by the collision, most of which is the emission due to charge exchange. Now, according to our analysis, showers produced by the latter process contribute unnegligible part of the showers experimentally observed. The contribution from the former process must be, therefore, at most of the same order as that from the latter. We shall now discuss several experimental results from this point of view.

(1) It was observed that in a mixed shower there appear often subshowers which emerge with larger angle with respect to the primary ray.⁵⁾ The nature of the subshowers is not clear but they might be due to the decay of neutral mesons, because otherwise the large angle could not be explained. If this interpretation is correct, the probability of mixed showers to be accompanied by

subshowers, a few percent, can give a rough estimation of the life of the neutral meson, the result being about 10^{-11} sec.

(2) If the extensive air shower has its origin solely in the soft rays produced by the decay of the neutral meson, its life must be shorter than 10^{-12} sec.⁶⁾ But if the life lies between 10^{-12} and 10^{-15} sec., we should observe an anomalous behaviour in the density-frequency relation in the observed region of the shower density, $10 \sim 10^4$ per m^2 . As we have found no such anomaly, the life should be either shorter than 10^{-15} sec. or longer than 10^{-12} sec. In the latter case the origin of the air showers must be sought in some other processes, for which the emission of photon in collision process will be responsible. If the contribution of the latter process is large enough, the above mentioned anomaly will be, even if exist, masked and may not be observed. But, in order to perform detailed analysis of this problem, it must be known what part of the incident energy is transferred to the mesons and what part to the photons.

(3) The large bursts under thick shield observed at high altitudes are mainly produced by nucleons⁷⁾. Since the rate of the burst frequency at high altitude to that of sea level calculated on the assumption that the bursts are solely due to the former process agrees in its order of magnitude well with experiments, the contribution from the latter process should be at most of the same order of magnitude as that from the former process. This requires that the life of the neutral meson can not be too short. If the life is of the order of 10^{-12} sec. both contributions are of the same order, but in this case the size-frequency relations of the bursts produced by the decay process shows a steeper decrease than observed. Thus in order not to contradict with observations, the life must be of the order of 10^{-11} sec.

(4) The soft component in the stratosphere can also give some informations. Its bulk

may be due to the decay of neutral mesons. It will be possible to infer the upper limit of the life of the neutral meson from the intensity-depth relation and the angular distribution of soft component. Here we remark that Taketani⁸⁾ has estimated the life of the neutral meson from the ratio of showers and hard rays in the upper atmosphere and found a life, longer than 10^{-9} sec. But the experiment cited by him has such an arrangement that it selects more meson showers than cascade ones, so that the actual life could not be estimated from this experiment.

The author should like to express his sincerest thanks to Prof. Tomonaga and Messrs. Fukuda and Miyamoto for their interest in this problem.

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- 2) H. Fukuda and Y. Miyamoto: *Prog. Theor. Phys.* in press.
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On the Nature of τ -Meson.

H. Fukuda, S. Hayakawa and Y. Miyamoto.
*Department of Physics, Tokyo University and
 Meteorological Research Institute.*

July 17, 1949.

The existence of so-called τ -meson has become more and more certain by increasing experimental evidences, and the transmutation of τ -meson into other mesons provides us a powerful clue to the theory of interaction of elementary particles. In this stage we venture to proceed to solve such a problem. For this purpose we must first know various

physical constants, masses of mesons, their coupling constants and their lives. Unfortunately available experimental data are yet very few, but it appears us not impossible to draw some conclusions about the nature of mesons and their interactions already in the present stage of our knowledge.

The mass of the τ -meson can be determined most accurately by considering the momentum balance in the process $\tau \rightarrow 3\pi$ observed by Brown *et al.*¹⁾ Though it is not concluded that the observed process is $\tau \rightarrow \pi + 2\mu$, the latter process is unlikely because the π -nucleon coupling is far stronger than the μ -nucleon coupling. Brown *et al* obtained thus the mass of τ -meson $m = 985 m_e$. We may hereafter put $m = 1000 m_e$, without introducing serious error.

The life can be estimated from the following three evidences: (a) The track length of the τ -meson found by Brown *et al* is longer than 3000μ , which corresponds to the life longer than 10^{-11} sec. (b) In order that the τ -meson can be captured by the nucleus before it disintegrates,²⁾ the life must be longer than 10^{-12} sec. The production of a star at the track end indicates that τ -meson interacts with the nucleon strongly. (c) The evidence found by Rochester and Butler³⁾ in the cloud chamber may be interpreted as the process $\tau^\pm \rightarrow \pi^\pm + \pi^0$ and $\tau^0 \rightarrow \pi^+ + \pi^-$. Then the life of τ -mesons here observed can be estimated, if we may assume that these are produced in the matter adjacent to the chamber. The life thus obtained is as long as 10^{-10} sec.

Now the coupling constants can be estimated in the following manner. The frequency of the occurrence of τ -mesons is so small that it is hard to get any precise knowledge about their production from experiments. Theoretically it may be expected that the production by mesonic interaction is the most predominant part, the creation by photo-mesonic process being negligible, only if the coupling constant G^2 of τ -meson with

nucleon is larger than 10^{-5} . The production rate of π -mesons to τ -mesons should be g^2/G^2 . The observed abundance ratio is, on the other hand, one τ -meson in 60 π -mesons, or 2 τ -mesons in 50 penetrating showers.³⁾ This means that g^2/G^2 is of the order of 100, from which we can infer that $G^2 \sim 10^{-3}$.

On the basis of these constants we can now theoretically calculate the life of the τ -meson assuming various possible types of mesons and forms of interaction. The most likely mode of decay is $\tau \rightarrow \pi + \pi$ and $\tau \rightarrow \pi + \gamma$, they are most likely because these are the processes of the lowest order in the coupling constants. For some types of mesons or some types of interaction these processes are forbidden, but if they are not forbidden the life is found to be so short as 10^{-14} sec, which is too short to be reconciled with experimental facts. These processes are forbidden when both τ - and π -mesons are scalar or pseudoscalar; the process $\tau \rightarrow \pi + \pi$ is forbidden by the selection rule and the process $\tau \rightarrow \pi + \gamma$ by the fact that one can not construct any invariant interaction Hamiltonian in this case. Also when the mesons are neither scalar nor pseudoscalar, so the processes are not forbidden in the above sense, it is possible to construct the theory such that the process does not occur. It is attained by adjusting the coupling constants for the possible types of interaction in such a way that the matrix elements for the transition compensate with each other. But in any of the case where the transition is forbidden, we can see that the process still takes place with a small probability, if we take into account the mass difference between proton and neutron.⁴⁾ The life is found to be of the order of 10^{-9} sec.

If the processes of the lowest order have so long lives, it is possible that the process of the next higher order takes place equally or more likely. In fact, our calculation of the life of the neutral π -meson into three photons suggests that the decay mode $\tau \rightarrow 3\pi$

gives the life of τ -meson of the order of 10^{-10} sec. Thus the process of the higher order can equally take place with the comparable or rather larger probability as the process of the lower one. As we think, the theory can thus give a satisfactory explanation for a somewhat puzzling fact why such a higher order process $\tau \rightarrow 3\pi$ can compete with lower order process.

If our interpretation is correct, the experimental evidences give us a clue to determine the type of τ - and π -mesons, provided that the interaction compensation as mentioned above does not correspond to the reality. Thus we can conclude that τ - and π -meson must be scalar or pseudoscalar; as is well known, the pseudoscalar nature of π -meson was argued by the evidence on the nuclear forces.

The authors thank to Prof. Tomonaga, Dr. Taketani and Mr. Nakamura for their effective suggestions on this work.

- 1) R. Brown, U. Camerini, P. H. Fowler, H. Muirhead, C. F. Powell and D. M. Riston; *Nature*, **163** (1949), 47, 82.
- 2) L. Leprince-Ringuet; *Rev. Mod. Phys.* **21** (1949), 42.
- 3) G. D. Rochester and C. C. Butler; *Nature* **169** (1947), 855.
- 4) Y. Tanikawa. The authors are indebted to him for suggesting this fact.

Selection Rule for Meson Problem.

H. Fukuda and Y. Miyamoto.
Physics Institute, Tokyo University.

July 17, 1949.

In a paper dealing with the γ -decay of a neutral meson through virtual creation and annihilation of nucleon pair we have referred to the selection rule which is known as Furry's theorem. We shall show here that similar rules exist also in the case where several Bose mesons interact with each other through virtual pairs of Dirac nucleons.

Let the general term in the interaction

energy density between Bose meson and nucleon fields be denoted by $\phi^\dagger \gamma_L \tau_L \phi U_L$, where ϕ is the spinor describing the nucleon field and U_L is the potential itself or some derivatives of the potential describing a meson field. The spin matrix γ_L is respectively 1, γ_4 , γ_{ij} , γ_{ijk} , or γ_5 (γ being Dirac matrices) according as U_L is a scalar, vector, tensor, pseudovector or pseudoscalar. The isotopic spin is either 1 or τ_3 when U_L belongs to a neutral meson field; it is 1 or τ_3 according as the latter field obeys the neutral theory or the symmetrical theory. When U_L describes a charged meson field, τ_L is τ_{PN} if U_L contains the creation (annihilation) operator of negative (positive) meson, and it is τ_{NP} if τ_L contains the creation (annihilation) operators of positive (negative) meson.

Now, according to Tomonaga-Schwinger theory, the interaction between various meson fields induced by the virtual emission and reabsorption of nucleon pairs are described by the effective Hamiltonian density of the form

$$\int_{-\infty}^{\infty} dX' \int_{-\infty}^{\infty} \dots \text{Spur} \left\{ \sum_{i=0, \dots, n} \bar{S}(X' - X') \gamma_L \bar{S}(X' - X'') \gamma_M \dots \gamma_P S_1(X^{(i)} - X^{(i+1)}) \gamma_Q \dots \gamma_R \bar{S}(X^{n-1} - X^n) \gamma_S \bar{S}(X^n - X) \gamma_T \right\} \cdot \tau_L \tau_M \dots \tau_P \tau_Q \dots \tau_R \tau_S \tau_T \cdot U_L U_M \dots U_P U_Q \dots U_R U_S U_T \quad (1)$$

when there are no nucleons before and after the interaction, and in the approximation in which we are not interested in the radiative correction (in this approximation U_L , U_M , ... may be assumed to commute with each other). The effective Hamiltonian gives rise to various modes of meson decay or various scattering processes between mesons U_L , U_M , ... We shall now show that if we denote the number of vector type coupling occurring in (1) by N_v , the number of tensor type coupling by N_t , and the number of τ_3 by N_s , the process is forbidden when $N_v + N_t + N_s = \text{an odd number}$. The proof is as follows.

Corresponding to the process described by

(1), there is also the process described by

$$\int_{-\infty}^{\infty} dX' \dots dX^n \text{Spur} \left\{ \sum_{i=0, \dots, n} \bar{S}(X' - X') \gamma_S \dots \gamma_M \bar{S}(X^{n-1} - X^n) \gamma_L \bar{S}(X^n - X) \gamma_T \dots \tau_S \tau_R \dots \tau_P \tau_Q \dots \tau_M \tau_L \tau_T \cdot U_L U_M \dots U_P U_Q \dots U_R U_S U_T \right\} \quad (2)$$

We first exclude the case where all of the mesons under consideration are neutral. Then it follows that

$$\tau_L \tau_M \dots \tau_P \tau_Q \dots \tau_R \tau_S \tau_T = (-1)^{N_s} \tau_S \tau_R \dots \tau_Q \tau_P \dots \tau_M \tau_L \tau_T \quad (3)$$

since

$$\tau_{PN} \tau_3 = -\tau_{PN} \quad \tau_{NP} \tau_3 = \tau_{NP} \quad \tau_3^2 = 1 \quad (4)$$

We notice further

$$\text{Spur} (\gamma_i \gamma_j \dots \gamma_p \gamma_q) = \text{Spur} (\gamma_q \gamma_p \dots \gamma_j \gamma_i)$$

Applying this relation to the spur of (2) and changing the integration variables x^i to x^{n+1-i} , we obtain the spur part of (2)

$$= \text{Spur} \left\{ \sum_{i=1, \dots, n} \bar{S}^*(X - X') \gamma_L^* \bar{S}^*(X' - X'') \gamma_M^* \dots \gamma_P^* S_1^*(X^i - X^{i+1}) \gamma_Q^* \dots \gamma_R^* \bar{S}^*(X^{n-1} - X^n) \gamma_S^* \bar{S}^*(X^n - X) \gamma_T^* \right\} \quad (5)$$

where $\gamma_L^* = \varepsilon \gamma_L$ $\varepsilon = 1$ for $\gamma_L = 1, \gamma_4, \gamma_5$ and $\varepsilon = -1$ for $\gamma_L = \gamma_{ij}, \gamma_{ijk}$

Thus (5) becomes

$$\text{Spur} (\bar{S}^*(X - X') \gamma_L \bar{S}^*(X' - X'') \gamma_M \dots \gamma_P S_1^*(X^i - X^{i+1}) \gamma_Q \dots \gamma_R \bar{S}^*(X^{n-1} - X^n) \gamma_S \bar{S}^*(X^n - X) \gamma_T) (-1)^{N_t + N_{pv}} \quad (5')$$

(N_{pv} denotes the number of pseudovector coupling occurring in this process.) Remembering the fact that spur of product containing odd number of γ 's vanishes and that

$$\bar{S}(X) = \left(\gamma \frac{\partial}{\partial X} - x \right) \bar{A}(X)$$

$$\bar{S}^*(X) = \left(-\gamma \frac{\partial}{\partial X} - x \right) \bar{A}(X)$$

$$S_1(X) = \left(\gamma \frac{\partial}{\partial X} - x \right) A^{(1)}(X)$$

$$S_1^*(X) = \left(-\gamma \frac{\partial}{\partial X} - x \right) A^{(1)}(X)$$

the spur part of (5') is simplified to

= Spur part of (1)

$$\times (-1)^{N_v + 2N_t + 3N_{pv} + 4N_{ps}}$$

$$= \text{Spur part of (1)} \times (-1)^{N_v + N_{pv}} \quad (6)$$

(N_{ps} denotes the number of pseudoscalar coupling occurring in this process) Combining (3), (5') and (6) we obtain finally

$$(2) = (-1)^{N_3} \times (-1)^{N_t + N_{pv}} \times (-1)^{N_v + N_{pv}} \times \quad (1)$$

$$= (-1)^{N_v + N_t + N_3} \times (1) \quad (7)$$

We see from (7) that the matrix element (1) and (2) cancel each other if $N_v + N_t + N_3 =$ an odd number, so that such a process is forbidden.

In the case where all the mesons are neutral, we may use

$$\tau_L \tau_M \cdots \tau_R \tau_S \tau_T = \tau_S \tau_R \cdots \tau_M \tau_L \tau_T \quad (3')$$

instead of (3), because in this case all the τ 's are 1 or τ_3 . In this case the selection rule is simply read. If $N_v + N_t = \text{odd}$ the process is forbidden.

We shall also consider the case where light quanta are participating in the process. In this case some of $U_L, U_M \cdots$ are electromagnetic potentials and, moreover, some of $\tau_L \tau_M \cdots$ are $\tau_P = \frac{1 + \tau_3}{2}$. In such a case we have no simple relations as (3) nor (3'), so that there is no simple selection rule. But if all mesons participating in the process are neutral we have again (3'), so obtain the rule: if $N_v + N_t = \text{odd}$, the process is forbidden.

In conclusion the authors wish to express their cordial thanks to Prof. S. Tomonaga for valuable discussion and encouragement in this work.

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On the Electron-Positron Pair Disintegration.

H. Fukuda and Y. Miyamoto.

Physics Institute, Tokyo University.

July 17, 1949.

Recently Prof. Sakata has suggested to us on the possibility of the neutral meson disintegrating into the electron-positron pair. In fact, we have found that the neutral vector meson with vector and tensor couplings can disintegrate into electron-positron pair much faster than into three quanta. This process is described in the perturbation theory as occurring through the following steps:

neutral meson \rightarrow proton + antiproton \rightarrow

photon \rightarrow electron + positron

(Møller interaction) (1)

The matrixelement for this process can be obtained by the same method as that used by Schwinger¹⁾ in the photon self energy problem, as follows: for the vector meson with vector and tensor couplings

$$\frac{1}{2\pi} \frac{e^2}{\hbar c} f_1 \psi^\dagger \gamma_j \psi U_j \times \left\{ \frac{2}{3} \int \frac{\cos \omega}{\omega} d\omega + \frac{2}{15} \left(\frac{\mu}{x} \right)^2 \cdots \right\} \quad (2)$$

$$\text{and } \frac{1}{2\pi} \frac{e^2}{\hbar c} f_t \left(\frac{x}{\mu} \right) \psi^\dagger \tau_j \psi U_j \times$$

$$\left\{ 2 \int \frac{\cos \omega}{\omega} d\omega + \frac{1}{3} \left(\frac{\mu}{x} \right)^2 \cdots \right\} \quad (3)$$

where U_j is the potential for the meson field, ψ the spinor for the electronic field, and $\mu(x)$ is the reciprocal compton wave length of the meson (nucleon). In the case of tensor coupling, we have employed the Yukawa equation $\partial_i U_{ij} = \mu^2 U_j$ and in the case of vector couplings, we have, following Schwinger,¹⁾ removed the non gauge invariant terms which appear in the vacuum polarization in the external field. As is seen in (2) and (3), there appear the divergent terms. Thus various kinds of lifetime can be given, according as which prescription is taken to

get rid of the divergent terms. The following prescriptions are possible; (I) Pauli's regulator²⁾ and (II) by adjusting both couplings in such a way that the divergent terms in (2) and (3) cancel each other, i.e. $f_1 = -\frac{\mu}{3\pi} f_2$. The table for the life time of the neutral meson will be given in the following.

where meson mass $\mu = 300m_e$, $\tau_0 = \left(\frac{\hbar c}{e^2}\right)^2 \left(\frac{\hbar c}{f^2}\right) \frac{\pi^2}{\mu c}$, and $f^2/\hbar c = \frac{1}{20}$ in I.

In II $f_2^2/\hbar c = \frac{1}{20}$ is tentatively assumed.

It is easily shown that this process of the neutral meson other than vector is forbidden in our approximation of $e^2/\hbar c$, by the Furry's theorem and from the other reason. The life time given in Table I is very short

Table I.
The lifetime of the neutral vector meson.

Coupling	I	II
Tensor	$108 \left(\frac{\kappa}{\mu}\right)^2 \tau_0$ $= 3 \times 10^{-14}$	$36 \times \tau_0$ in I $= 2.5 \times 10^{-10}$
Vector	$675 \left(\frac{\kappa}{\mu}\right)^2 \tau_0$ $= 7 \times 10^{-12}$	

compared with those by the three quanta disintegration, which will be described in the following letter below. Thus, from the arguments of the previous letter³⁾ and the present one, it can be concluded that the neutral scalar and pseudoscalar meson disintegrate into two quanta, and on the other hand the neutral vector meson, into electron-positron pair.

The γ beam of 70Mev. observed in the cyclotron at Berkley may be reasonably interpreted as due to the two quanta disintegration of the neutral, scalar or pseudoscalar meson³⁵⁾ if the experimental arrangement is so constructed as to discriminate between γ -ray and electron-positron beam which is the decay product of the neutral vector meson.

The authors wish to express their cordial thanks to Prof. S. Sakata for suggesting this important process, and also to Prof. S. Tomonaga for his encouragement in this work.

- 1) J. Schwinger Phys. Rev. **75** (1949) 651.
- 2) Pauli and Villars Rev. Mod. Phys. in press.
- 3) H. Fukuda and Y. Miyamoto Prog. Theor. Phys. in press.
- 4) J. Steinberger Phys. Rev. in press. The electron-positron pair disintegration has also been found by Dr. Steinberger independently of us and was stated in his letter to us.
- 5) Private communication of Prof. H. Yukawa (at Princeton) to us.

The Three Quanta Disintegration of the Neutral Meson.

H. Fukuda and Y. Miyamoto.
Physics Institute, Tokyo University.

July 17, 1949.

In the previous letter¹⁾ we pointed out that there appear non-gauge invariant terms in the matrixelement for the two quanta disintegration of the neutral meson, and that, if these terms are removed, the lifetime of the neutral meson becomes considerably longer than has hitherto been believed. In the meanwhile we had the opportunity of learning Pauli's²⁾ regularization method which makes it possible not only to remove the non-gauge invariant term in the photon self energy, but also to indicate how to dispose of the divergence and ambiguity occurring in the present theory. But it is found that Pauli's procedure can remove the non-gauge invariant terms in our problem, but either at the sacrifice of Dyson's identity, or preserving the identity by imposing a somewhat problematic extra condition on the regulator.³⁴⁾ Thus in this problem the puzzling character of the present theory becomes so much pronounced that the reasonable interpretation could not be made without the drastic change of the concept of the present

theory of wave fields.

Here we shall investigate the gauge invariance problem in the three quanta disintegration of the neutral meson. Though this problem turns out to be less physically significant, since, as stated in the above letter, it is found that the electron-positron pair disintegration of the neutral vector meson occurs considerably faster than the three quanta disintegration, we think that this problem still retains some formal interest.

The matrixelement for our process is described as follows; for the vector (scalar) meson with vector coupling

$$ifU_i(\varphi^\dagger\gamma_i\varphi)_s \quad if\partial_iU(\varphi^\dagger\gamma_i\varphi)_s \quad (1)$$

and for the vector (pseudovector) meson with tensor coupling

$$\frac{if}{2\mu}U_{st}(\varphi^\dagger\gamma_s\varphi)_t \quad \frac{if}{2\mu}(\varepsilon_{pqst}U_{pq})(\varphi^\dagger\gamma_s\varphi)_t \quad (2)$$

where U is the potential for meson field, φ , the spinor for protonic field, f , the coupling constant, and $\mu(\mathbf{x})$, the reciprocal Compton wave length of the meson (proton). $(\varphi^\dagger L\varphi)_s$ is the vacuum expectation value with respect to the protonic field, and explicitly it is

$$\left(\frac{ie}{\hbar c}\right)^3 \int \int \int \int^{C'''} dX'' dX''' dX'''' \times \\ \langle [H'''' [H'' [H', \varphi^\dagger L\varphi]]] \rangle_0 \quad (3)$$

where H is the interaction energy between protonic and electromagnetic fields and is defined by $H = -ie\varphi^\dagger\gamma_i\varphi A_i$, $(\varphi^\dagger L\varphi)_s$ has two important properties; (I) $(\varphi^\dagger L\varphi)_s$ is gauge invariant, that is, invariant under the transformation $A_i \rightarrow A_i + \partial_i A$, and (II) charge conservation law $\partial_i(\varphi^\dagger\gamma_i\varphi)_s = 0$. According to Furry's theorem, the three quanta disintegration of the neutral meson with scalar, pseudovector, and pseudoscalar coupling is forbidden. From II this mode of decay of the neutral scalar meson with vector coupling is also forbidden.

The results are:⁵⁾

$$(\varphi^\dagger\gamma_i\varphi)_s = -\frac{i}{24\pi^2} \left(\frac{e^2}{\hbar c}\right)^3 \left[A^2 A_i + \frac{2}{15\pi^2} \times \right.$$

$$\left. \{ \partial_j \{ F_{ki} (F_{12} F_{34} + F_{24} F_{13} + F_{23} F_{14}) \} \times \right. \\ \left. 7\varepsilon_{ijkl} + F_{ij} \partial_j (F_{kl})^2 \} \right] \quad (4)$$

$$(\varphi^\dagger\gamma_s\varphi)_s = -\frac{i}{12\pi^2 x^3} \left(\frac{e}{\hbar c}\right)^3 \times \\ \{ F_{12} (F_{34} + F_{23} F_{14} + F_{31} F_{24}) \varepsilon_{1234} \} \quad (5)$$

The first term in (4) is not only non-gauge invariant in contradiction with (I), but also destroys (II), whereas it does not so for the second term in (4) and the first term in (5). Thus it is expected that, in future theory, the first term in (4) should be zero. Really this term is found to be cancelled out by Pauli's regulator.²⁾ Further, (4), multiplied by $-ieA_i/4$, is the matrixelement for the scattering of light by light which was worked out by Euler.⁶⁾ The first term in (4) was removed by him with recourse to Heisenberg's subtraction method. The second term in (4), is in complete agreement with Euler's result.

In the following, the table for the lifetime of the neutral vector meson by this process will be given.

Table :

The lifetime of the neutral vector meson.

tensor coupling	$\tau_0 \left(\frac{\kappa}{\mu}\right)^6 1.5 \times 10^5 \sim 5.5 \times 10^{-5} \text{ sec.}$
vector coupling	$\bullet \tau_0 \left(\frac{\kappa}{\mu}\right)^8 2.8 \times 19^5 \sim 4 \times 10^{-3} \text{ sec.}$

where $\tau_0 = \frac{\hbar c}{f^2} \left(\frac{\hbar c}{e^2}\right)^3 \frac{\pi^5}{\mu c} \frac{f^2}{\hbar c} = \frac{1}{20}$, and $\mu = 300 m_e$. In our approximation the decay of the pseudovector meson with tensor coupling is forbidden. Comparing the above table with that of the above letter, we shall see that the life time by the three quanta disintegration is much longer than that of the electron-positron pair disintegration.⁷⁾ Further $\tau \rightarrow 3\pi$ decay which has been discovered by Powell, will probably be of the same mechanism as γ -decay of neutral meson. The life of τ -meson is about $10^{-9} \sim 10^{-10}$ sec. from our estimation from three quanta dis-

integration of neutral meson, which is in agreement with experiment.

We should like to express our gratitude to Prof. S. Tomonaga for his kind encouragements in this work.

- 1) H. Fukuda and Y. Miyamoto Prog. Theor. Phys. in press.
- 2) Pauli and Villars Rev. Mod. Phys. in press.
- 3) H. Fukuda, Y. Miyamoto T. Miyazima, and S. Tomonaga. Prog. Theor. Phys. in press. S. Ozaki. *ibid.*
- 4) J. Steinberger Phys. Rev. in press. The authors wish to express their cordial thanks for sending them his unpublished results.
- 5) J. Schwinger Phys. Rev. **75** (1949) 651,
- 6) H. Euler Ann. der Phys. **26** (1936) 338,
- 7) R. J. Finkelstein Phys. Rev. **72** (1947) 414, S. Nakamura. Su-Butsu-Kaishi **16** (1942) 201, They obtained very short lifetime, which seems to us due to their inadequate treatments.

Note on the Wave-equation in the Non-localizable System.*

H. Suzuki.

Department of Physics, Kyoto University.

July 20, 1949.

In the non-localizable system the field potential A_λ of the electromagnetic field satisfies the following equation¹⁾

$$[p_\mu, [p^\mu A_\lambda]] = 0 \quad (1)$$

$$[x_\mu, [x^\mu A_\lambda]] = 0 \quad (2)$$

$$[p_\mu, x_\nu] = i\hbar\delta_{\mu\nu}, [p_\mu, p^\nu] = 0, [x_\mu, x^\nu] = 0 \quad (3)$$

where p_μ and x_μ are the operators, corresponding to the energy-momentum and coordinates of the space-time of the particle respectively.

Now from the equation (1) follow the next equations:

$$0 = [x_\nu, [p_\mu, [p^\mu A_\lambda]]] = [p_\mu, [p^\mu, [x_\nu A_\lambda]]] \quad (4)$$

Therefore considering (1), the solution of (4) is

$$[x_\nu A_\lambda] = a_{\nu\lambda\mu} A_\mu \quad (5)$$

and from (2) and (4) the constants $a_{\lambda\mu\nu}$ satisfy the relation

$$a_{\nu\lambda\mu} + a_{\nu\mu\tau} = 0 \quad (6)$$

Similarly for p_ν the same relation holds:

$$[p_\nu A_\lambda] = b_{\nu\lambda\mu} A_\mu, \quad (7)$$

where the constants $b_{\nu\lambda\mu}$ satisfy the relation

$$b_{\nu\lambda\mu} + b_{\nu\mu\tau} = 0. \quad (8)$$

Then if the relation (7) can be written in the following:

$$[p_\nu A_\lambda] = \hbar \sum_s k_{\nu s} A_{\lambda s}; \quad \sum_\nu k_{\nu s} k_{\nu s} = 0 \quad (9)$$

which means that the field A_λ can be expanded by Fourier series i.e.

$$A_\lambda = \sum_s A_{\lambda s} = \sum_s \{ A_{\lambda s}^+(p_\nu, k_{\nu s}) e^{i|k_{\nu s}|x_\nu} + A_{\lambda s}^-(p_\nu, k_{\nu s}) e^{-i|k_{\nu s}|x_\nu} \} \quad (10)$$

where $k_{\nu s}$ are constants (c-number). Thereupon from (9), the equation

$$\hbar \sum_s k_{\nu s} [x_\mu A_{\lambda s}] = [x_\mu, [p_\nu A_\lambda]] = [p_\nu, [x_\mu A_\lambda]]$$

is deduced. The solution of this equation becomes

$$[x_\mu A_{\lambda s}] = c^\pm k_{\mu s} A_{\lambda s} \quad (11)$$

where c is a constant or zero.

Then from (9), commutation relations

$$[x_\mu, A_{\lambda s}^\pm] = c^\pm k_{\mu s} A_{\lambda s}^\pm; \quad [p_\mu, A_{\lambda s}^\pm] = 0$$

are deduced, which have been introduced by Markow.²⁾ Accordingly the nonlocalizable field contains the Markow-type field. As to be seen in this example the non-localizable field interaction can give the clue to set up the non-linear theory and remove the divergence difficulties of the present theory.

* The content of this paper was read at the annual meeting of the Physical Society of Japan held at Kyoto University on May 1948.

1) H. Yukawa: Prog. Theor. Phys. **3**. (1949) 209.

2) M. Markow: Jour. Phys. **2**. (1940) 453.

High Energy Photo-Disintegration of the α -Particle.

Y. Nishida and M. Nogami.

Department of Physics, Kyushu University.

July 12, 1949.

Nuclear phenomena in high energy region are essential to the exact determination of the nuclear properties. Especially high energy photo-disintegration of nuclei is valuable for this purpose because of its purity. In this letter as a typical nucleus we took up α -particle possessing the largest specific mass defect, and studied the behavior of its disintegration by 100~250 MeV gamma rays, and compared the results with those of deuteron, and finally expected the manner of the transition of disintegration phenomena with increase of mass number. Numerical calculations were performed about neutron emission and proton emission. We assumed the non-relativistic treatment and plane wave approximation to the emitted particle. We could neglect the existence of the excited states¹⁾ of the α -particle in our treatments of the high energy phenomena of very light nucleus. We used the Volz type nuclear forces and neglected the tensor forces.* In both emissions the total cross section σ is the sum of charge part σ_c and spin part σ_s . The relations between these cross sections and gamma ray energy are illustrated in Fig. 1 and Fig. 2. $\sigma_c(n)$ rises from the Heisenberg part only of the nuclear forces contrast with others, and it falls off with increase of gamma ray energy more steeply

than others. If P, P' mean the momenta of emitted particle and residual nucleus in the laboratory frame, respectively, the above difference is also distinguished by the fact $\sigma_c(n)$ depends on P' only, others P, P' . In our calculations $\sigma_s(p)/\sigma_s(n) = \mu_p^2/\mu_n^2$ (μ : magnetic moment); and $\sigma(p)/\sigma(n)$ becomes larger with increase of force range. As expected, the increase of force range reduces the total cross sections $\sigma(p), \sigma(n)$, and this tendency becomes remarkable with increase of gamma ray energy. This is due to the fact that with increase of gamma ray energy retardation effect becomes more essential, and requires the exact introduction of nuclear dimension. In our approximations nuclear dimension is supposed to be roughly proportional to force range. Angular distributions in both emissions are roughly uniform in the range 100~150 MeV, and then gradually forward emission becomes increasing, and at 250 MeV this is the whole. We calculated the cross section of the deuteron using this Volz type nuclear forces with the same procedures as Rose and Goertzel.²⁾ It falls off with increase of gamma ray energy more steeply and is smaller than the case of square well potential for the deuteron and our results for the α -particle. Therefore the cross sections of α -particle by the square well potential will

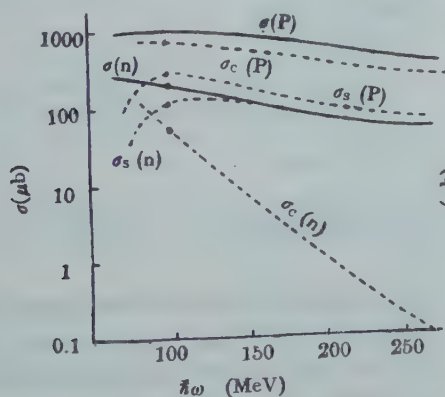


Fig. 1, $r_0 = 2.0 \times 10^{-13}$ cm.

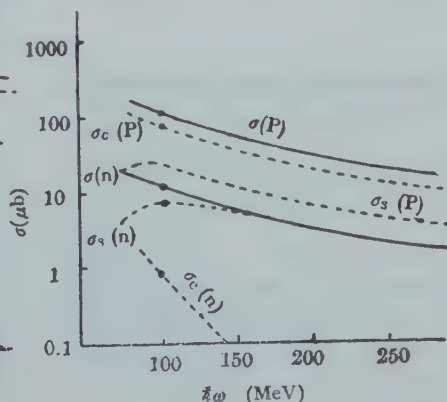


Fig. 2, $r_0 = 2.8 \times 10^{-13}$ cm.

Cross sections of the α -particle in μb (10^{-30}cm^2) on a logarithmic scale versus gamma ray energy. r_0 : force range.

be perhaps more slowly decreasing and larger than our results. In general the dependence of nuclear dimension upon force range is more sensitive by Gaussian potential than square well. Therefore the force range-dependence of the nuclear properties by the square well will be looser than our results, and this tendency will become eminently with increase of gamma ray energy. It may be noticed that the total cross sections of the α -particle have more slowly decreasing gradient than the deuteron. This is *chiefly* due to the increase of mass number and *merely* the greatness of mass defect. And from the fact that in the intermediate nuclei oscillator model is the adequate approximation, we can expect the fair slowness of the gradient of these nuclei. The increase of mass number tends to smooth the undulation of the angular distribution.

1) H. Bethe and R. Bacher: Rev. Mod. Phys. 8, A (1936), 147.

2) M. Rose and G. Goertzel: Phys. Rev. 72 (1947), 749.

* Tensor forces-conclusions are indefinite now. Cf. Discussions in the physical meeting at Birmingham Sept. 1948, p. I.

On the Self-Energy of the Neutretto.

Y. Katayama.

Dept. of Phys., Kyoto University.

July 25, 1949.

In the previous letter¹⁾, we reported that the self-energies of C-meson calculated by the method of Pauli²⁾ diverge logarithmically.* However, whether there are diverging terms of other types or not in the expression of the self-energy critically depends on the method used in the calculation and is therefore related to the applicability of this method.

In this connection, we have calculated the self-energy of pseudoscalar neutretto due to the nucleon field in order to examine the merit of these two methods. In this case, the identity which holds in the free field³⁾

$$\frac{\partial}{\partial x_\mu} (\psi^\dagger \gamma_5 \gamma_\mu \psi) = -2M(\psi^\dagger \gamma_5 \psi) \quad (1)$$

plays a very important role as the gauge-invariance of the photon self-energy problems.

In this problem, we have also to make use of the canonical transformation in order to reduce the second order Schrödinger equation. That is, by making use of the transformation $\Psi[C] = U[C] \Psi_1[C]$, we have the following equation:

$$\begin{aligned} \frac{\partial}{\partial x_\mu} (U \psi^\dagger \gamma_5 \gamma_\mu \psi U^{-1}) &= -2M(U \psi^\dagger \gamma_5 \psi U^{-1}) \\ &+ i(U [\psi^\dagger \gamma_5 \gamma_\mu \psi N_\mu, \int_C H_1(X') dF'] U^{-1}) \end{aligned} \quad (2)$$

where $i \frac{\delta U}{\delta C} = H_1 U$. In our case, taking

$$H_1 = \begin{cases} H_{f_{ps}} = i f_{ps} \psi^\dagger \gamma_5 \psi \cdot \phi & (3.1) \end{cases}$$

$$H_{f_{pv}} = i \frac{f_{pv}}{\mu} \psi^\dagger \gamma_5 \gamma_\mu \psi \frac{\partial \phi}{\partial x_\mu} \quad (3.2)$$

we get the identity (1) in the case of (3.2) in virtue of the commutability of $H_{f_{pv}}$, but not in (3.1). Judging from these circumstances, we can see that if we only consider $H_{f_{pv}}$ as the transformation function, this identity still holds after the transformation. In fact, we have

$$\begin{aligned} i \langle \psi^\dagger \gamma_5 \psi \rangle &= 4 \frac{M}{\mu} f_{pv} \int_{-\infty}^{+\infty} \bar{L}_\lambda(X-X') \frac{\partial \phi(X')}{\partial X'_\lambda} dX' \\ i \langle \psi^\dagger \gamma_5 \gamma_\mu \psi \rangle &= -4 \frac{f_{pv}}{\mu} \int_{-\infty}^{+\infty} \bar{K}_{\mu\nu}(X-X') \frac{\partial \phi(X')}{\partial X'_\nu} dX' \end{aligned} \quad (4)$$

where L_λ and $K_{\mu\nu}$ are bilinear formulae of $\Delta^{(1)}(X)$ and $\bar{\Delta}(X)$ and their derivatives, and M, μ are the masses of nucleon and neutretto, respectively. From these equations, we get

$$\frac{\partial}{\partial X_\mu} \langle \psi^\dagger \gamma_5 \gamma_\mu \psi \rangle = -2M \langle \psi^\dagger \gamma_5 \psi \rangle$$

$$-4if_{pv} \int_{-\infty}^{+\infty} \delta(X-X') \frac{\partial A^{(1)}(X-X')}{dX_\lambda} \frac{\partial \phi(X')}{\partial X'_\lambda} dX' \quad (5)$$

Therefore, in this case, the circumstance is the same as in the case of the gauge invariance problem of the self-energy of photon and this also promises to effectiveness of Pauli's methods. Furthermore, after the calculations we get the following equation equivalent to (5):

$$ip_\mu \tilde{K}_{\mu\nu}(p) = 2M^2 \tilde{L}_\nu(p) - \frac{f_{pv}}{16\pi^2} p_\nu R'(0) \quad (5')$$

Here we have replaced $R'(z)$ with $iM^2 R(z)$. From this it seems clear that if we take $R'(0)=0$ the identity does hold.

In the above considerations, we can utilize this method only to drop the indeterminate terms in the calculations. Then we get the self-energy of neutretto taking into account both Hf_{ps} and Hf_{pv} ,

$$-\delta L = A\phi\phi + \frac{1}{\mu^2} B \frac{\partial \phi}{\partial x_\lambda} \frac{\partial \phi}{\partial x_\lambda} \quad (6)$$

and

$$A = -\frac{f_{ps}}{4\pi} \left(f_{ps} + \frac{2M}{\mu} f_{pv} \right) \times \frac{\mu^2}{2\pi} \left[\log rw_0 + 2 \int_0^1 \frac{y^2 dy}{(4\delta^2 - 1) + y^2} \right]$$

$$+ \frac{f_{ps}^2}{4\pi} \frac{M^2}{\pi} \left[\log rw_0 + \frac{1}{6} \frac{1}{\delta^2} \right] \\ B = \frac{f_{pv}}{4\pi} \left(f_{ps} + \frac{2M}{\mu} f_{pv} \right) \times \frac{M\mu}{\pi} \left[\log rw_0 + 2 \int_0^1 \frac{y^2 dy}{(4\delta^2 - 1) + y^2} \right] \\ \delta = \frac{M}{\mu} \quad (7)$$

These results show that in the present case the divergence is also logarithmic.

We wish to thank Prof. M. Kobayasi and Messrs. S. Takagi and K. Sawada for their kind interests and discussions.

- 1) Y. Katayama and K. Sawada, Prog. Theor. Phys. **4** (1949) 377.
- 2) W. Pauli and F. Villars, Rev. Mod. Phys. in press.
- 3) E. Nelson, Phys. Rev. **60** (1941) 830, F. J. Dyson, Phys. Rev. **73** (1948) 929.
- 4) On the contrary, in the Hf_{ps} case, we get $\frac{\partial}{\partial x_\mu} \langle \psi^+ \gamma_5 \gamma_\mu \psi \rangle = -2M \langle \psi^+ \gamma_5 \psi \rangle - 4iMf_{ps}$ $\times \int_{-\infty}^{+\infty} \delta(X-X') A^{(1)}(X-X') \phi(X') dX'$ and cannot remove the last term.

* In the previous letter, we also utilize the alternative method proposed by Miyazima. But this method has no consistent way of calculations and many difficulties arise. See, Y. Katayama and S. Hori. Prog. Theor. Phys. in press.

Errata (Vol. IV, No. 2)

P. 232 Phenomenological Treatment on the Production of Cosmic-Ray Mesons.

In equ. (7), for $B=1$, read $B=\frac{1}{2}$
Eight lines under equ. (7), for higher, read lower.

On the Method of the Third Quantization. II.*

Yôichirô NAMBU

Department of Physics, Tokyo University

(Received May 16, 1949)

4. The wave equation in the q-3 formalism (*Continued*)

Our next task is expressing the various quantities mentioned above by the q -3 operators. To conform to the hole theory, it is necessary to decompose the electron field into electron (positive energy) and positron (negative energy) parts, and make necessary changes of order. We put accordingly

$$\psi^{\dagger} = \psi_p^{\dagger} + \psi_n, \quad \psi = \psi_p + \psi_n^{\dagger}, \quad (4.25)$$

where p and n refer to the positive and negative energy respectively. The free particle energy density

$$H_0 = -\frac{1}{2} \left(\psi^{\dagger} \gamma_4 \frac{\partial}{\partial x_4} \psi - \frac{\partial}{\partial x_4} \psi^{\dagger} \gamma_4 \psi \right), \text{ or } \sum_{i=1}^3 \psi^{\dagger} \gamma_i \frac{\partial}{\partial x_i} \psi + \kappa \psi^{\dagger} \psi \quad (4.26)$$

becomes then

$$\begin{aligned} H = & \psi_p^{\dagger} \left(\gamma_i \frac{\partial}{\partial x_i} + \kappa \right) \psi_p + \psi_p^{\dagger} \left(\gamma_i \frac{\partial}{\partial x_i} + \kappa \right) \psi_n^{\dagger} + \psi_n \left(\gamma_i \frac{\partial}{\partial x_i} + \kappa \right) \psi_p \\ & - \psi_n^{\dagger} \left(-\gamma_i' \frac{\partial}{\partial x_i} + \kappa \right) \psi_n, \end{aligned} \quad (4.27)$$

where γ_i' means the transposed of γ_i . We have avoided the symmetrization with respect to the electron and positron adopted by Schwinger,¹⁾ for the sake of simplicity and in accordance with the present scheme in which the lowest state (vacuum) is not defined by supplementary conditions.

Similarly for the electromagnetic field we get

$$A_{\mu} = a_{\mu}^* + a_{\mu}, \quad (4.28)$$

where a_{μ}^* is the emission, a_{μ} the absorption operator. The free radiation field (including provisionally the longitudinal part) is also understood to be rid of its zero-point fluctuation by rearrangement of a_{μ}^* and a_{μ} . The commutation relations are

*) Continued from Prog. Theor. Phys. 4 (1949), 331.

$$\begin{aligned}
\{\phi_{p\alpha}(x), \phi_{p\beta}^\dagger(x')\} &= S_{\alpha\beta}^{(+)}(x-x') \equiv i(\gamma_\mu \nabla_\mu - x)_{\alpha\beta} \mathcal{A}^{(+)}(x-x'), \\
\{\phi_{n\alpha}^\dagger(x), \phi_{n\beta}(x')\} &= S_{\alpha\beta}^{(-)}(x-x') \equiv i(\gamma_\mu \nabla_\mu - x)_{\alpha\beta} \mathcal{A}^{(-)}(x-x'), \\
\{\phi_p, \phi'_n\} &= \{\phi_p^\dagger, \phi_n^\dagger\} = \{\phi_p^\dagger, \phi'_n\} = \{\phi_p, \phi_n^{\dagger\dagger}\} = 0, \\
[a_\mu, a_\nu^*] &= R_{\mu\nu}^{(+)}(x-x') \equiv -i\delta_{\mu\nu} D^{(+)}(x-x'), \\
[a_\mu, a_\nu] &= [a_\mu^*, a_\nu^*] = 0,
\end{aligned} \tag{4.29}$$

where

$$\mathcal{A}^{(\pm)}(x) = \frac{1}{2} \left(\frac{1}{2\pi} \right)^3 \int \frac{e^{\pm i(-Et+k\cdot r)}}{\mp iE_k} d\mathbf{k}, \quad D^{(\pm)}(x) = \frac{1}{2} \left(\frac{1}{2\pi} \right)^3 \int \frac{e^{\pm i(-Et+k\cdot r)}}{\mp ik} d\mathbf{k}. \tag{4.30}$$

The left and right-multiplication become according to the formulae in Section 2 as follows:

$$\begin{aligned}
\vec{\phi}^\dagger &= \phi_p^\dagger + \phi_n^\dagger + \bar{\phi}_n^\dagger = \phi^\dagger + \frac{1}{2}(\bar{\psi} - \bar{\psi}_1), \\
\vec{\phi} &= \phi_p + \phi_n + \phi_n^\dagger = \phi + \frac{1}{2}(\bar{\psi}^\dagger + \bar{\psi}_1), \\
\bar{\phi}^\dagger &= \bar{\phi}_p + \bar{\phi}_n + \bar{\phi}_n^\dagger = \bar{\phi}^\dagger + \frac{1}{2}(\bar{\psi} + \bar{\psi}_1), \\
\bar{\phi} &= \bar{\phi}_p + \bar{\phi}_n^\dagger + \bar{\phi}_n = \bar{\phi} + \frac{1}{2}(\bar{\psi}^\dagger - \bar{\psi}_1),
\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}
\vec{A}_\mu &= a_\mu^* + a_\mu + \bar{a}_\mu^* = A_\mu + \frac{1}{2}(\bar{A}_\mu + \bar{A}_{1\mu}), \\
\bar{A}_\mu &= \bar{a}_\mu^* + \bar{a}_\mu + a_\mu = A_\mu - \frac{1}{2}(\bar{A}_\mu - \bar{A}_{1\mu}),
\end{aligned} \tag{4.32}$$

where

$$\begin{aligned}
\bar{\phi}^\dagger &\equiv \phi_p^\dagger + \phi_n^\dagger, & \bar{\phi} &\equiv \phi_p + \phi_n^\dagger, \\
\phi_i^\dagger &\equiv \phi_p^\dagger - \phi_n^\dagger, & \phi_i &\equiv \phi_p - \phi_n^\dagger, \\
\vec{A}_\mu &\equiv a_\mu^* + a_\mu, & \bar{A}_\mu &\equiv \bar{a}_\mu^* - \bar{a}_\mu, \\
\vec{A}_{1\mu} &\equiv a_\mu^* - a_\mu, & \bar{A}_{1\mu} &\equiv \bar{a}_\mu^* + \bar{a}_\mu.
\end{aligned} \tag{4.33}$$

They obey the commutation relations

$$\begin{aligned}
\{\bar{\phi}_{p\alpha}^\dagger(x), \phi_{p\beta}^\dagger(x')\} &= \{\phi_{p\alpha}^\dagger(x), \bar{\phi}_{p\beta}(x')\} = S_{\alpha\beta}^{(+)}(x-x'), \\
\{\phi_{n\alpha}^\dagger(x), \bar{\phi}_{n\beta}(x')\} &= \{\bar{\phi}_{n\alpha}(x), \phi_{n\beta}^\dagger(x')\} = S_{\alpha\beta}^{(-)}(x-x'), \\
\{\bar{\phi}_\alpha^\dagger(x), \phi_\beta^\dagger(x')\} &= \{\phi_\alpha^\dagger(x), \bar{\phi}_\beta(x')\} = S_{\alpha\beta}(x-x') \equiv i(\gamma_\mu \nabla_\mu - x) \mathcal{A}(x-x'),
\end{aligned}$$

$$\begin{aligned}
\{\bar{\phi}_{1\alpha}^\dagger(x), \phi_{1\beta}^\dagger(x')\} &= \{\bar{\phi}_{1\alpha}^\dagger(x), \bar{\phi}_{1\beta}(x')\} = S_{\alpha\beta}(x-x'), \\
\{\bar{\phi}_\alpha^\dagger(x), \phi_{1\beta}^\dagger(x')\} &= \{\bar{\phi}_\alpha^\dagger(x), \phi_\beta^\dagger(x')\} = S_{\alpha\beta}^{(1)}(x-x') \equiv -(\gamma_\mu \nabla_\mu - x)_{\alpha\beta} \mathcal{A}^{(1)}(x-x'), \\
\{\phi_\alpha^\dagger(x), \bar{\phi}_{1\beta}(x')\} &= \{\phi_\alpha^\dagger(x), \bar{\phi}_\beta(x')\} = S_{\alpha\beta}^{(1)}(x-x'), \\
[\bar{A}_\mu(x), \bar{A}_\nu^\dagger(x')] &= [\bar{A}_{1\mu}(x), \bar{A}_{1\nu}^\dagger(x')] = R_{\mu\nu}(x-x') \equiv -i\delta_{\mu\nu} D(x-x'), \\
[\bar{A}_{1\mu}(x), \bar{A}_\nu^\dagger(x')] &= [\bar{A}_{1\mu}(x), \bar{A}_{1\nu}^\dagger(x')] = R_{\mu\nu}^{(1)}(x-x') \equiv \delta_{\mu\nu} D^{(1)}(x-x'), \\
\mathcal{A}^{(1)} &\equiv -i(\mathcal{A}^{(+)} - \mathcal{A}^{(-)}), \quad D^{(1)} \equiv -i(D^{(+)} - D^{(-)}).
\end{aligned} \tag{4.34}$$

The quantities $\bar{\phi}_i^\dagger, \bar{\phi}_i$ and the function A_i are different from the corresponding ones used by Schwinger by a factor i . (This choice is for the sake of avoiding explicit occurrence of i in the subsequent expressions.)

The effect of operating the above defined quantities on a vector is, for example,

$$\begin{aligned}
\phi_\alpha^\dagger \cdot \phi_\beta^\dagger \phi_\gamma' &= \phi_\alpha^\dagger \phi_\beta^\dagger \phi_\gamma', \quad \phi_\alpha^\dagger \cdot \phi_\beta^\dagger \phi_\gamma' = -\phi_\beta^\dagger \phi_\alpha^\dagger \phi_\gamma', \\
\bar{\phi}_\alpha^\dagger \cdot \phi_\beta^\dagger \phi_\gamma' &= -\phi_\beta^\dagger \phi_\alpha^\dagger \phi_\gamma', \quad \bar{\phi}_\alpha^\dagger \cdot \phi_\beta^\dagger \phi_\gamma' = \phi_\beta^\dagger \phi_\alpha^\dagger \phi_\gamma', \\
\bar{\phi}_\alpha^\dagger \cdot \phi_\beta^\dagger \phi_\gamma' &= S_{\alpha\beta}(x-x') \phi_\gamma', \quad \bar{\phi}_\alpha^\dagger \cdot \phi_\beta^\dagger \phi_\gamma' = -\phi_\beta^\dagger S_{\alpha\beta}(x'-x) \phi_\gamma', \\
\bar{\phi}_\alpha^\dagger \cdot \phi_\beta^\dagger \phi_\gamma' &= -S_{\alpha\beta}(x-x') \phi_\gamma', \quad \bar{\phi}_\alpha^\dagger \cdot \phi_\beta^\dagger \phi_\gamma' = \phi_\beta^\dagger S_{\alpha\beta}(x'-x) \phi_\gamma', \\
\bar{A}_\mu \cdot \bar{A}_\nu' &= A_\mu A_\nu', \quad \bar{A}_\mu \cdot \bar{A}_\nu' = R_{\mu\nu}(x-x').
\end{aligned} \tag{4.35}$$

The results of the operation which are given on the right-hand side should be so understood that they are automatically put in the right order with respect to the creation and annihilation operators though they are apparently not so. It is for the sake of simplicity that we do not explicitly separate the creation and annihilation operators, but write, for instance, merely as $\phi^\dagger \phi$ instead of $\phi_p^\dagger \phi_p + \phi_p^\dagger \phi_n^\dagger + \phi_n \phi_p - \phi_n^\dagger \phi_n$. This convention does not cause trouble in any iterated application of the formulae (4.35). In calculating according to the above rules, we need not be nervous about putting the operators in their right place, but have only to consider every quantity as either commuting or anticommuting. The separation of the fluctuation terms is automatically taken care of in the q -3 expressions.

The equation of motion for the transformation function $U[CC_0]$ can be written as

$$\begin{aligned}
i \frac{\partial}{\partial C} U[CC_0] &= \vec{H} U[CC_0] = -\vec{j}_\mu \vec{A}_\mu U[CC_0] \\
&= -ei (\phi_p^\dagger \gamma_\mu \phi_p + \phi_p^\dagger \gamma_\mu \phi_n^\dagger + \phi_n \gamma_\mu \phi_p - \phi_n^\dagger \gamma_\mu \phi_n) (a_\mu^* + a_\mu) U[CC_0]
\end{aligned}$$

$$\begin{aligned}
&= -\varepsilon i \{ \psi_p^\dagger \gamma_\mu (\psi_p + \bar{\psi}_p^\dagger) + \psi_p^\dagger \gamma_\mu \psi_n^\dagger + (\psi_n + \bar{\psi}_n^\dagger) \gamma_\mu (\psi_p + \bar{\psi}_p^\dagger) \\
&\quad - \psi_n^\dagger \gamma_\mu' (\psi_n + \bar{\psi}_n^\dagger) \} (a_\mu^* + a_\mu + \bar{a}_\mu^*) U[CC_0] \\
&= -\varepsilon i \{ (\psi_p^\dagger \gamma_\mu \psi_p + \psi_p^\dagger \gamma_\mu \psi_n^\dagger + \psi_n \gamma_\mu \psi_p - \psi_n^\dagger \gamma_\mu' \psi_n) \\
&\quad + (\psi_p^\dagger \gamma_\mu \bar{\psi}_p^\dagger + \bar{\psi}_n^\dagger \gamma_\mu \psi_p + \psi_n \gamma_\mu \bar{\psi}_p^\dagger - \bar{\psi}_n^\dagger \gamma_\mu' \bar{\psi}_n^\dagger) \\
&\quad + \bar{\psi}_n^\dagger \gamma_\mu \bar{\psi}_p^\dagger \} (a_\mu^* + a_\mu + \bar{a}_\mu^*) U[CC_0]. \tag{4.36}
\end{aligned}$$

Thus there arise several kinds of terms which are different in the effect they cause on the number of "particles" ψ^\dagger, ψ and a^*, a present. Some terms create two ψ 's (or two ψ^\dagger 's or a ψ and a ψ^\dagger), simultaneously annihilating a a_μ^* , and so on. We denote these terms symbolically as $(+2, -1)$ etc., the first number referring to the electron, the second to the radiation field.

The same result can be obtained more easily by writing simply

$$\begin{aligned}
\overrightarrow{j_\mu A_\mu} &= \varepsilon i \vec{\psi}^\dagger \gamma_\mu \vec{\psi} \vec{A}_\mu = \varepsilon i \left[\left\{ \psi^\dagger + \frac{1}{2} (\bar{\psi} - \bar{\psi}_1) \right\} \gamma_\mu \left\{ \psi + \frac{1}{2} (\bar{\psi}^\dagger + \bar{\psi}_1^\dagger) \right\} \right. \\
&\quad \left. - Tr \{ S^{(0)}(0) \gamma_\mu \} \right] \left\{ \bar{A}_\mu + \frac{1}{2} (\bar{A}_\mu + \bar{A}_{1\mu}) \right\}. \tag{4.37}
\end{aligned}$$

The trace here is necessary to compensate the zero-point fluctuation which should be subtracted on changing $\psi_n \gamma_\mu \psi_n^\dagger$ to $-\psi_n^\dagger \gamma_\mu' \psi_n$. The adjoint equation to (4.36) becomes similarly

$$\begin{aligned}
-i \frac{\partial}{\partial C_0} U[CC_0] &= \overleftarrow{H}(x_0) U[CC_0] = -\overleftarrow{j_\mu A_\mu}(x_0) U[CC_0] \\
&= -\varepsilon i \left[\left\{ \bar{\psi} + \frac{1}{2} (\bar{\psi}^\dagger - \bar{\psi}_1^\dagger) \right\} \gamma_\mu' \left\{ \bar{\psi}^\dagger + \frac{1}{2} (\bar{\psi} + \bar{\psi}_1) \right\} \right. \\
&\quad \left. - Tr \{ S^{(-)}(0) \gamma_\mu \} \right] \left\{ \bar{A}_\mu - \frac{1}{2} (\bar{A}_\mu - \bar{A}_{1\mu}) \right\} (x_0) U[CC_0]. \tag{4.38}
\end{aligned}$$

These equations serve to determine the change of the $U[CC_0]$ accompanying that of the surfaces C and C_0 . But they appear to be too complicated, and moreover, they have no symmetry between the creation (+) and annihilation (−) operators, which would be required in order that the operators be Hermitian. To remedy these defects, we note that, in case of the left-operators, the quantities with asterisk (or dagger) do show the above mentioned symmetry with respect to + and −, while the quantities without asterisk do not; in case of the right-operators the reverse is true. In the first case, then, we eliminate all the quantities without asterisk and retain only the quantities with asterisk. This can be done by the transformation

$$\vec{\psi}^\dagger = S^{-1} \vec{\phi}^\dagger S, \quad \vec{\psi} = S^{-1} \vec{\phi} S, \quad \vec{A}_\mu = S^{-1} V_\mu^* S, \tag{4.39}$$

where
$$S = \Pi \exp(-\bar{\chi}_\nu^\dagger \phi_n^+ - \bar{\chi}_a^\dagger \phi_p^+) \cdot \exp(-\bar{a}_\mu^+ \bar{b}_\mu^*), \quad (4.40)$$

and χ, b shall satisfy the commutation relations

$$\{\bar{\chi}_{p\alpha}^\dagger, \phi_{p\beta}^+\} = \{\bar{\chi}_{n\alpha}^\dagger, \phi_{n\beta}^+\} = \delta_{\alpha\beta} \delta^4(x-x'), \quad [\bar{b}_\mu^*, \bar{a}_\nu^+] = \delta_{\mu\nu} \delta^4(x-x'), \quad (4.41)$$

otherwise commuting or anticommuting; finally $\phi^\dagger, \bar{\phi}^\dagger$ and V^* are defined by

$$\phi^\dagger = \phi_p^\dagger + \phi_n^\dagger, \quad \bar{\phi}^\dagger = \bar{\phi}_p^\dagger + \bar{\phi}_n^\dagger, \quad V_\mu^* = \bar{a}_\mu^* + \bar{b}_\mu^*. \quad (4.42)$$

These transformed quantities are indeed symmetric with respect to $+$ and $-$, and obey the same commutation rules as those for the original field. As a consequence of this transformation of the operator, the operand U also undergoes the transformation

$$U[CC_0] = S U_1[CC_0] S^{-1}. \quad (4.43)$$

U_1 contains no more the quantities without asterisk. The equation of motion becomes then

$$i \frac{\partial}{\partial C} U_1[CC_0] = H_1(x) U_1[CC_0],$$

in which
$$H_1 = -\epsilon i \phi^\dagger \gamma_\mu \bar{\phi}^\dagger V_\mu^*. \quad (4.44)$$

Similar procedure applies also to the adjoint equation (4.38). This time we eliminate the quantities with asterisk by means of

$$\begin{aligned} \phi^\dagger &= \tilde{S}^{-1} \tilde{\phi} \tilde{S}, & \bar{\phi}^\dagger &= \tilde{S}^{-1} \tilde{\bar{\phi}} \tilde{S}, \\ \bar{A} &= \tilde{S}^{-1} \bar{V}_\mu \tilde{S}, \end{aligned} \quad (4.45)$$

with
$$\tilde{S} = \Pi \exp(\tilde{\chi}_n \phi_n^\dagger + \tilde{\chi}_p \phi_p^\dagger) \cdot \exp(\tilde{a}_\mu^+ \tilde{b}_\mu^+),$$

$$\{\tilde{\chi}_{n\alpha}, \phi_{n\beta}^\dagger\} = \{\tilde{\chi}_{p\alpha}, \phi_{p\beta}^\dagger\} = \delta_{\alpha\beta} \delta^4(x-x'), \quad [\tilde{b}_\mu^+, \tilde{a}_\nu^+] = -\delta_{\mu\nu} \delta^4(x-x'). \quad (4.46)$$

The equation of motion becomes

$$-i \frac{\partial}{\partial C_0} \tilde{U}_1[CC_0] = \tilde{H}_1(x_0) \tilde{U}_1[CC_0],$$

where
$$\tilde{U}_1 = \tilde{S} U \tilde{S}^{-1}, \quad \tilde{H} = -\epsilon i \tilde{\phi} \gamma_\mu' \tilde{\bar{\phi}} V_\mu. \quad (4.47)$$

The operators H_1 and \tilde{H}_1 are in fact Hermitian, so that U_1 remains unitary. To show this we must explain the apparent asymmetry in the defining equations (2.9) and (2.15). There we considered as if every field quantity at different space-time point represented a different and independent state. This is not true, as was mentioned in Section 3, and a field quantity at an arbitrary point can be expressed by those on an arbitrary but fixed surface C_0 . Indeed,

$$\psi(x) = \int_{C_0} S(x-x') \gamma_\mu \psi' df'_\mu, \quad \psi(x) = \int_{C_0} \psi'^{\dagger} \gamma_\mu S(x'-x) df'_\mu,$$

$$A_\mu(x) = - \int_{C_0} \left\{ \frac{\partial}{\partial x'_\nu} D(x-x') A_\mu(x') - D(x-x') \frac{\partial}{\partial x'_\nu} A_\mu(x') \right\} df'_\nu. \quad (3.48)$$

(See Schwinger, *op. cit.* in the preceding part.) Therefore,

$$\psi^\dagger(x) = \int S(x-x') \gamma_\mu \psi'^{\dagger} df'_\mu, \quad \psi^\dagger(x) = \int \psi'^{\dagger} \gamma_\mu S(x'-x) df'_\mu,$$

$$A_\mu^\dagger(x) = - \int \left\{ \frac{\partial}{\partial x'_\nu} D(x-x') A_\mu^\dagger(x') - D(x-x') \frac{\partial}{\partial x'_\nu} A_\mu^\dagger(x') \right\} df'_\nu. \quad (4.49)$$

In this form, the symmetry seems to have been recovered at least partially. However, the decomposition of a field into creation and annihilation operators is only possible by considering its behavior over the entire space-time, for it is closely connected with the Fourier decomposition with respect to frequency; and the idea of creation and annihilation is reasonable only when we mean by it a change of state that exists between the past ($C < C_0$) and the future ($C > C_0$). Thus we decompose the fields into Fourier components:

$$\psi_\alpha(x) = \sum_k \sum_{r=1}^2 \{ \psi_p^r(k) u_\alpha^r(k) e^{+ik_\mu x^\mu} + \psi_n^{*r}(k) \bar{v}_\alpha^r(k) e^{-ik_\mu x^\mu} \}, \quad k_\mu^2 = -x^2,$$

$$\psi_\alpha^*(x) = \sum_k \sum_{r=1}^2 \{ \psi_p^{*r}(k) \bar{u}_\alpha^r(k) e^{-ik_\mu x^\mu} + \psi_n^r(k) v_\alpha^r(k) e^{+ik_\mu x^\mu} \},$$

$$A_\mu(x) = \frac{1}{\sqrt{2}} \sum_{\tilde{k}} \frac{1}{\sqrt{\tilde{k}}} \{ a_\mu(\tilde{k}) e^{i\tilde{k}_\mu x^\mu} + a_\mu^*(\tilde{k}) e^{-i\tilde{k}_\mu x^\mu} \}, \quad \tilde{k}^2 = 0, \quad (4.50)$$

and introduce the third quantization for the amplitudes u, u^*, v, v^* and a, a^* by means of

$$\vec{\psi}_p^*(k) = \psi_p^*(k), \quad \vec{\psi}_p(k) = \psi_p^{\dagger}(k) + \bar{\psi}_p^*(k)$$

$$\vec{a}_\mu^*(\tilde{k}) = a_\mu^*(\tilde{k}), \quad \vec{a}_\mu(\tilde{k}) = a_\mu^{\dagger}(\tilde{k}) + \bar{a}_\mu^*(\tilde{k}), \quad \text{etc.}$$

$$\{ \bar{\psi}_p^{*r}(k), \psi_p^{*s}(l) \} = \{ \bar{\psi}_p^r(k), \psi_p^s(l) \} = \delta_{rs} \delta(\mathbf{k} - \mathbf{l}),$$

$$\{ \bar{\psi}_n^{*r}(k), \bar{\psi}_n^{*s}(l) \} = \{ \bar{\psi}_n^r(k), \bar{\psi}_n^s(l) \} = \delta_{rs} \delta(\mathbf{k} - \mathbf{l}), \quad (4.51)$$

$$[\bar{a}_\mu^*(\tilde{k}), a_\nu^{\dagger}(\tilde{l})] = [\bar{a}_\mu(\tilde{k}), a_\nu^{\dagger}(\tilde{l})] = \delta_{\mu\nu} \delta(\tilde{\mathbf{k}} - \tilde{\mathbf{l}}).$$

Transforming these q-3 operators defined in the momectum space back to the space-time representation, we obtain the corresponding space-time field, and at the same time it establishes the fact that the $+$ operators are really adjoint to the $-$ operators. In order to obtain the original transformation function $U[CC_0]$ from $U_1[CC_0]$, we have only to replace in the latter all the quantities like $\psi_p^{\dagger}, \psi_n^{\dagger}$ and a_μ^* by $\psi_p^{\dagger} + \psi_n, \psi_n^{\dagger} + \psi_p$ and $a_\mu^* + a_\mu$ respectively. The conservation of the norm

of $U[CC_0]$ holds then for that part of U which contains only the quantities with asterisk, or alternatively, without asterisk (using the adjoint equation). In other words, if originally $\|U[C_0C_0]\|=1$, and for C

$$U[CC_0] = \int f(x, x') \psi^\dagger(x) \psi(x') dx dx' + \dots + \int g(x, x') A_\mu(x) A_\mu(x') dx dx' + \dots, \quad (4.52)$$

where the integration extends over the three-dimensional surface C , then necessarily

$$1 = \int |f(x, x')|^2 dx dx' + \dots + \frac{1}{2} \int |g(x, x') + g(x', x)|^2 dx dx' + \dots \quad (4.53)$$

This means that the transformation function conserves its length if we define the norm of the elementary vectors simply by

$$\|\psi^\dagger\| = \|\psi\| = \|A\| = 1, \text{ etc.} \quad (4.54)$$

Actually the norm of $U[CC_0]$ is infinite owing to the continuous degrees of freedom and also to the delta-function type coupling of the elementary vectors in the Hamiltonian (the localizable system!).

Our last but main problem is that of the reaction of the interaction on various quantities including the self-energy. To handle this problem it is more suitable to use the Heisenberg equation for Q :

$$-i \frac{\partial}{\partial C} Q[C] = [H] Q[C]. \quad (4.55)$$

The operator $[H]$ can be written as

$$[H] = [-j_\mu A_\mu] = -\frac{1}{2} ([j_\mu] \{A_\mu\} + \{j_\mu\} [A_\mu]). \quad (4.56)$$

Using the expressions (4.31) to (4.33) and the relation (2.7) between the left- and right-operators, the following formulae hold:

$$\begin{aligned} \{\psi_\alpha^\dagger \psi_\beta\} &= 2\psi_\alpha^\dagger \psi_\beta + \psi_\alpha^\dagger \bar{\psi}_{1\beta}^\dagger + \psi_\beta^\dagger \bar{\psi}_{1\alpha} + \frac{1}{2} (\bar{\psi}_{1\alpha} \bar{\psi}_{1\beta}^\dagger - \bar{\psi}_\alpha \bar{\psi}_\beta^\dagger), \\ [\psi_\alpha^\dagger \psi_\beta] &= \psi_\alpha^\dagger \bar{\psi}_\beta^\dagger - \bar{\psi}_\beta \psi_\alpha^\dagger + \frac{1}{2} (\bar{\psi}_\alpha \bar{\psi}_{1\beta}^\dagger - \bar{\psi}_{1\alpha} \bar{\psi}_\beta^\dagger), \\ \{A_\mu\} &= 2\bar{A}_\mu^\dagger + \bar{A}_{1\mu}, \\ [A_\mu] &= \bar{A}_\mu. \end{aligned} \quad (4.57)$$

Then (4.56) becomes

$$\begin{aligned} [H] &= -\frac{\varepsilon i}{2} \left[\{2\psi_\mu^\dagger \gamma_\mu \psi^\dagger + \psi_\mu^\dagger \gamma_\mu \bar{\psi}_1^\dagger + \psi^\dagger \gamma_\mu \bar{\psi}_1 + \frac{1}{2} (\bar{\psi}_1 \gamma_\mu \bar{\psi}_1^\dagger - \bar{\psi} \gamma_\mu \bar{\psi}^\dagger)\} \bar{A}_\mu \right. \\ &\quad \left. + \{\psi_\mu^\dagger \gamma_\mu \bar{\psi}^\dagger - \bar{\psi} \gamma_\mu \psi^\dagger + \frac{1}{2} (\bar{\psi} \gamma_\mu \bar{\psi}_1^\dagger - \bar{\psi}_1 \gamma_\mu \bar{\psi}^\dagger)\} \{2\bar{A}_\mu^\dagger + \bar{A}_{1\mu}\} \right]. \end{aligned} \quad (4.58)$$

The trace term appearing in the Eqs. (4.37) and (4.38) has dropped in the above result. To obtain the Heisenberg representation of a quantity Q , or rather the radiative correction for it, we substitute the above expressions in the formula (4.24):

$$Q[C] = (1 + i \int_{C_0}^C [H'] dx' - \int_{C_0}^C [H'] dx' \int_{C_0}^{C'} [H''] dx'' - \dots) Q[C_0], \quad (4.59)$$

and take the limit $C \rightarrow +\infty$.

The self-energy in the second approximation is contained in the expression

$$\frac{i}{2} \left[\int_{-\infty}^C H' dx', H(x) \right] = \frac{i}{2} \int_{-\infty}^C [H'] dx' \cdot H(x). \quad (4.60)$$

We first divide the expression (4.57) into several groups:

$$\begin{aligned} [H] = & -\varepsilon i \psi^\dagger \gamma_\mu \bar{\psi} \bar{A}_\mu & (+2, -1) \\ & -\varepsilon i (\psi^\dagger \gamma_\mu \bar{\psi} - \bar{\psi}^\dagger \gamma_\mu \psi) \bar{A}_\mu & (0, +1) \\ & -\frac{\varepsilon i}{2} \{ (\psi^\dagger \gamma_\mu \bar{\psi}_1^\dagger + \bar{\psi}^\dagger \gamma_\mu \psi_1) \bar{A}_\mu + (\psi^\dagger \gamma_\mu \bar{\psi}^\dagger - \bar{\psi}^\dagger \gamma_\mu \psi) \bar{A}_{1\mu} \} & (0, -1) \\ & -\frac{\varepsilon i}{2} (\bar{\psi} \gamma_\mu \psi_1^\dagger - \bar{\psi}_1 \gamma_\mu \psi^\dagger) \bar{A}_\mu & (-2, +1) \\ & -\frac{\varepsilon i}{4} \{ (\bar{\psi}_1 \gamma_\mu \bar{\psi}_1^\dagger - \bar{\psi} \gamma_\mu \bar{\psi}^\dagger) \bar{A}_\mu + (\bar{\psi} \gamma_\mu \bar{\psi}_1^\dagger - \bar{\psi}_1 \gamma_\mu \bar{\psi}^\dagger) \bar{A}_{1\mu} \}. & (-2, -1) \end{aligned} \quad (4.61)$$

The different rôles played by these terms are symbolized on the right side. Operated on $H = -\varepsilon i \psi^\dagger \gamma_\mu \bar{\psi}$, they give rise to the following terms:

$$\begin{aligned} (+2, -1) \cdot H(x) &= -\frac{\varepsilon^2 i}{2} \int_{C_0}^C \psi^\dagger \gamma_\mu \bar{\psi}' R(x' - x) \psi^\dagger \gamma_\mu \bar{\psi} dx', \\ (0, +1) \cdot H(x) &= -\frac{\varepsilon^2 i}{2} \int_{C_0}^C (\psi^\dagger \gamma_\mu S(x' - x) \gamma_\nu \bar{\psi} - \bar{\psi}^\dagger \gamma_\nu S(x - x') \gamma_\mu \bar{\psi}') A_\mu' A_\nu dx', \\ (0, -1) \cdot H(x) &= -\frac{\varepsilon^2 i}{2} \int_{C_0}^C \{ \psi^\dagger \gamma_\mu S^{(n)}(x' - x) \gamma_\mu \bar{\psi} + \psi^\dagger \gamma_\mu S^{(n)}(x - x') \gamma_\mu \bar{\psi}' \} R(x' - x) \\ &\quad + \{ \bar{\psi}^\dagger \gamma_\mu S(x' - x) \gamma_\mu \bar{\psi} + \bar{\psi}^\dagger \gamma_\mu S(x - x') \gamma_\mu \bar{\psi}' \} R^{(n)}(x' - x) dx', \quad (4.62) \\ (-2, +1) \cdot H(x) &= -\frac{\varepsilon^2 i}{4} \int_{C_0}^C Tr \{ \gamma_\mu S^{(n)}(x' - x) \gamma_\nu S(x - x') \\ &\quad - \gamma_\mu S(x' - x) \gamma_\nu S^{(n)}(x - x') \} A_\mu' A_\nu dx', \\ (-2, -1) \cdot H(x) &= -\frac{\varepsilon^2 i}{8} \int_{C_0}^C [Tr \{ \gamma_\mu S^{(n)}(x' - x) \gamma_\mu S^{(n)}(x - x') \\ &\quad - \gamma_\mu S(x' - x) \gamma_\mu S(x - x') \} R(x' - x) \end{aligned}$$

$$+ Tr \{ \gamma_\mu S^{(1)}(x' - x) \gamma_\mu S(x - x') - \gamma_\mu S(x' - x) \gamma_\mu S^{(1)}(x - x') \} R^{(1)}(x' - x)] dx'.$$

Thus, they correspond respectively to the Møller interaction between the electrons, the Compton interaction between the electron and the radiation, the self-energy of the electron, the self-energy of the radiation, and a constant term independent of the field quantities. The last term may safely be subtracted away since it does not cause any observable effect. The interpretation of the self-energy terms and their relativistic evaluation will not be discussed here as we are occupied in the present paper only with mathematical formalism.

The equation (4.59) can be applied to any quantity. For example, the behavior of the elementary vector ψ in the course of time is given by

$$\begin{aligned} \underline{\psi}(x) &= (1 + i \int_0^\infty [H'] dx' - \int_0^\infty [H''] dx'' \int_0^{c''} [H'] dx' - \dots) \psi(x) \\ &= \psi(x) + \varepsilon \int S(x - x') \gamma_\mu \psi' A_\mu' dx' \\ &\quad - \varepsilon^2 \int \psi''' \gamma_\mu \psi'' R(x'' - x') S(x - x') \gamma_\mu \psi' dx' dx'' \\ &\quad + \varepsilon^2 \int S(x - x') \gamma_\mu S(x' - x'') \gamma_\nu \psi'' A_\nu'' A_\mu' dx' dx'' \\ &\quad - \frac{\varepsilon^2}{2} \int \{ S(x - x') \gamma_\mu S^{(1)}(x' - x'') \gamma_\mu R(x'' - x') \\ &\quad - S(x - x') \gamma_\mu S(x' - x'') \gamma_\mu R^{(1)}(x'' - x') \} \psi'' dx' dx''. \end{aligned} \quad (4.63)$$

The last term is a correction due to the self-energy effect, being a result of the operation $\int [0, -1] \int [0, +1] \psi$, and hence may be renormalized into the original ψ .

By an exactly analogous procedure, the corrected behavior of the radiation field turns out as follows:

$$\begin{aligned} \underline{A}_\mu(x) &= A_\mu(x) - \varepsilon \int \psi''' \gamma_\mu \psi' R(x' - x) dx' \\ &\quad + \varepsilon^2 \int \{ \psi''' \gamma_\nu S(x'' - x') \gamma_\mu \psi' - \psi''' \gamma_\mu S(x' - x'') \gamma_\nu \psi'' \} A_\nu'' R(x' - x) dx' dx'' \\ &\quad + \frac{\varepsilon^2}{2} \int Tr \{ \gamma_\nu S^{(1)}(x'' - x') \gamma_\mu S(x' - x) - \gamma_\nu S(x'' - x') \gamma_\mu S(x' - x'') \} \\ &\quad \times A_\mu'' R(x' - x) dx' dx''. \end{aligned} \quad (4.64)$$

The second and third term are the field induced by the (real) current distribution, while the last term corresponds to the self-energy of the photon as caused by the virtual fluctuation of the electron-positron field.

The correction to a quantity bilinear in ψ and ψ^\dagger like the current vector j_μ is easily obtained in a similar way. To eliminate the self-energy effects we have to pick up in the second order correction operator only those terms which simul-

taneously affect both ψ and ψ^\dagger . We shall not do it here, but adopt another interesting, if somewhat more laborious, method which leads to the same result. That is, we construct directly the current vector j_μ using the corrected field ψ and its adjoint ψ^\dagger :

$$\begin{aligned}\psi^\dagger(x) = & \psi^\dagger(x) - \varepsilon \int \psi^\dagger \gamma_\mu S(x' - x) A_\mu' dx' \\ & + \varepsilon^2 \int \psi^{\dagger''} \gamma_\mu S(x' - x) \psi^{\dagger'''} \gamma_\mu \psi'' R(x'' - x') dx' dx'' \\ & + \varepsilon^2 \int \psi'' \gamma_\nu S(x'' - x') \gamma_\mu S(x' - x) A_\nu'' A_\mu' dx' dx'' \\ & + \frac{\varepsilon^2}{2} \int \psi^{\dagger''} \{ \gamma_\mu S^{(1)}(x'' - x') \gamma_\mu S(x' - x) R(x'' - x') \\ & \quad + \gamma_\mu S(x'' - x') \gamma_\mu S(x' - x) R^{(1)}(x'' - x') \} dx' dx'', \\ j_\mu = & i \varepsilon \psi^\dagger \gamma_\mu \psi.\end{aligned}\tag{4.65}$$

The ψ and ψ^\dagger are of course well-ordered quantities, but the product of them is not so. Accordingly we take the expectation value with respect to the electron and radiation field in the state $(1, 0)$. Remembering the formulae (4.31), (4.32) and (4.34), we see that

$$\begin{aligned}\langle A_\mu' \times A_\nu'' \rangle_0 = & \frac{1}{2} \{ R_{\mu\nu}(x' - x'') + R_{\mu\nu}^{(1)}(x' - x'') \}, \\ \langle \psi^\dagger \times \psi^{\dagger''} \gamma_\mu \psi'' \cdot \psi' \rangle_1 = & \frac{1}{2} \{ \psi^{\dagger''} \gamma_\mu (-S(x'' - x) + S^{(1)}(x'' - x)) \cdot \psi' \\ & + \psi^{\dagger''} \gamma_\mu \psi'' \cdot (S(x' - x) - S^{(1)}(x' - x)), \\ \langle \psi^{\dagger''} \cdot \psi^{\dagger'''} \gamma_\mu \psi'' \times \psi \rangle_1 = & \frac{1}{2} \{ -\psi' \cdot (S(x - x'') - S^{(1)}(x - x'')) \gamma_\mu \psi'' \\ & + (S(x - x') - S^{(1)}(x - x')) \cdot \psi^{\dagger''} \gamma_\mu \psi''.\end{aligned}\tag{4.66}$$

Inserting these results, the correction in the order ε^2 turns out to be

$$\begin{aligned}-\frac{\varepsilon^2}{2} \int_c^\infty \int_c^\infty & \psi^{\dagger''} \gamma_\nu S(x' - x) \gamma_\mu S(x - x'') \gamma_\nu \psi'' \{ R(x' - x'') + R^{(1)}(x' - x'') \} dx' dx'' \\ -\frac{\varepsilon^2}{2} \int_c^\infty \int_c^\infty & [-\psi^{\dagger''} \gamma_\nu \{ S(x'' - x) - S^{(1)}(x'' - x) \} \gamma_\mu S(x - x') \gamma_\nu \psi' R(x'' - x') \\ & + \psi^{\dagger''} \gamma_\nu S(x' - x) \gamma_\mu \{ S(x - x'') - S^{(1)}(x - x'') \} \gamma_\nu \psi'' R(x'' - x')] dx' dx'' \\ -\frac{\varepsilon^2}{2} \int_c^\infty \int_c^\infty & \psi^{\dagger''} \gamma_\nu \psi'' R(x'' - x') \text{Tr} [S(x - x') \gamma_\nu \{ S(x' - x) - S^{(1)}(x' - x) \} \gamma_\mu \\ & - \gamma_\mu \{ S(x - x') - S^{(1)}(x - x') \} \gamma_\nu S(x' - x)] dx' dx''\end{aligned}$$

$$\begin{aligned}
 & -\frac{\varepsilon^2}{2} \int_c^\infty \int_c^{c''} \{ \phi^{i''} \gamma_\nu S(x'-x) \gamma_\mu \{ S(x-x'') R^{(1)}(x'-x'') \\
 & \quad + S^{(1)}(x-x'') R(x'-x'') \} \gamma_\nu \phi'' + \phi^{i''} \gamma_\nu \{ S^{(1)}(x''-x) R(x''-x') \\
 & \quad + S(x''-x) R^{(1)}(x''-x') \} \gamma_\mu S(x'-x) \phi' \} dx' dx'' \\
 & -\frac{\varepsilon^2}{2} \int_c^\infty \int_c^{c''} \phi^{i''} \gamma_\nu \phi'' \text{Tr} [S(x'-x) \gamma_\nu S^{(1)}(x'-x) \gamma_\mu + \gamma_\mu S^{(1)}(x'-x) \gamma_\nu S(x'-x)] \\
 & \quad R(x''-x') dx' dx''. \tag{4.67}
 \end{aligned}$$

This agrees with the result given by Schwinger.¹⁾ The subtraction of the self-energy has been carried out already in the equations (4.63) and (4.64). The present method corresponds in the non-relativistic one-body theory to the transformation of Pauli and Fierz in which the matrix element for an external potential is changed from $V(\mathbf{r})$ to $V\left(\mathbf{r}-\frac{\varepsilon}{x}\mathbf{Z}\right)$ where \mathbf{Z} is the Hertz vector for the radiation field.

Besides further applications to various problems on the radiative corrections, in particular to higher order corrections, there remains yet much to be done. Here are some examples: what is the physical meaning of the $q-3$ operator and the state in the $q-3$ space; what is the classical analog ($\hbar \rightarrow 0$) of the present theory; how is its application to many-body problems; and finally, what can a statistical ensemble of those "particles" mean? All such questions lie somewhat beyond the scope of the present paper, and their answer, if any, will have to await future opportunities.

5. Some remarks on other fields of applications

It may be perhaps worth while to point out other possible domains in which the present method appears to be promising. The eigenvalue problem in Section 3 for an operator X rather than for the state Ψ enabled one to obtain the transformation or displacement operator which transforms one eigenvector into another. When this operator happens to be of particularly simple structure, the solution is easily obtained. For example, let H be the harmonic oscillator Hamiltonian:

$$H = \frac{1}{2} (p^2 + q^2), \quad \text{with} \quad [p, q] = -i. \tag{5.1}$$

If we consider the well-ordered quantity of the form $\sum f(p)g(q)$, the multiplication formulae read:

$$\begin{aligned}
 \vec{p} &= \overset{+}{p}, & \overset{\leftarrow}{p} &= \overset{+}{p} + \overset{-}{q}, \\
 \vec{q} &= \overset{+}{q} + \overset{-}{p}, & \overset{\leftarrow}{q} &= \overset{+}{q},
 \end{aligned} \tag{5.2}$$

$$\text{with} \quad [\overset{-}{p}, \overset{+}{p}] = [\overset{-}{q}, \overset{+}{q}] = i.$$

The eigenvalue equation then becomes

$$\begin{aligned}
 [H]X &= \lambda X, \\
 [H] &= \vec{H} - \overleftarrow{H} = -\frac{1}{2} \{ (\overset{+}{p}^2 + (\overset{+}{q} + \overleftarrow{p})^2) - (\overleftarrow{p} + \overset{+}{q})^2 + \overset{+}{q}^2 \} \\
 &= -\frac{1}{2} (\overleftarrow{p}^2 + 2\overset{+}{p}\overleftarrow{q}) - \frac{1}{2} (\overleftarrow{q}^2 + 2\overset{+}{p}\overleftarrow{q}).
 \end{aligned} \tag{5.3}$$

A particular solution of this equation is found at once by assuming X to be a linear function of $\overset{+}{p}$ and \overleftarrow{q} :

$$X_{\pm} = \overset{+}{p} \pm i\overleftarrow{q}, \quad \text{with } \lambda = \pm 1. \tag{5.4}$$

Any power of X_{\pm} is also a solution:

$$X_{\pm}^n = (\overset{+}{p} \pm i\overleftarrow{q})^n, \quad \text{with } \lambda = \pm n. \tag{5.5}$$

As the relation (5.1) is satisfied by putting $\overset{+}{p} = -id/d\overleftarrow{q}$, X_{\pm}^n is nothing but the generating operator for the Hermite polynomial:

$$i^n (\overset{+}{p} + i\overleftarrow{q})^n = \left(\frac{d}{d\overleftarrow{q}} - \overleftarrow{q} \right)^n = e^{\overleftarrow{q}^2/2} \frac{d^n}{d\overleftarrow{q}^n} e^{-\overleftarrow{q}^2/2}. \tag{5.6}$$

In connection with this, we shall consider the so-called generalized wave equation which was proposed independently by Snyder³⁾ and Yukawa⁴⁾. Laying aside the physical considerations which led them to setting up the equation, it is written typically as

$$\begin{aligned}
 [\overset{+}{p}_{\mu}, [\overset{+}{p}^{\mu}, A]] &= 0, \\
 [\overleftarrow{q}_{\mu}, [\overleftarrow{q}^{\mu}, A]] &= 0,
 \end{aligned} \tag{5.7}$$

for the electromagnetic potential A . $\overset{+}{p}$ and \overleftarrow{q} are the momenta and the coordinates regarded as q -numbers related by $[\overset{+}{p}_{\mu}, \overleftarrow{q}_{\mu}] = -i\delta_{\mu\nu}$, and the bracket operation replaces the ordinary definition:

$$[\overset{+}{p}_{\mu}] \sim \frac{1}{i} \frac{\partial}{\partial \overleftarrow{q}_{\mu}}, \quad \overleftarrow{q}_{\mu} \sim \overleftarrow{q}_{\mu} \tag{5.8}$$

in the ordinary wave equation. Substitution of the relation (5.2) in (5.8) shows that

$$[\overset{+}{p}_{\mu}] = -\overleftarrow{q}_{\mu}, \quad [\overleftarrow{q}_{\mu}] = \overset{+}{p}_{\mu}. \tag{5.9}$$

Hence

$$\begin{aligned}
 [\overset{+}{p}_{\mu}, [\overset{+}{p}^{\mu}, A]] &= [\overset{+}{p}_{\mu}]^2 \cdot A = \overleftarrow{q}_{\mu}^2 \cdot A = 0, \\
 [\overleftarrow{q}_{\mu}, [\overleftarrow{q}^{\mu}, A]] &= [\overleftarrow{q}_{\mu}]^2 \cdot A = \overset{+}{p}_{\mu}^2 \cdot A = 0.
 \end{aligned} \tag{5.10}$$

But the relation $[\overset{+}{p}, \overleftarrow{p}] = [\overleftarrow{q}, \overset{+}{q}] = i$ is satisfied again by putting

$$\bar{p}_\mu = i \frac{\partial}{\partial p_\mu}, \quad \bar{p}_\mu = p_\mu, \quad \bar{q}_\mu = q_\mu, \quad \bar{q}_\mu = i \frac{\partial}{\partial q_\mu}, \quad (5.11)$$

so that the wave equations (5.10) reduce simply to

$$\frac{\partial^2}{\partial p_\mu^2} A = 0, \quad \frac{\partial^2}{\partial q_\mu^2} A = 0, \quad (5.12)$$

that is, the ordinary equations. The fundamental solution of them is given by

$$\begin{aligned} A(\vec{N}, \vec{M}) &= f_{\vec{N}}^*(p) \cdot g_{\vec{M}}^*(q), \\ f_{\vec{N}}^*(p) &= \exp(iN_\mu p_\mu), \quad g_{\vec{M}}^*(q) = \exp(iM_\mu q_\mu), \\ N_\mu^2 &= M_\mu^2 = 0, \end{aligned} \quad (5.13)$$

and the general solution by

$$A = \sum_{NM} a_{NM} A(\vec{N}, \vec{M}). \quad (5.14)$$

Another set of solutions are obtained by interchanging the order of the factor f and g , though they are not independent of the first.

The meaning of the solution (5.13) is as follows. $A(\vec{N}, \vec{M})$ transforms an arbitrary wave function at an arbitrary point \vec{x} with energy-momentum \vec{k} into one at $\vec{x} + \vec{N}$ with energy-momentum $\vec{k} + \vec{M}$. In fact, writing $\vec{q} = \vec{x}$, $\vec{p} = \frac{1}{i} \text{grad}$ as usual and operating $A(\vec{N}, \vec{M})$ on the function $e^{i(k, x)}$

$$\begin{aligned} A(\vec{N}, \vec{M}) \cdot e^{i(k, x)} &= e^{i(N, p)} e^{i(M, x)} e^{i(k, x)} \\ &= e^{i(N, \text{grad})} e^{i(M+k, x)} = e^{i(M+k, x+N)}. \end{aligned} \quad (5.15)$$

This is just the relation from which Yukawa has deduced the equations (5.7). These considerations are but of preliminary character, and detailed investigations in this direction will be done elsewhere.

In concluding the author wishes to express heartfelt thanks to Prof. Tomonaga for his interest and encouragement toward the present work. Useful advices and remarks received from the author's colleagues should also be appreciated to the full.

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A Divergence Free Field Theory.

Katurō SAWADA

Department of Physics, Kyoto University

(Received June 1, 1949)

A property of commutation relations between field quantities, especially that of Δ -functions appearing on the right hand side of the commutator of field variables suggests us to generalize a relativistic cut-off procedure developed recently by Feynman¹⁾ and surveyed seriously by Pauli and Villars²⁾. The formal simplifications can be attained by this generalization than the introduction of the regulator in the course of the evaluation of integrals. The electromagnetic and scalar mesonic self-energies of Dirac particle are calculated according to our prescription and they turned out to be finite.

§ 1. General Remarks on the Self-Energy

Recent developement of quantum electrodynamics enables us to classify the reaction effects consistently, but the reaction effects themselves, for instance self-energies, radiative corrections to the cross-sections and vacuum polarization effects, remain infinite. It is the well known fact that these difficulties are due to the peculiar singularities appearing in Pauli's Δ -functions, the introduction of them is necessary to demonstrate the relativistic invariance of the formalism.

We, therefore, first observe a sort of singularities appearing in the course of the calculation concerning the behavior of Δ -function, and which brings the current theory inconsistent in some respects.

We discuss, here, for simplicity, only scalar meson field interacting with a Dirac particle.

Usually, the commutation relations between field variables are given by the following, which is derived directly from the fact that the system is written in canonical form;

$$[C(X), C(X')]_- = i\Delta_{\mu^2}(X - X') \quad (1)$$

where $C(X)$ is the meson variable at the world point X , Δ_{μ^2} means Δ -function with mass factor μ^2 , and μ is the mass of the meson. The right hand side of this commutation relation has the property, which was used by Stückelberg and Rivier³⁾, that it becomes to vanish if the mass of the field becomes to infinite, since Δ -function vanishes in this limit. Then, in this limit, all the field variables commute each other, and hence all quantum fluctuations and its combined effects

due to this field should vanish, such as self-energy of Dirac particle interacting with this field. But, unfortunately, this cannot be checked by usual calculations. Usual calculations show that the self-energy of a Dirac particle interacting with this field becomes infinite in this limit depending logarithmically on the mass factor appearing on the right hand side of the commutator (1). To show this fact, it is convenient to distinguish the meson mass μ and the mass factor μ appearing in Δ -function of the commutator, so we denote the latter by λ . Then taking the interaction Hamilton density as follows; (only f-interaction i.e. weaker coupling) :

$$H=f\psi^\dagger\psi C \quad (2)$$

one obtains as the mass correction of Dirac particle in order f^2 following the calculation of Schwinger⁹ :

$$\begin{aligned} \frac{\delta m}{m} &= -\frac{f^2}{16\pi} \int_0^\infty \int_{-1}^1 \frac{1}{z} \cos\left(m^2 \frac{(1-y)^2}{4} z + \lambda^2 \frac{1+y}{2} z\right) \cdot (y+3) \cdot dy dz \\ &= -\frac{f^2}{16\pi} \int_{-1}^1 \log \frac{1}{z_0 \cdot \gamma \cdot \left(m^2 \frac{(1-y)^2}{4} + \lambda^2 \frac{1+y}{2}\right)} \Big|_{z_0 \rightarrow 0} \cdot (y+3) \cdot dy \end{aligned} \quad (3)$$

where m is the mass of Dirac particle. This expression diverges logarithmically as the mass factor λ becomes to infinite, apart from the numerical divergence which is independent of λ .

Now, to remove this inconsistency, if we subtract from the self-energy with meson mass μ the expression (3) with mass factor λ infinite, this just corresponds to Feynman's relativistic cut-off procedure.

He started with the following commutation relation ;

$$[C(X), C(X')]_- = i \{ \Delta_{\mu^2}(X-X') - \Delta_{\lambda_0^2}(X-X') \} \Big|_{\lambda_0 \rightarrow \infty} = i \Delta_{\mu^2}(X-X') \quad (4)$$

and using the middle expression for Δ -function, calculating the reaction effects and after the calculations went over to the limit of infinite auxiliary mass $\lambda_0 \rightarrow \infty$; since Δ -function vanishes in the limit of infinite mass factor, the middle expression is in fact trivial, but in this way of calculation the consistency of vanishment of the reaction effects in the limit of infinite mass $\mu \rightarrow \infty$ on the right hand side of the commutator (1) was secured.

We can see also from the equivalent expression to (4) ;

$$[C(X), C(X')] = i \int_{-\infty(-\lambda_0)}^{\infty(+\lambda_0)} \frac{1}{2} \sigma(\mu-\lambda) \cdot \frac{\partial}{\partial \lambda} \cdot \Delta_{\lambda^2}(X-X') \cdot d\lambda \quad (4')$$

where

$$\sigma(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

the vanishment of the relativistically non-covariant self-energy in Feynman's

calculation along usual perturbation theory⁵⁾ is secured by the appearance of the differential factor $\partial/\partial\lambda$, since this operator only selects terms which depend on λ . But the self-energy or reaction corrections diverge in this procedure also.

Now we try to generalize the above statement, e.g. the reaction effects should vanish in the limit of infinite field mass in consistent way with the canonical commutator between field variables, to the field variable itself.

To formulate the statement that the field variable should vanish in the limit of infinite mass, we can take for $C(X)$ analogous form as (4');

$$C(X) = \int_{-\infty}^{\infty} \frac{1}{2} \sigma(\mu - \lambda) \cdot \frac{\partial}{\partial \lambda} \cdot C^{(\lambda)}(X) \cdot d\lambda \quad (5)$$

If we assume that $C^{(\lambda)}(X)$ behaves regularly with respect to λ and vanishes in the limit of λ infinite, this expression (5) gives only the following identification;

$$C(X) = \left[\frac{1}{2} \sigma(\mu - \lambda) \cdot C^{(\lambda)}(X) \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\partial}{\partial \lambda} (\mu - \lambda) \cdot C^{(\lambda)}(X) \cdot d\lambda = C^{(\mu)}(X)$$

We assume $C^{(\lambda)}(X)$ has this property, i.e. vanishes in the limit of infinite mass, only when it operates on the state functional giving real effects. The contribution in the virtual processes should be given by the commutator between them.

This restriction is obviously stronger than the (mathematical) condition imposed on (1) concerning the vanishment of $\Delta_{\mu 2}$ in the limit of infinite field mass. And the restriction (5) makes free field energy vanish in this limit.

Of course we should retain the commutation relation (1), since it does not show any peculiar property itself in the limit of infinite mass and also since it comes from the canonical formalism;

$$\begin{aligned} [C(X), C(X')]_{-} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma(\mu - \lambda) \cdot \frac{1}{2} \sigma(\mu' - \lambda') \cdot \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda'} \times \\ &[C^{(\lambda)}(X), C^{(\lambda')}(X')] \cdot d\lambda d\lambda' = [C^{(\mu)}(X), C^{(\mu')}(X')]_{-} - \frac{1}{2} \{ [C^{(\infty)}(X), C^{(\mu)}(X')]_{-} + \\ &+ [C^{(-\infty)}(X), C^{(\mu)}(X')]_{-} + [C^{(\mu)}(X), C^{(\infty)}(X')]_{-} + [C^{(\mu)}(X), C^{(-\infty)}(X')]_{-} \} + \\ &+ \frac{1}{4} \{ [C^{(\infty)}(X), C^{(\infty)}(X')]_{-} + [C^{(-\infty)}(X), C^{(\infty)}(X')]_{-} + [C^{(\infty)}(X), C^{(-\infty)}(X')]_{-} + \\ &+ [C^{(-\infty)}(X), C^{(-\infty)}(X')]_{-} \} = i\Delta_{\mu 2}(X - X') \quad (6) \end{aligned}$$

This relation can be satisfied if one assumes the following commutation relations between $C^{(\lambda)}(X)$'s;

$$[C^{(\lambda)}(X), C^{(\lambda')}(X')]_{-} = i\Delta_{\lambda\lambda'}(X - X') \quad (7)$$

This restriction also determines the operational property of $C^{(\lambda)}(X)$ in the virtual processes.

So that the commutation relations between field variables now becomes;

$$[C(X), C(X')]_- = i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma(\mu-\lambda) \cdot \frac{1}{2} \sigma(\mu-\lambda') \cdot \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda'} \cdot \mathcal{A}_{\lambda\lambda'}(X-X') \cdot d\lambda d\lambda' \quad (8)$$

Comparing this form with the Pauli's regulator²⁾, one can see that in this case the regulator is not only the function of auxiliary mass spectrum but also contains the differential operator with respect to the auxiliary mass. And the commutator is regularized by the double expansion to mass spectrum.

Parallel to this, from (7), we have for the expression for $\mathcal{A}^{(1)}$ -function;

$$\langle [C(X), C(X')]_+ \rangle_{vac} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma(\mu-\lambda) \cdot \frac{1}{2} \sigma(\mu-\lambda') \cdot \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda'} \cdot \mathcal{A}_{\lambda\lambda'}^{(1)}(X-X') \cdot d\lambda d\lambda' \quad (9)$$

Of course, one can say that the relation such as (8) is as trivial as that of Feynman, e.g. (4') or (4), since \mathcal{A} -function is regular with respect to mass factor; But in the course of the calculation, as is shown in (3), the singularity appears which contradicts with the regular property of \mathcal{A} -function, so that we should maintain the regulator $\frac{1}{2} \sigma(\mu-\lambda) \cdot \partial/\partial\lambda$ throughout the calculations and finally operate it as done by Feynman (His procedure corresponds to the λ -integral in (4') after the calculations).

Now, by the condition imposed on the field variables $C(X)$ and the assumption of the commutator (7) and (9), which is the generalization of Feynman's commutator (4'), we shall show that these simple assumptions can bring the self-energy finite for Dirac particle.

§ 2. Scalar Mesonic Self-Energy of Dirac Particle

We firstly calculate the self-energy of Dirac particle interacting with the scalar meson field (C-meson).

For the interaction Hamilton density, we can take the following which was obtained from (2) by setting the expression (5);

$$H = f \psi_+ \psi_- \int_{-\infty}^{\infty} \frac{1}{2} \sigma(\mu-\lambda) \cdot \frac{\partial}{\partial \lambda} \cdot C^{(\lambda)} \cdot d\lambda \quad (10)$$

From this, following Schwinger and Pauli¹⁾²⁾, calculating the mass correction in order f^2 , we obtain

$$\frac{\delta m}{m} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma(\mu-\lambda) \cdot \frac{1}{2} \sigma(\mu-\lambda') \cdot \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda'} \cdot \left(-\frac{f^2}{16\pi} \right) \cdot \int_{-1}^1 \log \frac{4}{s_0 \gamma m^2 (1-y)^2 + 2 \frac{\lambda\lambda'}{m^2} (1+y)} \Big|_{s_0 \rightarrow 0} \cdot (y+3) \cdot dy \cdot d\lambda d\lambda' \quad (11)$$

By the operation of differential operator with respect to λ and λ' , the infinite

term which gives usual diverging self-energy

$$\log \frac{4}{\varepsilon_0 \gamma m^2} \quad \varepsilon_0 \rightarrow 0$$

vanishes identically, and produces the following from the remaining which depend on λ and λ' ;

$$\begin{aligned} \frac{\delta m}{m} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma(\mu-\lambda) \cdot \frac{1}{2} \sigma(\mu-\lambda') \cdot \left(-\frac{f^2}{16\pi} \right) \cdot \\ &\quad \int_{-1}^1 \frac{2}{m^2} \frac{(y+1)(y-1)^2(y+3)}{(y-1)^2 + 2 \frac{\lambda\lambda'}{m^2} (y+1)} dy \cdot d\lambda d\lambda' \\ &= \frac{f^2}{16\pi} \int_{-1}^1 \log \frac{(y-1)^2 + 2\delta^2(y+1)}{2\delta^2(y+1)} \cdot (y+3) \cdot dy \end{aligned} \quad (11')$$

which is now finite and gives;

$$\begin{aligned} \frac{\delta m}{m} &= \frac{f^2}{16\pi} \left[2\delta^2 - 9 - 2(\delta^4 - 6\delta^2 - 6) \log \delta + \right. \\ &\quad \left. + \begin{cases} 2\delta(4-\delta^2)^{3/2} \cdot \cos^{-1}(\delta/2) & \delta < 2 \\ 2\delta(\delta^2-4)^{3/2} \cdot \log \frac{\sqrt{\delta^2-4} + \delta}{2} & \delta > 2 \end{cases} \right] \end{aligned} \quad (12)$$

where $\delta = \mu/m$ is the mass ratio between meson and Dirac particle.

So that our prescription gives the finite result for the self-energy.

§ 3. Electromagnetic Self-Energy of Dirac Particle

We then treat the electromagnetic self-energy of Dirac particle on the same prescription described in § 1. and § 2.

In the preceding section, we assumed that the meson mass is finite, but from (12) we can see that for the vanishing meson mass $\mu=0$ this expression diverges again logarithmically, this is due to the vanishing of the term $\log \lambda\lambda'$ which served in Feynman's procedure to hold the whole expression finite in the limit of $\mu=0$; and also this circumstance comes from the fact that the expression (6) with relation (7) becomes meaningless in the case of mass zero field; since for $\mu=0$;

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma(-\lambda) \cdot \frac{1}{2} \sigma(-\lambda') \cdot \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda'} \cdot [C^{(\lambda)}(X), C^{(\lambda')}(X')] \cdot d\lambda d\lambda' \\ &= i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma(-\lambda) \cdot \frac{1}{2} \sigma(-\lambda') \cdot \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda'} \Delta_{\lambda\lambda'}(X-X') \cdot d\lambda d\lambda' \end{aligned}$$

$$= i \int_{-\infty}^{\infty} \frac{1}{2} \sigma(-\lambda) \cdot \frac{\partial}{\partial \lambda} \left\{ \mathcal{A}_0(X-X') - \frac{1}{2} \mathcal{A}_{\infty}(X-X') - \frac{1}{2} \mathcal{A}_{-\infty}(X-X') \right\} \cdot d\lambda = 0 \quad (13)$$

So that the case of mass zero field can only be treated in the alternative way to impose the condition on the electron wave function; analogous considerations lead to the following;

$$\phi(X) = \int_{-\infty}^{\infty} \frac{1}{2} \sigma(m-\lambda) \cdot \frac{\partial}{\partial \lambda} \cdot \phi^{(\lambda)}(X) \cdot d\lambda \quad (14)$$

$$[\phi^{(\lambda)}(X), \phi^{(\lambda')}(X')]_+ = \frac{1}{i} \left(\gamma^\lambda \frac{\partial}{\partial X_\lambda} - m \right) \mathcal{A}_{\lambda\lambda'}(X-X') \quad (15)$$

where also when $\phi^{(\lambda)}$'s *really* operates on the state functional, it is assumed to be regular with respect to λ , and the operational property in the *virtual* processes is determined by (15). Similarly;

$$\langle [\phi^{(\lambda)}(X), \phi^{(\lambda')}(X')] \rangle_{\text{vac}} = - \left(\gamma^\lambda \frac{\partial}{\partial X_\lambda} - m \right) \mathcal{A}_{\lambda\lambda'}^{(0)}(X-X')$$

By making use of these and with the following interaction Hamilton density between electromagnetic field and Dirac particle;

$$H = -ic\psi^\dagger \gamma^\mu \psi A_\mu \quad (16)$$

where ψ 's are given by (14); we have for the mass correction in order c^2 ;

$$\begin{aligned} \frac{\delta m}{m} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma(m-\lambda) \cdot \frac{1}{2} \sigma(m-\lambda') \cdot \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda'} \cdot \frac{e^2}{8\pi} \int_0^\infty \int_{-1}^1 \cos\left(\frac{m^2}{4}(y^2-1)\tau\right) \\ &\quad - \frac{\lambda\lambda'}{2}(y-1)\tau) \cdot \frac{(3-y)}{\tau} dy d\tau \cdot d\lambda d\lambda' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2} \sigma(m-\lambda) \cdot \frac{1}{2} \sigma(m-\lambda') \cdot \\ &\quad \cdot \frac{\partial}{\partial \lambda} \frac{\partial}{\partial \lambda'} \cdot \frac{e^2}{8\pi} \int_{-1}^1 \log \frac{4}{\varepsilon_0 \gamma m^2 (y-1) \left(y+1-2 \frac{\lambda\lambda'}{m^2} \right)} \cdot (3-y) dy \cdot d\lambda d\lambda' \quad (17) \end{aligned}$$

Then, by the differentiation and integration over λ and λ' , there results;

$$\begin{aligned} \frac{\delta m}{m} &= \frac{e^2}{8\pi} \int_{-1}^1 (3-y) \cdot \log\left(\frac{2}{y-1}\right) \cdot dy \\ &= \frac{5}{8\pi} e^2, \quad (18) \end{aligned}$$

which is the finite and positive correction.

By calculating the scalar mesonic self-energy in this alternative way, the mass correction was found to be;

$$\frac{\delta m}{m} = \left(\frac{\delta m}{m} \right)_{(12)} + \frac{f^2}{16\pi} (2 - 12 \log \delta) \quad (19)$$

which contains no divergence in the limit of mass zero field, $\mu=0$, contrary to (12). This difference comes from the fact that the expression (12) was obtained by imposing the condition on $C(X)$'s and (18) was obtained from conditioned ψ . The former self-energy vanishes in the limit of large meson mass in consistent way with the vanishment of the commutator (1) in this limit, and the latter vanishes when the mass factor of the Dirac particle appearing in the Δ -function in the commutator becomes to infinite.

It seems that the latter procedure is preferable since it can treat mass zero field as well as mass finite field.

We can now add a note to the mass defect of proton calculated by C -meson theory⁶⁾, which is given by adding (18) and (19) with the relation $f^2=2e^2$, which becomes;

$$\begin{aligned} \frac{\delta m}{m} = & -\frac{e^2}{4\pi} \left[1 - \delta^2 + \delta^2(\delta^2 - 6) \log \delta - \right. \\ & \left. - \begin{cases} \delta(4 - \delta^2)^{3/2} \cdot \cos^{-1}(\delta/2) & \delta < 2 \\ \delta(\delta^2 - 4)^{3/2} \cdot \log \frac{\sqrt{\delta^2 - 4} + \delta}{2} & \delta > 2 \end{cases} \right], \end{aligned} \quad (20)$$

which is just $\frac{e^2}{4\pi}$ smaller than the usual perturbation calculation (in absolute value; usually the constant term is $-\frac{1}{2\pi}e^2$). The same result can be obtained by the method of relativistic cut-off along Feynman⁷⁾ if one takes his alternative method of modifying the electron density. This circumstance also depends on the condition that the Δ_m^2 function should vanish in the limit of infinite mass factor appearing in the Δ -function. Formally this is related to the appearance of differentiation and integration in (14) and hence in the commutator between ψ 's; In Feynman's case we can write the commutator in the following form;

$$[\psi(X), \psi^\dagger(X')]_+ = \frac{1}{i} \left(\gamma^\lambda \frac{\partial}{\partial X_\lambda} - m \right) \int_{-\infty}^{\infty} \frac{1}{2} \sigma(m - \lambda) \cdot \frac{\partial}{\partial \lambda} \Delta_{\lambda^2}(X - X') \cdot d\lambda$$

Although the mass correction (17) cannot explain the mass defect of proton, the interaction of the nucleon with charged meson field with its dissociation probability above 0.5 brings the sign of mass difference correctly, so that we may expect the strong coupling between π -meson and nucleon.

§ 4. Conclusions

We have shown that by imposing the condition on the wave function and assuming the commutator not contradicting with the canonical form, the self-

energies of particles of Dirac type becomes to the finite value.

The formal developement to the vacuum expectation value of charge-current density and the problem of photon self-energies can be treated on the same line.

The formal analogy seems to exist between our assumption (5) or (14) with five-dimensional formalism. But, our main purpose of this paper is to show that the generalization of Feynman's procedure brings the theory convergent. And, whether such formal analogy has any physical meaning or not is now being examined.

In conclusion I say much thanks to Prof. M. Kobayasi for his kind interest throughout this work.

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An Improvement on the Integrations appearing in Perturbation Theory.

Hiroomi UMEZAWA and Rokuo KAWABE

Institute of Theoretical Physics, Nagoya University

(Received June 9, 1949)

The customary perturbation method is not only ambiguous as to its relativistic covariance, but often actually gives the results evidently destroying the relativistic covariance. This can be seen for instance, from the fact, indicated by Pais¹⁾ and others, that a result with the transformation property of mass cannot be obtained from perturbation calculations for the self-energy of a moving electron due to an electromagnetic field. This circumstance has also made the calculation of integrals extremely difficult, making it almost impossible to calculate various diverging integrals fully, up to their finite terms.

But, since the system of quantum field theory is of a relativistically invariant structure, it must be concluded that the cause of the failure of relativistic covariance lays in the process of the perturbation calculations. We intend to propose a method to remedy this defect of perturbation calculation.

We shall confine the following arguments to calculations in the momentum space. It appears that the cause of the failure of relativistic covariance in the perturbation theory lies in the following circumstance: We consider the case when two particles 1 and 2 are created in the intermediate state, and denote their momentum-energy four-vectors by $p^{(1)} = (\mathbf{P}^{(1)}, E_{\mathbf{P}}^{(1)})$, $p^{(2)} = (\mathbf{P}^{(2)}, E_{\mathbf{P}}^{(2)})$ respectively. Then an integral of the following form occurs in the perturbation calculation;

$$\iint F(p^{(1)}, p^{(2)}) d\mathbf{P}^{(1)} d\mathbf{P}^{(2)} \quad (1)$$

Where $F(p^{(1)}, p^{(2)})$ is a certain function of $p^{(1)}$ and $p^{(2)}$. Owing to the momentum conservation law, (1) can be rewritten as

$$\int G(\mathbf{P}^{(1)}) d\mathbf{P}^{(1)} \quad (2)$$

Hitherto, the method has been employed of taking a sphere as the integration domain for $\mathbf{P}^{(1)}$ in (2) and making its radius tend to infinity. For instance, in the calculation for the self-energy of the electron, the integration for the momentum \mathbf{l} of the photon created in the intermediate state has been carried out within an infinitely large sphere of \mathbf{l} .

This method destroys the relativistic covariance in the following two ways:

(i) A sphere is not a relativistically invariant domain. (ii) A momentum-

conservation relation among particles which do, not satisfy the energy-conservation law is not a relativistically invariant relation. That is, two particles inter-related so as to satisfy momentum conservation do not in general retain this relation after undergoing a Lorentz transformation. For example, if an electron of momentum \mathbf{P} emitted a photon of momentum \mathbf{l} in the intermediate state and had its momentum decreased to $\mathbf{P}-\mathbf{l}$, these three particles generally no longer satisfy the momentum-conservation law after undergoing a Lorentz transformation. This is because the energy needs not be conserved in an intermediate state.

(a) In order to remove the difficulty (ii), an integration domain must be prescribed for the integral (1). (b) Further, in order to remove the difficulty (i), the sphere must be replaced by domain enclosed by a surface on which the momentum-energy space scalar quantity w takes a constant value, and the integration performed within this domain which is then allowed to become infinitely large. (c) Furthermore, we place the condition that the above domain becomes a momentum-space sphere when referred to an appropriate system of coordinates, so that results identical with those previously obtained are given as special case. This is necessary, as will later be seen, as a boundary condition determining the limits of the integral, that is, the value which w takes on the enclosing surface.

As the four-dimensional scalar quantity w satisfying the above conditions, we may take

$$f(w) = (p', p)$$

where $f(w)$ is an arbitrary function of w , and (p', p) the four-dimensional scalar product of the momentum-energy four-vectors of the two particles created in the intermediate state.

We next consider the above-mentioned determining of the integral domain in its two-dimensional analogue for the sake of facility.

Let the shaded part in Fig. 1 represent the domain in the P', P space enclosed by the surface $w=a, b$ (both constants). The integral (1) will be integrated in this domain. We select a sub-domain AB in which the momentum-conservation law holds. This, then, is the integration-domain for the integral (2). On performing a Lorentz-transformation, the domain satisfying the momentum-conservation law shifts from AB to $A'B'$. This is the integration domain for (2) in the transformed system of coordinates. But, since the shaded domain is relativistically invariant, the integration domain of (1) does not change due to the Lorentz-transformation. Therefore, the limits of the integral (2) are determined by $w=a, b$ irrespective of the Lorentz transformation.

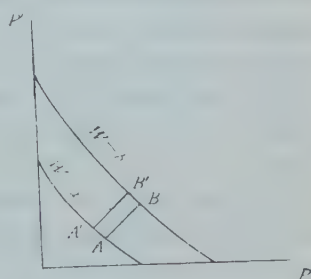


Fig. 1

Condition (c) requires that, as a special case, AB coincide with the sphere which was the integration-domain occurring in previous calculations. That is, the boundaries $w=a, b$, must be so drawn as to pass the boundaries, A,B of this special sphere.

The actual method of calculation is as follows: We perform the transformation of variables

$$|\mathbf{P}| \rightarrow w \quad (4)$$

in the integral (2), rewriting it as an integration for w . We next set the bounds of the integration-domain at $w=a, b$. The values of a and b are so determined that when the domain $w=\text{const.}$ becomes a sphere, the results will be identical with those of previous calculations. As result if the integrand comes to depend solely on w , the value of the integral becomes relativistically invariant. By transforming the variable into w , in this way we can see the transformation property of the integral.

In general, when there are n particles 1, 2, n in the intermediate state, the following $(n-1)$ scalar quantities must be employed:

$$(p^{(1)}, p^{(k)}) = w^{(k)} \quad (k=2, \dots, n) \quad (e')$$

It can be ascertained, by actual calculations in many problems such as the self-energy of the electron, various quantities appearing in vacuum polarization, etc., that the results having the correct relativistic covariance can be obtained by applying this method to the perturbation theory. Detailed reports of various results thus obtained will be given in this issue.

This method is not confined to the perturbation theory, but may be applied to any case involving an integration in the momentum space. Furthermore, although we restricted the above arguments to integrals in the momentum space there is danger of destroying the relativistic covariance of integrals in the space of other quantities, also unless similar precautions are taken.

In such cases, too, a scalar quantity w must be taken, and the integration must be performed in the domain determined by methods similar to (a), (b) and (c).

In conclusion, we would like to express our sincere thanks to Prof. S. Sakata and Dr. Y. Tanikawa for their interest in this work.

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Some General Formulae relating to Vacuum Polarization

Hiroomi UMEZAWA and Rokuo KAWABE

Institute of Theoretical Physics, Nagoya University.

(Received June 9, 1949)

§ 1. Introduction.

Among the divergence difficulties appearing in quantum electrodynamics, there are two types, one the self-energy of charged particles, and the other the type hitherto called vacuum polarization. In calculations for various collision processes, too, many divergences appear when higher approximations of perturbation theory are taken. Detailed examinations of various cases of these divergences have recently been performed by many authors. As a result, the divergences appearing in various processes have respectively been traced down to certain operators in the Hamiltonian function. For instance, the self-energy of the electron, and a part of the divergences appearing in the elastic scattering of an electron have been accounted for by the electronic mass-term of the form $\delta m \int \psi^* \beta \psi d\tau$. Discussions of these diverging terms have been made particularly by the method of contact transformation in the perfectly relativistic form proposed by Tomonaga and Schwinger.¹⁾

These authors further proposed theories dissolving the divergence difficulties by subtracting beforehand in the Hamiltonian function various separated terms responsible for the divergences. This was based on the idea that the subtraction of diverging terms would merely cause changes in the mass and charge of the charged particle. As a result experimental facts such as the energy level shift of the normal state of the hydrogen atom from levels predicted by the Dirac equation for a bound electron, and the hyperfine structures of various atomic spectra were satisfactorily accounted for.

In being interpreted in terms of the difference of the self-energies of electrons in a bound and in a free state, these experimental facts show the reality of the self-energy appearing in higher order perturbation calculations. Therefore, it is necessary simultaneously to find other means of dissolving the divergence difficulties besides making discussions based on the phenomenological subtraction of diverging terms. For this purpose, we must search into the physical nature of the divergences, and find substantialistic solutions appropriate at each stage of theory.

Hitherto, the C-meson theory²⁾ has been proposed as an attempt to dissolve

the difficulty of the self-energy of the electron, and to provide a footing of the future theory. This was put forward by Prof. Sakata with the idea of analysing the divergence difficulties through considering synthetically the existence of various fields. Prof. Sakata³⁾ has explained the relation between his theory and that of Tomonaga and Schwinger which appeared subsequently as a relation between concrete and abstract.

We desired to extend the C-meson theory and find substantialistic solutions possible at the present stage by making a synthetic investigation covering the whole range of quantum electrodynamics with various fields considered. For this purpose, we first of all examined many divergences by simultaneously considering various charged particle fields in the domain of vacuum polarization.⁴⁾ As a result, it was found that, among the electromagnetic divergences appearing in our processes, those of the self-energies of photons and charged particles could be provisionally dissolved by simultaneously considering the C-meson and charged particles.

However, in our calculations up to now, detailed discussions inclusive of finite terms could not be made, owing to the imperfectness of perturbation calculation from the relativistic standpoint. We now intend to make detailed discussions by means of the improved perturbation calculation put forward in a separate paper.⁵⁾ For this purpose, it is convenient first of all to obtain some general formulae for various quantities appearing in vacuum polarization regardless of the type of charged particles. This is most readily done by perturbation calculation, so we shall, in the following, derive the formulae by this method.

§ 2. The Current Induced by an External Current.

1) Perturbation Calculation in the Interaction Representation. The calculation for the current induced by a current J existing in vacuum is performed on the perturbation theory. But, in order to obtain results whose spatio-temporal symmetry is self-evident, we start from the wave equation in the interaction representation. That is, denoting the state vector by Ψ

$$i\hbar \frac{\partial \Psi}{\partial t} = \lambda H' \Psi \quad (1)$$

where $\lambda H'$ stands for the interaction, and λ is the interaction constant. Let Ψ be expanded in power series of λ , and the state a free one Ψ_0 until the instant $t=t_0$ when the interaction is introduced.

$$\Psi = \Psi_0 + \lambda \Psi_1 + \lambda^2 \Psi_2 + \dots \quad (2)$$

In the interaction representation, the wave function of each field satisfies the free equation of motion. Furthermore, the state vector Ψ_0 for $\lambda \rightarrow 0$ is independent of t , while Ψ_1, Ψ_2, \dots involve t .

The t -dependencies are determined by (1). Also, the operators $\lambda H'$, etc. depend to t .

We next expand Ψ_n in terms of the free state vector

$$\Psi_n = \sum_R C_{nR}(t) \phi_{0R} \quad (3)$$

The matrix element of $\lambda H'$ corresponding to a transition $B \rightarrow B'$ between two free states (whose energies are $E_R, E_{R'}$, respectively) can be written, in the interaction representation, as

$$(B' | \lambda H' | B) = \sum h_{BR'} e^{\frac{i}{\hbar} (E_R - E_{R'}) t} \quad (4)$$

This is readily obtained by expanding $\lambda H'$ in a Fourier series in the space-time $(\mathbf{r}, t) = X$ (taking account of the fact that the various wave functions involved satisfy, in this representation, the free field equation), and then taking the matrix element corresponding to the transition $B \rightarrow B'$. Here $h_{RR'}$ is independent of t .

Substituting (4) and (3) into (1),

$$i\hbar \frac{\partial C_{1B}}{\partial t} = h_{AB} e^{-\frac{i}{\hbar} (E_A - E_B) t} \quad (5)$$

$$i\hbar \frac{\partial C_{nB}}{\partial t} = \sum h_{BB'} e^{-\frac{i}{\hbar} (E_{B'} - E_B) t} C_{B'} \quad (n \geq 2) \quad (6)$$

From (5) and (6), we conclude the following fact: In perturbation calculation in the interaction representation, we need only replace $(B' | \lambda H' | B)$ by $h_{RR'}$ in the calculation in Schrödinger representation.

Consequently,

$$C_{1B} = \frac{h_{AB} \{ e^{-\frac{i}{\hbar} (E_A - E_B) t} - 1 \}}{E_A - E_B} \quad (7)$$

$$C_{nB} = \frac{h_{n-1n}, h_{n-2, n-1}, \dots, h_{0,1}}{(E_0 - E_n)(E_0 - E_{n-1}) \dots (E_0 - E_1)} \left\{ e^{-\frac{i}{\hbar} (E_0 - E_n) t} - 1 \right\} \\ + \sum \left\{ \text{terms in which contain } \exp \left(-\frac{i}{\hbar} (E_n - E_{n'}) t \right) \right\} \quad (8)$$

(7) and (8) can readily be obtained from the result of perturbation calculation in the Schrödinger representation by considering the fact that this representation goes over to the interaction representation on making the contact transformation,

$$\Psi \rightarrow e^{\frac{i}{\hbar} H_0 t} \Psi \quad (9)$$

where H_0 is the Hamiltonian function not involving the interaction.

For the perturbation formula for stationary states, too, we obtain, by performing the transformation (9) on that in the Schrödinger representation,

$$C_{nn} = \sum \frac{(n|h|n-1)(n-1|h|n-2)\cdots(1|h|0)}{(E_0-E_n)(E_0-E_{n-1})\cdots(E_0-E_1)} \quad (8)$$

Considering that the state is approximately a free stationary one, it is natural that this expression is independent of t . In this case, too, we need only replace $\lambda H'$ by λh in the Schrödinger representation.

2) Calculation of the Induced Current.

Let a current $J_\mu^{(i)} = (\mathbf{J}^{(i)}, \rho^{(i)})$ exist in vacuum. Hereunder μ and k will always denote suffixes denoting the components of four and three-dimensional vectors respectively ($\mu=1, 2, 3, 4, k=1, 2, 3$), while boiefaced letters stand for three-dimensional vectors. We calculate the current induced by $J^{(i)}$.

We write the current produced by charged particles in the intermediate state as $J^{(v)} = (\mathbf{J}^{(v)}, \rho^{(v)})$, regardless of the type of particle. Hereunder (i) and (v) denote pairs of charged particles existing in vacuum and created in the intermediate state, respectively.

The interaction between electromagnetic field and charged particle can, in general, be written in the following form.

$$H^{(i)} = cH_1^{(i)} + c^2H_2^{(i)} = -\int (\mathbf{A}\mathbf{J}_1^{(i)}) dv - \frac{1}{2} \int (\mathbf{A}\mathbf{J}_2^{(i)}) dv \quad (10)$$

$$H^{(v)} = cH_1^{(v)} + c^2H_2^{(v)} = -\int (\mathbf{A}\mathbf{J}_1^{(v)}) dv - \frac{1}{2} \int (\mathbf{A}\mathbf{J}_2^{(v)}) dv \quad (11)$$

Besides these, there are the Coulomb potentials $V^{(iv)} = \int v \rho^{(i)} \rho^{(v)} dv$

$V^{(ii)} = \int v \rho^{(i)} \rho^{(i)} dv$, $V^{(vv)} = \int v \rho^{(v)} \rho^{(v)} dv$, $\left(v = \frac{1}{r}\right)$. \mathbf{A} is the transversal part of the electromagnetic vector potential, cH_1 and c^2H_2 the first and second order interactions respectively, and J_1, J_2 those parts of the current proportional respectively to e and e^2 . Of course,

$$\mathbf{J}^{(i)} = \mathbf{J}_1^{(i)} + \mathbf{J}_2^{(i)}, \quad \mathbf{J}^{(v)} = \mathbf{J}_1^{(v)} + \mathbf{J}_2^{(v)}$$

We denote the state in which a single particle (i) exists in vacuum in the state A by the vector Ψ_{0A} . From (2), (1) the state Ψ at the time t is,

$$\Psi_A = \Psi_{0A} + e\Psi_1 + e^2\Psi_2 + \cdots \quad (12)$$

$$i\hbar \frac{\partial \Psi_A}{\partial t} = \left(H^{(i)} + H^{(v)} + V^{(iv)} + \frac{1}{2} V^{(vv)} + \frac{1}{2} V^{(ii)} \right) \Psi_A \quad (13)$$

In the following, we calculate only up to the e^2 approximation. Expanding

the operators $J^{(i)}, J^{(n)}, H^{(i)}, H^{(n)}$, etc. related to the field quantities in Fourier integrals,

$$J_{\mu}^{(i)}(X) = \int j_{\mu}^{(i)}(l) e^{i(l, X)} dl \quad J_{\mu}^{(n)}(X) = \int j_{\mu}^{(n)}(l) e^{i(l, X)} dl \quad (14)$$

Similarly, $\Psi_n(X)$ is analysed into

$$e\Psi_0(X) = \int \eta_0(\mathbf{P}) e^{i(\mathbf{P}, \mathbf{r})} d\mathbf{P}, \quad e^n \Psi_n(X) = \int \eta_n(\mathbf{P}, t) e^{i(\mathbf{P}, \mathbf{r})} d\mathbf{P} \quad (n \geq 1) \quad (15)$$

where $X = (\mathbf{r}, t)$, $l = (\mathbf{l}, E_{\mathbf{P}} - E_{\mathbf{P}-\mathbf{l}})$

The current is

$$J = J^{(i)} + J^{(n)} \quad (16)$$

$$\begin{aligned} \langle J(X) \rangle &= \Psi_A^*(X) J(X) \Psi_A(X) = \Psi_{0A}^*(X) J(X) \Psi_{0A}(X) \\ &+ e \Psi_1^*(X) J(X) \Psi_0(X) + e \Psi_0^*(X) J(X) \Psi_1(X) \\ &+ e^2 \Psi_1^*(X) J(X) \Psi_1(X) + e^2 \Psi_2^*(X) J(X) \Psi_0(X) \\ &+ e^2 \Psi_0^*(X) J(X) \Psi_2(X) \end{aligned} \quad (17)$$

The induced current $\delta \cdot J$ is

$$\langle \delta J \rangle = \langle J(X) \rangle - \Psi_{0A}^*(X) J(X) \Psi_{0A}(X) - \langle J(X) \rangle_{\text{vac.}} \quad (18)$$

Where $\langle J(X) \rangle_{\text{vac.}}$ is the current induced in vacuum.

Among (17), the term proportional to e involves A linearly, so that it has no matrix element between states free from photons, and consequently does not contribute to $\langle J_1(X) \rangle$, but it does to $\langle J_2(X) \rangle$.

Using the perturbation formula (8),

$$\begin{aligned} \eta_i(\mathbf{P}-\mathbf{l}, t) &= \frac{(\mathbf{P}_i - \mathbf{l}, \mathbf{l}^- | - (\overline{A} J_1^{(i)}) | \mathbf{P}_i)}{E - l} \eta_0(\mathbf{P}_i) \\ &+ \frac{\left(\mathbf{P}_i - \frac{\mathbf{l}}{2}, -\mathbf{P}_i - \frac{\mathbf{l}}{2}, \mathbf{l}^- | - (\overline{A} J_1^{(i)}) | \text{vac.} \right)}{\left(-E_{\mathbf{P}_i - \frac{\mathbf{l}}{2}}^{(i)} - E_{-\mathbf{P}_i - \frac{\mathbf{l}}{2}}^{(i)} - l \right)} \eta_0(\mathbf{P}_i - \mathbf{l}) \end{aligned} \quad (19)^*$$

$$(E = E_{\mathbf{P}_i}^{(i)} - E_{\mathbf{P}_i - \mathbf{l}}^{(i)})$$

In the calculation for η_2 , the following point must be borne in mind: Since A^2 is involved in the e^2 approximation, the system can return to its initial state. Denoting the component of Ψ_2 which is equal to the initial state A by Ψ_{2A} , we have

*) II denotes the three-dimensional space integral $\int h d\mathbf{v}$ of the quantity h appearing in the relation (4).

$$\Psi_{2A} = C_{2A}(t) \Psi_{0A} \quad (20)$$

The coefficient c_2 cannot be determined from (8'). It is determined by the normalization condition

$$\int (\Psi_A^*, \Psi_A) dv = 1$$

Substituting (12) in this relation and comparing the coefficients of ϵ^2 on either side,

$$C_{2A}(t) = -\frac{1}{2} \sum_B \frac{(A | \epsilon H_1 | B) (B | \epsilon H_1 | A)}{(E_A - E_B)^2} \quad (22)^{(b)}$$

From (8') and (22)

$$\begin{aligned} & \gamma_2(P_t - l) \\ &= \frac{\left(P_r + \frac{l}{2}, -P_r + \frac{l}{2} \middle| -(\overline{AJ_1^{(n)}}) \right. \left. (P_t - l, l^2 - (\overline{AJ_1^{(t)}}) \middle| P_t \right)}{(E - E_{P_r + \frac{l}{2}}^{(n)} - E_{P_r - \frac{l}{2}}^{(n)}) (E - l)} \gamma_0(P_t) \quad (23.1) \end{aligned}$$

$$\begin{aligned} & + \frac{\left(P_t - l \middle| -(\overline{AJ_1^{(t)}}) \middle| P_t, -l^2 \right) \left(P_r + \frac{l}{2}, P_r + \frac{l}{2}, -l^2 \middle| -(\overline{AJ_1^{(n)}}) \middle| \text{vac} \right)}{(E - E_{P_r + \frac{l}{2}}^{(n)} - E_{P_r - \frac{l}{2}}^{(n)}) (-E_{P_r + \frac{l}{2}}^{(n)} - E_{P_r - \frac{l}{2}}^{(n)} - l)} \times \\ & \gamma_0(P_t) \quad (23.2) \end{aligned}$$

$$\begin{aligned} & + \frac{\left(P_r + \frac{l}{2}, -P_r + \frac{l}{2} \middle| \rho^{(n)} \middle| \text{vac} \right) (P_t - l \middle| \rho^{(t)} \middle| P_t)}{(E - E_{P_r + \frac{l}{2}}^{(n)} - E_{P_r - \frac{l}{2}}^{(n)})} \gamma_0(P_t) \quad (23.3) \end{aligned}$$

$$\begin{aligned} & - \frac{1}{2} \frac{1}{l'} \frac{(P_t - l - (\overline{AJ_1^{(t)}}) \middle| P_t - l - l', l'^2 - (\overline{AJ_1^{(t)}}) \middle| P_t - l)}{(E_{P_t - l}^{(t)} - E_{P_t - l - P - l'}^{(t)})^2} \\ & \times \gamma_0(P_t - l) \quad (23.4) \end{aligned}$$

$$\begin{aligned} & - \frac{1}{2} \sum_{P_n} \frac{(\text{vac} - (\overline{AJ_1^{(n)}}) \middle| P_r + \frac{l}{2}, -P_r + \frac{l}{2}, -l^2) (P_r + \frac{l}{2}, -P_r + \frac{l}{2}, -l^2)}{(-E_{P_r + \frac{l}{2}}^{(n)} - E_{P_r - \frac{l}{2}}^{(n)} - l)^2} \\ & \frac{-(\overline{AJ_1^{(n)}}) \middle| \text{vac})}{\gamma_0(P_t - l)} \quad (23.5) \end{aligned}$$

$$\begin{aligned} & + \frac{(P_r, -P_r \middle| -(\frac{1}{2} \overline{AJ_1^{(n)}}) \middle| \text{vac})}{(-2E_{P_r}^{(n)})} \quad (23.6) \end{aligned}$$

In the case of vacuum, only the terms (23.5) and (23.6) remain for γ_2 and only (23.6) for γ_1 . Using this γ_{vac} , we can calculate

$$\langle J \rangle_{\text{vac}} = \langle J^{(i)} + J^{(v)} \rangle_{\text{vac}}.$$

We first construct $\langle J_1 \rangle_{\text{vac}}$ and subtract this from $\langle J_1 \rangle$, whence by (18),

$$\langle \delta J_1 \rangle = \int \eta_0^*(\mathbf{P}_i - \mathbf{l}) e^{-i(\mathbf{P}_i - \mathbf{l}, \mathbf{r})} f(\mathbf{l}) e^{i(\mathbf{l}, \mathbf{r})} \eta_0(\mathbf{P}_i) e^{i(\mathbf{P}_i, \mathbf{r})} d\mathbf{l} d\mathbf{P} \quad (24)$$

$$f(\mathbf{l}) = \left[\sum_{\mathbf{P}_v} \frac{(\text{vac} | -(\overline{\mathbf{A}}\mathbf{J}_1^{(v)}) | \mathbf{P}_v - \frac{\mathbf{l}}{2}, -\mathbf{P}_v - \frac{\mathbf{l}}{2}, \mathbf{l}^-) \times}{\left(-l - E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(v)} - E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(v)} + \frac{l}{2} \right) (E - l)} \right. \\ \left. \times \frac{(\mathbf{P}_v - \frac{\mathbf{l}}{2}, -\mathbf{P}_v - \frac{\mathbf{l}}{2} | J_1^{(v)} | \text{vac}) (\mathbf{P}_i - \mathbf{l}, \mathbf{l}^- | -(\overline{\mathbf{A}}\mathbf{J}_1^{(v)}) | \mathbf{P}_i)}{(-l - E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(v)} - E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(v)} + \frac{l}{2}) (E - l)} \right] \quad (25.1)$$

$$+ \left[\sum_{\mathbf{P}_v} \frac{(\text{vac} | J_1^{(v)} | \mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2}) (\mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2} | -\overline{\mathbf{A}}\mathbf{J}_1^{(v)} | \mathbf{l}^-)}{(E - E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(v)} - E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(v)}) (E - l)} \right. \\ \times \frac{(\mathbf{P}_i - \mathbf{l}, \mathbf{l}^- | -(\overline{\mathbf{A}}\mathbf{J}_1^{(i)}) | \mathbf{P}_i)}{(-l - E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(v)} - E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(v)} + \frac{l}{2}) (E - l)} \\ + \sum_{\mathbf{P}_v} \frac{(\text{vac} | J_1^{(v)} | \mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2}) (\mathbf{P}_i - \mathbf{l} | -(\overline{\mathbf{A}}\mathbf{J}_1^{(v)}) | \mathbf{P}_i, -\mathbf{l}^-)}{(E - E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(v)} - E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(v)}) (-E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(v)} - E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(v)} - l)} \times \\ \left. \times \frac{(\mathbf{P}_v + \frac{\mathbf{l}}{2} - \mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{l}^- | -(\overline{\mathbf{A}}\mathbf{J}_1^{(v)}) | \text{vac})}{(-l - E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(v)} - E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(v)} + \frac{l}{2}) (E - l)} \right] \quad (25.2)$$

$$+ \left[\sum_{\mathbf{P}_v} \frac{(\text{vac} | J_1^{(v)} | \mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2}) (\mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2} | \rho^{(v)} | \text{vac})}{(E - E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(v)} - E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(v)})} \times \right. \\ \left. \times \frac{(\mathbf{P}_i - \mathbf{l} | \rho^{(i)} | \mathbf{P}_i)}{(-l - E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(v)} - E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(v)} + \frac{l}{2}) (E - l)} \right] \quad (25.3)$$

+ Hermite conj,

$$+ f'(\mathbf{l}). \quad (25)$$

The first bracket of (25) is the contribution from $\epsilon^2 \Psi_1^* J \Psi_1$, while the second and third brackets are those from (23.1), (23.2) and (23.3) among the contributions from $\epsilon^2 \Psi_0 J \Psi_0$. The "Hermitian conjugate" does not imply that of the above four terms themselves, but that of the matrix elements of

$(\gamma_0^*(\mathbf{P}_i) | \gamma_0(\mathbf{P}_i - \mathbf{l}))$ corresponding to these four terms. This is evident from (24), in which we need only replace the E involved in the above four terms by $-E$. $f'(l, t)$ does not take the form $\sum_{\mathbf{P}_r}$, and we shall discuss this later. Hereunder we drop $f'(l, t)$ out.

We next transform the numerators appearing in (25), for which purpose we utilize the continuity equation.

$$d\mathbf{l}/dt \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (26)$$

whence

$$(\mathbf{P}_i - \mathbf{l} | \mathbf{l} \cdot \mathbf{J}_i^{(v)} | \mathbf{P}_i) = \dot{E}(\mathbf{P}_i - \mathbf{l} | \rho^{(v)} | \mathbf{P}_i), \quad (26')$$

$$\begin{aligned} \left(\mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2} | (\mathbf{l} \cdot \mathbf{J}_1^{(v)}) | \text{vac.} \right) &= (E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}} + E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}) \times \\ &\times \left(\mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2} | \rho^{(v)} | \text{vac.} \right) \end{aligned} \quad (26'')$$

$$\begin{aligned} (a) \cdots (\text{vac.} | (\overline{\mathbf{A}} \cdot \mathbf{J}_1^{(v)}) | \mathbf{P}_i - \frac{\mathbf{l}}{2}, -\mathbf{P}_v - \frac{\mathbf{l}}{2}, t^-) &\left(\mathbf{P}_v - \frac{\mathbf{l}}{2}, -\mathbf{P}_v - \frac{\mathbf{l}}{2} | J_1^{(v)} | \text{vac.} \right) \times \\ &\times (\mathbf{P}_i - \mathbf{l}, t^- | -(\overline{\mathbf{A}} \cdot \mathbf{J}_1^{(v)}) | \mathbf{P}_i) \\ &= \frac{2\pi}{l} (\text{vac.} | \mathbf{e}_1 \cdot \mathbf{J}^{(v)} | \mathbf{P}_v - \frac{\mathbf{l}}{2}, -\mathbf{P}_v - \frac{\mathbf{l}}{2}) \left(\mathbf{P}_v - \frac{\mathbf{l}}{2}, -\mathbf{P}_v - \frac{\mathbf{l}}{2} | J_1^{(v)} | \text{vac.} \right) \times \\ &\times (\mathbf{P}_i - \mathbf{l} | \mathbf{e}_1 \cdot \mathbf{J}_1^{(v)} | \mathbf{P}_i) \end{aligned} \quad (27)$$

where \mathbf{e}_1 is the unit vector in the direction of the vector potential \mathbf{A} , so that it is perpendicular to \mathbf{l} , while \mathbf{e} is the unit vector in the direction of \mathbf{l} .

But, in general,

$$(\text{vac.} | \mathbf{e} \cdot \mathbf{J}_1^{(v)} | \mathbf{P}_v - \frac{\mathbf{l}}{2}, -\mathbf{P}_v - \frac{\mathbf{l}}{2}) \propto \cos \varphi$$

and when considering the component of $J_\mu^{(v)}$ in the direction of \mathbf{e} , φ is involved in no other way besides this, so that it vanishes on being integrated over φ . This can be proved directly for various fields. Thus, (27) can be rewritten as follows:

$$\begin{aligned} \frac{2\pi}{l} (\text{vac.} | \mathbf{e} \cdot \mathbf{J}_1^{(v)} | \mathbf{P}_v - \frac{\mathbf{l}}{2}, -\mathbf{P}_v - \frac{\mathbf{l}}{2}) &\left(\mathbf{P}_v - \frac{\mathbf{l}}{2}, -\mathbf{P}_v - \frac{\mathbf{l}}{2} | \mathbf{e}_1 \cdot \mathbf{J}_1^{(v)} | \text{vac.} \right) \times \\ &\times (\mathbf{P}_i - \mathbf{l} | (\mathbf{e} \cdot \mathbf{J}_1^{(v)}) \mathbf{e}_1 | \mathbf{P}_i) \end{aligned} \quad (27')$$

$$\begin{aligned} (b) \cdots (\text{vac.} | J_1^{(v)} | \mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2}) &\left(\mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2} | \rho^{(v)} | \text{vac.} \right) \times \\ &\times (\mathbf{P}_i - \mathbf{l} | \rho^{(v)} | \mathbf{P}_i) \end{aligned} \quad (28)$$

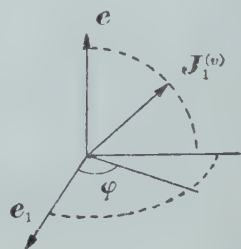


Fig. 2

Exactly as in (a), the contribution from the component $(\mathbf{e}_1 \mathbf{J}_1^{(n)}) \mathbf{e}$ of $\mathbf{J}^{(n)}$ vanishes on being integrated over all directions of \mathbf{P}_v . The contribution from $(\mathbf{e} \mathbf{J}_1) \mathbf{e}$ is, using (26') (26''):

$$\frac{1}{E} \left\{ E - \left(E - E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(n)} - E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(n)} \right) \right\} \left(\text{vac.} | \rho^{(n)} | \mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2} \right) \times \\ \times \left(\mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2} | \rho^{(n)} | \text{vac.} \right) (\mathbf{P}_i - \mathbf{l} | (\mathbf{e} \mathbf{J}_1^{(i)}) \mathbf{e} | \mathbf{P}_i) \quad (29)$$

Since the contribution to (25) from the second term in the bracket is proportional to $\frac{1}{E}$, it cancels out with the contribution from its Hermitian conjugate. Thus, the term among (29) which contributes to (25) is

$$\left(\text{vac.} | \rho^{(n)} | \mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2} \right) \left(\mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2} | \rho^{(n)} | \text{vac.} \right) \times \\ \times (\mathbf{P}_i - \mathbf{l} | (\mathbf{e} \mathbf{J}_1^{(i)}) \mathbf{e} | \mathbf{P}_i) \quad (28')$$

Also, the contributing term from $\mathbf{J}_1^{(n)} = \rho^{(n)}$ among (28) is:

$$\left(\text{vac.} | \rho^{(n)} | \mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2} \right) \left(\mathbf{P}_v + \frac{\mathbf{l}}{2}, -\mathbf{P}_v + \frac{\mathbf{l}}{2} | \rho^{(n)} | \text{vac.} \right) \times \\ \times (\mathbf{P}_i - \mathbf{l} | \rho^{(i)} | \mathbf{P}_i) \quad (28'')$$

(28) can thus be replaced by (28') + (28'').

Replacing each numerator in (25.1) and (25.2) by (27'), rearranging the denominators suitably, and adding the Hermitian conjugates, we obtain expressions of the form

$$\left\{ f_{11}(l) \frac{2l}{\square} + f_2(l) + f_3(l) \right\} (\mathbf{P}_i - \mathbf{l} | -(\mathbf{e}_1 \mathbf{J}_1^{(i)}) \mathbf{e}_1 | \mathbf{P}_i) \quad (30)$$

$$f_{11}(l) = \sum_{\mathbf{P}_v} \left[2 \left(E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}} + E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}} \right) \right] \left\{ l^2 - \left(E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(n)} + E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(n)} \right)^2 \right\} g_1(l) \quad (31)$$

$$f_2(l) = \sum_{\mathbf{P}_v} \left[-4l \left(E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(n)} + E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(n)} \right) \right] \left\{ l^2 - \left(E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(n)} + E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(n)} \right)^2 \right\} g_1(l) \quad (32)$$

$$f_3(l) = \sum_{\mathbf{P}_v} \frac{4l \left(E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}}^{(n)} + E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}}^{(n)} \right)}{\left\{ l^2 - \left(E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}} + E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}} \right)^2 \right\}} \cdot \frac{\square}{\left\{ l^2 - \left(E_{\mathbf{P}_v - \frac{\mathbf{l}}{2}} + E_{\mathbf{P}_v + \frac{\mathbf{l}}{2}} \right)^2 + \square \right\}} \times \\ \times g_1(l) \quad (33)$$

where

$$g_1(l) = \frac{2\pi}{l} \left(\text{vac.} | \mathbf{e}_1 \mathbf{J}_1^{(n)} | \mathbf{P}_v - \frac{\mathbf{l}}{2}, -\mathbf{P}_v - \frac{\mathbf{l}}{2} \right) \left(\mathbf{P}_v - \frac{\mathbf{l}}{2}, -\mathbf{P}_v - \frac{\mathbf{l}}{2} | \mathbf{e}_1 \mathbf{J}_1^{(n)} | \text{vac.} \right) \quad (34)$$

$$\square = E^2 - l^2. \quad (35)$$

Similarly replacing the numerators in (25.3) and its Hermitian conjugate by (28'), (28''), and suitably rearranging the denominators, we obtain

$$(f_2'(l) + f_2''(l)) (P_i - l - (eJ_1^{(0)} e | P_i) + (f_2'(l) + f_3'(l)) (P_i - l | \rho^{(i)} | P_i) \quad (36)$$

$$f_2'(l) = \sum_{P_v} \frac{2(E_{P_v}^{(v)} - \frac{l}{2} + E_{P_v}^{(v)} + \frac{l}{2})}{\{l^2 - (E_{P_v}^{(v)} - \frac{l}{2} + E_{P_v}^{(v)} + \frac{l}{2})^2\}} g_2(l) \quad (37)$$

$$f_3'(l) = \sum_{P_v} \frac{-2(E_{P_v}^{(v)} - \frac{l}{2} + E_{P_v}^{(v)} + \frac{l}{2})}{\{l^2 - (E_{P_v}^{(v)} - \frac{l}{2} + E_{P_v}^{(v)} + \frac{l}{2})^2\}} \cdot \frac{\square}{\{l^2 - (E_{P_v}^{(v)} - \frac{l}{2} + E_{P_v}^{(v)} + \frac{l}{2})^2 + \square\}} \times \\ \times g_2(l) \quad (38)$$

$$g_2(l) = i(\text{vac.} | \rho^{(n)} | P_v - \frac{l}{2}, -P_v - \frac{l}{2}) (P_v - \frac{l}{2}, -P_v - \frac{l}{2} | \rho^{(n)} | \text{vac.}) \quad (39)$$

$$= (\text{vec.} | I^{(m)} | \text{vac.})$$

and finally,

$$f(l, t) = \{f_1(l) + f_2(l) + f_3(l)\} (P_i - l | - (eJ_1^{(0)} e | P_i) \\ + \{f_2'(l) + f_3'(l)\} (P_i - l | - (eJ_1^{(0)} e | P_i) \\ + \{f_2'(l) + f_3'(l)\} (P_i - l | \rho^{(i)} | P_i) \\ + f'(l, t) \quad (40)$$

We next calculate $\langle J_2 \rangle = \langle J_2^{(i)} + J_2^{(n)} \rangle$, which is obtained from $i\psi_1^*(x)J(x)\psi_0(x)$ + Hermitian conj, in (17). Constructing $\langle J_2 \rangle$ and using (18)

$$\langle \delta J_2 \rangle = \left\{ \int \eta_0^* (P_i - l) e^{-i(P_i - l, r)} d(l, t) e^{i(lr)} \eta_0(P_i) e^{i(P_i, r)} dl dP \right. \\ \left. d(l, t) = \frac{(\text{vac.} | J_2^{(n)} | \bar{U}) (P_i - l, \bar{U} | - \bar{A} J_1^{(0)} | P_i) + \text{Hennite Conj.}}{E - l} \right. \\ \left. + d'(l, t) \right\} \quad (41)$$

The Hermitian conjugate is obtained simply by replacing E by $-E$ in the original expression.

$$(c) \dots (\text{vac.} | J_2^{(n)} | \bar{U}) (P_i - l, \bar{U} | \bar{A} J_1^{(0)} | P_i) \quad (42)$$

Considering the effect of vacuum, the following relation holds:

$$2(|A| \bar{U}) (\bar{U} | A) = (\bar{U} | AA | \bar{U}) \quad (43)$$

Further, $J_2^{(n)}$ is of the form $A\phi^*\phi$, if we write ϕ for the wave function of the

matter field, disregarding the Lorentz-transformability. As to $(\text{vac.} | J_2^{(v)} | \bar{\mathcal{U}})$, no component of $J_2^{(v)}$ other than that in the direction of \mathbf{A} makes any contribution, since the same components of ϕ and ϕ^* are involved in the form of a product, whereas these stand in a relation such that a particle pair created in ϕ is annihilated in ϕ^* . This fact can also be proved directly by constructing the matrix elements of $(\text{vac.} | (26) | \bar{\mathcal{U}})$, using the continuity equation (26), thus, (42) can be rewritten as

$$\frac{1}{2} (\bar{\mathcal{U}} | \overline{\mathbf{A} J_2^{(v)}} | \bar{\mathcal{U}}) (\mathbf{P}_i - \mathbf{l} | (\mathbf{e}_i J_1^{(v)}) \mathbf{e}_i | \mathbf{P}_i) \quad (42')$$

Hence

$$d(l, t) = f_{12}(l) (\mathbf{P}_i - \mathbf{l} | -(\mathbf{e}_i J_1^{(v)}) \mathbf{e}_i | \mathbf{P}_i) \frac{2l}{\square} \quad (44)$$

$$f_{12}(l) = (\bar{\mathcal{U}} | -\frac{1}{2} \overline{\mathbf{A} J_2^{(v)}} | \bar{\mathcal{U}}) = (\bar{\mathcal{U}} | e_2 H_2^{(v)} | \bar{\mathcal{U}}) \quad (45)$$

Substituting this in (41), adding (24), and using (14),

$$\begin{aligned} \langle \partial J \rangle &= \langle \partial J_1 \rangle + \langle \partial J_2 \rangle \\ &= \int \gamma_0^* (\mathbf{P}_i - \mathbf{l}) e^{-i(\mathbf{P}_i - \mathbf{l}, \mathbf{r})} \left[\left\{ (f_{11}(l) + f_{12}(l)) \frac{2l}{\square} + (f_2(l) + f_3(l)) \right\} \mathbf{e}_i \mathbf{J}_1^{(v)}(l) \right. \\ &\quad \left. + (f_2'(l) + f_3'(l)) (\mathbf{e} \mathbf{J}_1(l)) \mathbf{e} + (f_2'(l) + f_3'(l)) J_4(l) \right] e^{i(\mathbf{l} \cdot \mathbf{r} - Et)} \eta_0 e^{i(\mathbf{P}_i, \mathbf{r})} d\mathbf{l} d\mathbf{P} \end{aligned} \quad (46)$$

Of course $J_4(l) = \rho(l)$.

Performing an inverse Fourier transformation of (14)

$$J_\mu^{(v)}(l) = \int J_\mu^{(v)}(X') e^{-i(l, X')} dX' \quad (47^*)$$

Substituting this in (46),

$$\begin{aligned} \langle \partial J \rangle &= \phi_0^*(X) \partial J(X) \phi_0(X) \\ &= \phi_0^*(X) \iint \left[\left\{ (f_{11}(l) + f_{12}(l)) \frac{2l}{\square} + (f_2(l) + f_3(l)) \right\} (\mathbf{e}_i \mathbf{J}_1^{(v)}(X')) \mathbf{e}_i \right. \\ &\quad \left. + (f_2'(l) + f_3'(l)) (\mathbf{e} \mathbf{J}_1^{(v)}(X)) \mathbf{e} + (f_2'(l) + f_3'(l)) J_4^{(v)}(X') \right] e^{i(l, X - X')} dX' dl \phi_0(X) \\ &\quad + (\text{contributions from } f'(l, t) \text{ and } d'(l, t))^{\dagger} \end{aligned} \quad (48)$$

Where \square may be understood as the d'Alembertian-operator. (48) involves only $\partial J_{\mu 1}^{(v)}$ and no $\partial J_{\mu 2}^{(v)}$. This is because $J_{\mu 2}^{(v)}$ involves \mathbf{A} linearly, so that it has no matrix element describing a transition between two states neither of which has

* $\int dX'$ is an integration over the four-dimensional space coordinates (x_1, x_2, x_3, t) .

† We get the anomalous magnetic moment of the electron from $f'(l, t)$ and $d'(l, t)$ terms.

a photon. If we take a matrix element of δJ which binds two states, one having and the other not having a photon, the calculation runs on exactly similar lines as that in this argument, and it is readily seen that the displacement of $J_{\mu 2}^{(i)}$ has a coefficient equal to that of $\langle \delta J_{\mu 1}^{(i)} \rangle / J_{\mu 1}$ in (48).

(48) is the general formula for the current induced by a charged particle. Actual calculations for various types of particles will be reported elsewhere.

The results of these actual calculations show that the following relations hold for the cases of Fermi, scalar and vector charged mesons alike:

$$f_2(l) = f_2'(l) \quad (49)$$

$$f_3(l) = f_3'(l) \quad (50)$$

Consequently, in these cases, (48) reduces to

$$\begin{aligned} \delta J_{\mu} = & \iint [\{f_{11}(l) + f_{12}(l)\} \frac{2l}{\square} (\mathbf{e}_1 \mathbf{J}^{(i)}(x')) \mathbf{e}_1 \\ & + \{f_2(l) + f_3(l)\} J_{\mu}^{(i)}(x')] e^{i(l, x-x')} dx' dl \end{aligned} \quad (48)$$

(3) Concerning $f'(l, t)$ and $d'(l, t)$;

Hitherto we have not considered $f'(l, t)$ and $d'(l, t)$.

In these expressions, the integrations are not for the charged particle \mathbf{P}_r but for the photon \mathbf{l}' , which fact gives them a nature totally different from that of (48), so that they can be better accounted for by an effect similar to the self-energy of a charged particle. They take the forms:

$$\begin{aligned} f'(l, t) = & \sum_{\mathbf{l}'} \left[\frac{(\mathbf{P}_i - \mathbf{l} | cH^{(i)} | \mathbf{P}_i - \mathbf{l} - \mathbf{l}', \mathbf{l}' | (\mathbf{P}_i - \mathbf{l} - \mathbf{l}' | J_1^{(i)} | \mathbf{P}_i - \mathbf{l}' | (\mathbf{l}', \mathbf{P}_i - \mathbf{l}' | cH^{(i)} | \mathbf{P}_i)}{(E_{\mathbf{P}_i}^{(i)} - E_{\mathbf{P}_i - \mathbf{l}' - \mathbf{l}'}^{(i)}) (E_{\mathbf{P}_i - \mathbf{l}}^{(i)} - E_{\mathbf{P}_i - \mathbf{l} - \mathbf{l}' - \mathbf{l}'}^{(i)})} \right. \\ & + \frac{(\text{vac.} | cH^{(i)} | \mathbf{l}', \mathbf{P}_i, -\mathbf{P}_i - \mathbf{l}') (-\mathbf{P}_i - \mathbf{l}' | J_1^{(i)} | -\mathbf{P}_i + \mathbf{l} - \mathbf{l}') \times}{(-E_{\mathbf{P}_i}^{(i)} - E_{\mathbf{P}_i + \mathbf{l}' - \mathbf{l}'}^{(i)})} \\ & \left. \times \frac{(\mathbf{l}', \mathbf{P}_i - \mathbf{l}, -\mathbf{P}_i + \mathbf{l} - \mathbf{l}' | cH^{(i)} | \text{vac.})}{(-E_{\mathbf{P}_i - \mathbf{l}}^{(i)} - E_{\mathbf{P}_i - \mathbf{l} + \mathbf{l}' - \mathbf{l}'}^{(i)})} \right] \\ & \pm \sum_{\mathbf{l}'} \left[\frac{(\text{vac.} | cH^{(i)} | \mathbf{P}_i, -\mathbf{P}_i - \mathbf{l}', \mathbf{l}' | (\mathbf{P}_i - \mathbf{l} | J_1^{(i)} | \mathbf{P}_i) (\mathbf{P}_i, -\mathbf{P}_i - \mathbf{l}', \mathbf{l}' | cH^{(i)} | \mathbf{P}_i)}{(-E_{\mathbf{P}_i}^{(i)} - E_{\mathbf{P}_i - \mathbf{l}' - \mathbf{l}'}^{(i)})^2} \right. \\ & + \frac{(\text{vac.} | cH^{(i)} | \mathbf{P}_i - \mathbf{l}, -\mathbf{P}_i + \mathbf{l} - \mathbf{l}', \mathbf{l}' | (\mathbf{P}_i - \mathbf{l} | J_1^{(i)} | \mathbf{P}_i) \times}{(-E_{\mathbf{P}_i - \mathbf{l}}^{(i)} - E_{\mathbf{P}_i - \mathbf{l} + \mathbf{l}' - \mathbf{l}'}^{(i)})^2} \\ & \left. \times \frac{(\mathbf{P}_i - \mathbf{l}, -\mathbf{P}_i + \mathbf{l} - \mathbf{l}' | cH^{(i)} | \text{vac.})}{(-E_{\mathbf{P}_i - \mathbf{l}}^{(i)} - E_{\mathbf{P}_i - \mathbf{l} + \mathbf{l}' - \mathbf{l}'}^{(i)})^2} \right] \\ & - \sum_{\mathbf{l}'} \left[(\mathbf{P}_i - \mathbf{l} | J_1^{(i)} | \mathbf{P}_i) \left\{ \frac{(\mathbf{P}_i | cH^{(i)} | \mathbf{P}_i - \mathbf{l}', \mathbf{l}' | (\mathbf{P}_i - \mathbf{l}' | \mathbf{l}' | cH^{(i)} | \mathbf{P}_i)}{2(E_{\mathbf{P}_i}^{(i)} - E_{\mathbf{P}_i - \mathbf{l}' - \mathbf{l}'}^{(i)})^2} \right. \right. \end{aligned}$$

$$\begin{aligned}
& \pm \frac{(\text{vac.} | eH^{(i)} | \mathbf{P}_i, -\mathbf{P}_i - \mathbf{l}', \mathbf{l}'') (\mathbf{P}_i, -\mathbf{P}_i - \mathbf{l}', \mathbf{l}'') | eH^{(i)} | \text{vac.})}{2(-E_{\mathbf{P}_i} - E_{\mathbf{P}_i - \mathbf{l}' - \mathbf{l}''})^2} \Big\} \\
& + \text{Hermite Conjugates of those in which substitute } \mathbf{P}_i \text{ by } \mathbf{P}_i - \mathbf{l} \Big] \quad (51) \\
& + \sum_{\mathbf{l}'} \Big[\frac{(\mathbf{P}_i | eH^{(i)} | \mathbf{P}_i - \mathbf{l} - \mathbf{l}', \mathbf{l}'') (\mathbf{l}' | eH^{(i)} | \mathbf{P}_i, -\mathbf{P}_i + \mathbf{l}')} {E_{\mathbf{P}_i - \mathbf{l}} - E_{\mathbf{P}_i} - E_{\mathbf{P}_i - \mathbf{l}' - \mathbf{l}''}} \\
& \times \frac{(\mathbf{P}_i - \mathbf{l} - \mathbf{l}', -\mathbf{P}_i + \mathbf{l}' | J_1^{(i)} | \text{vac.})}{(E_{\mathbf{P}_i - \mathbf{l}} - E_{\mathbf{P}_i - \mathbf{l}' - \mathbf{l}''})} + \frac{(\text{vac.} | eH^{(i)} | \mathbf{P}_i, -\mathbf{P}_i - \mathbf{l}', \mathbf{l}')} {(-E_{\mathbf{P}_i} - E_{\mathbf{P}_i - \mathbf{l}' - \mathbf{l}''})} \\
& \times \frac{(\mathbf{P}_i - \mathbf{l}, \mathbf{l}' | eH^{(i)} | \mathbf{P}_i - \mathbf{l} + \mathbf{l}')}{(-E_{\mathbf{P}_i} + E_{\mathbf{P}_i - \mathbf{l}} - E_{\mathbf{P}_i - \mathbf{l}' - \mathbf{l}''} - E_{\mathbf{P}_i - \mathbf{l} + \mathbf{l}'})} | J_1^{(i)} | \text{vac.}) \\
& + \tau' \frac{(\mathbf{P}_i - \mathbf{l} | \rho^{(i)} | \mathbf{P}_i - \mathbf{l} - \mathbf{l}' \times \text{vac.} | \rho^{(i)} | \mathbf{P}_i - \mathbf{P}_i + \mathbf{l}')} {(E_{\mathbf{P}_i - \mathbf{l}} - E_{\mathbf{P}_i} - E_{\mathbf{P}_i - \mathbf{l}' - \mathbf{l}''} - E_{\mathbf{P}_i - \mathbf{l} + \mathbf{l}'})} \\
& \times (\mathbf{P}_i - \mathbf{l} - \mathbf{l}', -\mathbf{P}_i + \mathbf{l}' | J_1^{(i)} | \text{vac.}) + \text{Hermite conj.} \Big] \\
d'(l, t) = & \frac{(\mathbf{P}_i - \mathbf{l} | J_2^{(i)} | \mathbf{P}_i - \mathbf{l}', \mathbf{l}'') (\mathbf{P}_i - \mathbf{l}', \mathbf{l}'') | \overline{A} J_1^{(i)} | \mathbf{P}_i)} {(E_{\mathbf{P}_i} - E_{\mathbf{P}_i - \mathbf{l}' - \mathbf{l}''})} + \text{Hermite conj.} \\
& + \frac{(\text{vac.} | J_2^{(i)} | \mathbf{P}_i, -\mathbf{P}_i + \mathbf{l} - \mathbf{l}', \mathbf{l}'') (\mathbf{P}_i - \mathbf{l}, -\mathbf{P}_i + \mathbf{l} - \mathbf{l}', \mathbf{l}'') | \mathbf{A} J_1^{(i)} | \text{vac.})}{(-E_{\mathbf{P}_i - \mathbf{l}} - E_{\mathbf{P}_i - \mathbf{l} + \mathbf{l}' - \mathbf{l}''})} + \text{H.C.} \quad (52)
\end{aligned}$$

the double-sign in the first of which depends on whether the charged particle (i) is a Bose or Fermi particle in which cases the positive and negative signs respectively must be taken.

It is seen from their forms that these quantities are effects of the interaction between initial current $J^{(i)}$ and the electromagnetic field, which fact can also be seen from the following considerations: if we calculate the current induced by the electromagnetic field A_μ ($\square A_\mu = -4\pi J_\mu$) produced by $J^{(i)}$ instead of the current induced directly by $J^{(i)}$, the result is identical with (48) except that J_μ is replaced by $-\frac{1}{4\pi} \square A$. Therefore, (51) and (52) make no contribution. This is because we pay attention only to the electromagnetic field and have no regard for the nature of the charge acting as its source.

Hereunder we discuss only the quantity contributing to the current induced by an external electric field, that is, only (48).

We shall discuss (51) and (52) elsewhere.

§ 3. General Discussions Concerning Vacuum Polarization.

Let us now make some general discussions on the basis of the general formulae (48) and (48').

For the sake of convenience, we separate (48) into three parts:

$$\delta J = \delta J_a + \delta J_b + \delta J_c,$$

with

$$\delta J_a = \iint \{f_{11}(l) + f_{12}(l)\} \frac{2l}{\square} (\mathbf{e}_1 \cdot \mathbf{J}^{(i)}(X')) \mathbf{e}_1 e^{i(l, \mathbf{x} - \mathbf{x}')} dX' dl, \quad (53)$$

$$\delta J_b = \iint [f_2(l) (\mathbf{e}_1 \cdot \mathbf{J}^{(i)}(X')) \mathbf{e}_1 + f_2'(l) \{(\mathbf{e} \cdot \mathbf{J}^{(i)}(X')) \mathbf{e} + J_4^{(i)}(X)\}] e^{i(l, \mathbf{x} - \mathbf{x}')} dX' dl, \quad (54)$$

$$\delta J_c = \iint [f_3(l) (\mathbf{e}_1 \cdot \mathbf{J}^{(i)}(X')) \mathbf{e}_1 + f_2'(l) \{(\mathbf{e} \cdot \mathbf{J}^{(i)}(X')) \mathbf{e} + J_4^{(i)}(X')\}] e^{i(l, \mathbf{x} - \mathbf{x}')} dX' dl. \quad (55)$$

Because δJ_b does not involve the \square , and further because $f_2(l)$ and $f_2'(l)$ are, as shown by actual calculations in a separate paper, equal and constant (cf. (49)),

$$\delta J_b = \iint f_2(l) J^{(i)}(X') e^{i(l, \mathbf{x} - \mathbf{x}')} dX' dl \quad (54')$$

Consequently, δJ_b is proportional to $J^{(i)}$, and can be amalgamated into the latter. We shall call this the charge renormalization term (c.r.). It can be seen from (32) that $f_2(l)$ have the sign “—” independently of the spin-properties of the particle (v).

δJ_c cannot be amalgamated into $J^{(i)}$ because it involves the \square . Consequently, this is a quantity which should be observable, so we call it the observable current term: (o.c.).

δJ_a is proportional to the component of $J^{(i)}$ in the direction of \mathbf{e}_1 , and not to $J^{(i)}$ itself. Also, this involves the \square . Furthermore, as readily seen from (31) and (45), $f_{11}(l) + f_{12}(l)$ is equal to the self-energy W of the photon with momentum l . This shows that δJ_a is closely related with the self-energy of a photon. This relation is made even more evident if δJ_a is rewritten as follows.

$$\delta J_a = \int \frac{2l W \mathbf{A}_1(X')}{4\pi} e^{i(l, \mathbf{x} - \mathbf{x}')} dX' dl \quad (56)$$

where \mathbf{A}_1 is the transversal part of the electromagnetic field at the position of $J^{(i)}$.

This appears in the Hamiltonian function as a term of the form $\int \frac{lW}{2(2\pi)} \mathbf{A}_1^2 d\mathbf{v}$, which is just the mass term of a photon.

Hereunder we shall call this the mass-type current (m.c.), in contrast to

which the (*o.c.*) and (*c.r.*) will together be called the interaction term.

According to (31), (32) and (33), the following general relation is seen to hold for the order of divergence of δJ_a , δJ_b and δJ_c if the order of divergence of that self-energy $f_{11}(l)$ of the photon among δJ_a which is contributed through $eH^{(1)}$, is denoted by n , those of δJ_b and δJ_c are $n-2$ and $n-4$ respectively.

Consequently, if $n < 4$, the (*o.c.*) δJ_c does not diverge, but it does if $n=4$ and there results a divergence that cannot be amalgamated into the current.

For a Fermi or scalar charged particle, $n=2$, so that the (*c.r.*) δJ_b diverges logarithmically, while the (*o.c.*) δJ_c , remains finite.

For a vector charged particle, however, $n=4$, so that δJ_b and δJ_c diverge quadratically and logarithmically, respectively, and since the (*o.c.*) δJ_c cannot be amalgamated into $J^{(1)}$, this seems to present a new difficulty to the Tomonaga-Schwinger theory. We shall discuss this in detail upon the basis of actual calculations, in this issue.

This δJ in (48) will make contributions in elastic scattering as an effect of vacuum polarization. δJ will change the electromagnetic field \mathbf{A} by an amount $\delta \mathbf{A}$, so that, in the case of polarization due to an electromagnetic field entering from without, vacuum polarization will make its contribution through $\delta \mathbf{A}$. But in the case of Compton scattering, $\square \mathbf{A} = 0$, and a quantity corresponding to δJ_c will presumably not make appearance. In fact, the results of direct calculations of the Compton scattering in fourth order perturbation show that vacuum polarization here contributes only through $\frac{IWA^2}{2(2\pi)}$ and δJ_b .

§ 4. The Contribution of Vacuum Polarization in Elastic Scattering

We write only those processes in which vacuum polarization makes a contribution in the ϵ^4 -approximation when an incident particle \mathbf{k}^0 is scattered by a charged particle \mathbf{P}_i through the medium of a Coulomb force and a photon.

(Fig. 1): The horizontal axes are those of time, proceeding from left to right, and consequently the processes occur in this sequence. The arrows denote the directions of the transitions. Of the vertical arrows, those in broken, unbroken and undulating lines denote the incident particle, source-particle, and the particle created in an intermediate state, respectively. The undulating lines running simultaneously upward and downward from the line denoting vacuum, and those running to is imply pair-creation and pair-annihilation, respectively. The symbols \bigcirc and \otimes denote that the process involves an emission and absorption, respectively, of a photon. Processes not marked (+) are those involving the photon \mathbf{l} , while those marked (−) involve the photon $-\mathbf{l}$.

In processes involving the Coulomb potential V^{iv} or V^{ov} only once, enters only in that part of the above scheme where \bigcirc and \otimes or \bigcirc and \otimes are found adjacent to each other. The only process which involves the Coulomb potential



Fig. 1

twice is (i)'. Processes involving V^{00} need not be considered since they are effects of vacuum.

Fig. 1 shows only those processes in which the transition $P_i \rightarrow P_i - l$ occurs before $k^0 \rightarrow k^0 + l$.

Processes in which this sequence is reversed of course total exactly the same number as those given above.

(i), (viii) and a part of (ii) make contributions through the photonic mass term $\frac{lll'}{4\pi} A_1^2 \equiv a A_1^2$. (v) and a part of (iv) give the effect of annihilation the photons $l, -l'$, while (xi) and a part of (xii) that of creating the same, both through $a A_1^2$.

(ix), (x) and another part of (ii) can be collected to express the effect of $\partial \cdot J_h$.

(iii), (vi), and the remaining parts of (ii), (iv) and (xii) together express the effect of $\partial \cdot J_e$.

Processes involving V^{iv} or V^{ov} only once make no contributions, for the same reason as in the calculation of (27) in subsection (2) of § 2. (i)' produces the effect of ∂J_e .

Thus, the above thirteen processes can be collected into the following form:

$$\begin{aligned}
& \frac{(k^0 + l | eH_1^0 | k^0, \bar{l}) (P_i - l, \bar{l} | \sum_{k=a,b} A \delta J_k^{(1)} | P_i)}{E - l} \\
& + (P_i - l, k^0 + l | \tau J_4 \delta J_4 | P_i, k^0) \\
& + \frac{(k^0 + l | eH_1^0 | k^0, \bar{l}) (\bar{l} | a \bar{A}_1^2 | \bar{l}) (P_i - l, \bar{l} | eH_1^{(e)} | P_i)}{(E - l)^2} \\
& + \frac{(k^0 + l | eH_1^0 | k^0, \bar{l}) (P_i - l | eH_1^{(v)} | P_i, -\bar{l}) (\bar{l}, -\bar{l} | a \bar{A}_1^2 | \text{vac.})}{(E - l)(-2l)} \\
& + \frac{(\text{vac.} | a \bar{A}_1^2 | \bar{l}, -\bar{l}) (-\bar{l}, k^0 + l | eH_1^0 | k^0) (\bar{l}, P_i - l | eH_1^{(e)} | P_i)}{(-2l)(E - l)}
\end{aligned} \quad (57)$$

where $eH_1^{(0)}$ is the interaction, to the first order of e , between the incident charged particle k^0 and the photon. The last three terms of (57), involving the photonic mass-term added together with the contributions from the three corresponding terms (obtained by replacing E by $-E$) arising from the processes in which the sequence of the transitions $P_i \rightarrow P_i - l_i$ and $k \rightarrow k^0 + l$ is reversed, are readily collected into the form δJ_a .

Thus, the whole set of twenty-six processes (both sequences considered) finally reduces into the following form:

$$2l \frac{(k^0 + l | eH_1^0 | k^0) (P_i - l | \sum_{k=a,b,c} A \delta J_k^{(e)} | P_i)}{(E^2 - l^2)} + (P_i - l, k^0 + l | \tau J_4 \delta J_4 | P_i, k^0). \quad (57')$$

§ 5. Contribution of Vacuum-Polarization to Compton Scattering

In the following we give only those processes due to vacuum polarization in the fourth order perturbation calculations of Compton scattering.

Let us suppose that the incident photon l is absorbed by the charged particle P_i , and l_f is emitted in its place. We shall consider only the fourth order process corresponding to one of the four processes of Compton scattering which appear in the second order perturbation, viz.

$$P_i, \bar{l} \rightarrow P_i + l \rightarrow P_i + l - l_f, \bar{l}_f$$

The processes due to vacuum-polarization in this case include those due to \bar{l} and those to \bar{l}_f , but in the following we write only those due to \bar{l} , since the other cases are exactly similar.

The unbroken and broken lines denote the processes $P_i \rightarrow P_i + l$ and $P_i + l \rightarrow P_i + l - l_f$ respectively, while the undulating lines mean that a charged particle-pair is created or annihilated in the intermediate state. \bigcirc , \otimes denote the emission and absorption, respectively, of \bar{l} , while \bigcirc , \otimes those of $-\bar{l}$. The four processes, (i), ..., (iv) do not involve $-\bar{l}$, while the others do. Besides these, there are the processes involving the Coulomb potential, but, for the same

reason as in the calculations of § 2, 2 (27), these make no contributions. $e^2 H_2^{(n)}$ contributes to (i), (ii), (iv), (v), (vi), and (ix), that is, only in place where the transitions $P_i \rightarrow P_i + l$ and $P_i + l \rightarrow P_i + l - l_f$ are formed adjacent to each other. $e^2 H_2^{(w)}$ contributes to (i), (ii), (v), (vi), (vii), and (viii), that is, where the creation and annihilation of charged particle pairs occur consecutively.

From (i), (ii), (iii), (iv), and the corresponding processes due to l_f^- , we obtain

$$\left. \begin{aligned} & - \frac{(P_i + l - l_f, l_f^- | e H_1^{(i)} | P_i + l) (P_i + l | e H_1^{(i)} | P_i, l)}{2(E_{P_i} - E_{P_i - l} + l)^2} \times \\ & \quad \times [(\bar{l} | \bar{a} A_1^2 | \bar{l}) + (l_f^- | \bar{a}^2 A_1^2 | l_f^-)] \\ & + \frac{(P_i + l - l_f, l_f^- | e H_1^{(i)} | P_i + l) (P_i + l | \bar{J}_1^{(i)} \delta A_b | P_i)}{(E_{P_i} - E_{P_i + l} + l)} \\ & + (P_i + l - l_f, l_f^- | \bar{J}_1^{(i)} \delta A_b | P_i, l) \end{aligned} \right\} \quad (58)$$

where δA_b is the field change due to the current induced by the radiation field A , so that $\delta A_b = \frac{\langle \delta J_b \rangle}{\langle J \rangle} A = f_2(l) A$ [cf. (32)], (vii), (x), and parts of (vi) and (viii) give:

$$\left. \begin{aligned} & \frac{(\text{vac.} | \bar{a} A_1^2 | \bar{l}, -\bar{l})}{(-2l)} \left[\frac{(P_f, l_f^- | e H_1^{(i)} | P_i + l) (P_i + l, -\bar{l} | e H_1^{(i)} | P_i)}{(E - l)} \right. \\ & \quad \left. + (P_f, l_f^-, -\bar{l} | e^2 H_2^{(i)} | P_i) \right] \\ & + \frac{(P_f, \bar{l} | e H_1^{(w)} | P_i + l) (\text{vac.} | \bar{a} A_1^2 | \bar{l}, -\bar{l}) (P_i + l, -\bar{l} | e H_1^{(i)} | P_i)}{(E - l)(E + l)} \end{aligned} \right\} \quad (59)$$

This is an effect of the photonic mass-term αA_1^2 . Others than this cancel each other out.

Thus, the part δJ_c of δJ which involves the factor \square does not appear in Compton scattering, as was stated at the end of § 3.

The third term of (58) expresses the effect in which a charged particle P_i enters into the place where a current $\delta J_b^{(i)}$ was previously induced by \bar{l} , and is scattered by δJ_c .

However, both in the cases of (57) and (58), when we consider the effect of the (c.r.)-term δJ_b not as the scattering of a charged particle by the current δJ induced by the electromagnetic field, but as the renormalization of the charge in the interaction term of the Hamiltonian. We must subject the charge to the renormalization

$$e \rightarrow e \left(1 + \frac{\langle \delta J_b^{(i)} \rangle}{2 \langle J^{(i)} \rangle} \right) \quad (60)$$

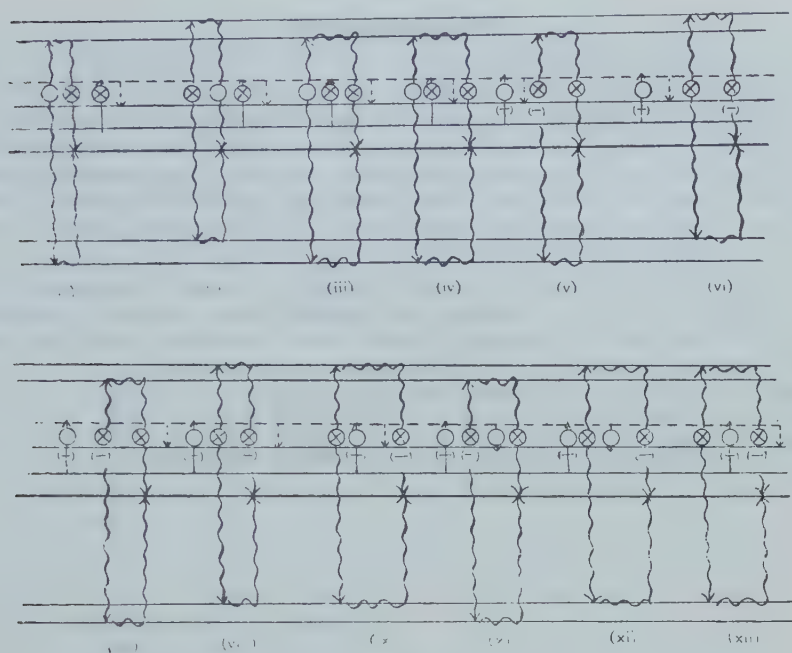


Fig. 2

that is, a renormalization with the factor $\frac{1}{2}$. This is because an alteration of e would affect not only $eH_1^{(i)}$, but eH_1^0 also, in (57'). Again, if we consider (57') as being a scattering due not directly to the current $\mathbf{J}^{(i)}$ but to the external field produced by it, the potential of the scattering field is proportional to e^2 , so that

$$e^2 \left(1 + \frac{\langle \partial \cdot \mathbf{J}_h^{(i)} \rangle}{\langle \mathbf{J} \rangle} \right) = e^2 \left(1 + \frac{\langle \partial \cdot \mathbf{J}_h^{(i)} \rangle}{2 \langle \mathbf{J} \rangle} \right)^2$$

showing the appropriateness of the renormalization (60).

In conclusion, we would like to express our sincere thanks to Prof. S. Sakata and Dr. Y. Tanikawa for their interest in this work.

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Note added in proof

We have two methods in mixture theory for interaction terms in vacuum polarization as follows: (a) mixture of various charged particles, (b) altering the interaction between charged particles and the electromagnetic field, (a) is impossible to remove the divergencies of (c.r.) and (o.c.) from the discussions in this paper. In (b) we must use an interaction which contains higher derivative than usual current. In this case higher derivative in the new current gives a more factor \square on the whole, so the first term of the general formula becomes (c.r.) and another terms (o.c.), and the self-energy of photon by this term is 0. As now $e^2 H_2$ can contribute to (c.r.), the sign of (c.r.) might be positive. Then, however, the order of the divergence of (c.r.) and (o.c.) is higher by two order than (c.r.) and (o.c.) from usual interaction respectively. Of course, there are many different points in the second quantization arising from the time derivative ∂_4 in the interactions, it is difficult to apply the general formula directly, and so discussion in this note based on the general formula is only a anticipation, but we can not expect to remove the divergences of (c.r.) and (o.c.) simultaneously by the method (b).

Recently D. Feldman studied actually the vacuum polarization considering the tensor coupling. His results are in agreement with our anticipation in this work. We would express our gratitude to Dr. D. Feldman for his kindness to send me his paper on the vacuum polarization.

Vacuum Polarization due to Various Charged Particles.

Hiroomi UMEZAWA and Rokuo KAWABE.

Institute of theoretical Physics, Nagoya University.

(Received June 9, 1949)

§ 1. Introduction.

We shall synthetically investigate the multifarious divergence difficulties appearing in vacuum polarization by considering the existence of various charged particles, thus observing the extent to which measures of dissolving them may be at the present stage. We shall also compare the result with those previously obtained by various methods, especially the Tomonaga-Schwinger theory, for the vacuum polarization due to an electron.

We previously obtained general formulae in vacuum polarization which did not depend on the type of charged particles.* We shall now apply these formulae to the cases of vacuum polarization due to charged vector, scalar and Fermi particles. The notations are the same as used in Paper I. When citing equations from (1) we shall affix "I" to the number of the equation.

Let a current $J_\mu^{(i)} = (\mathbf{J}^{(i)}, \rho^{(i)})$ exist in vacuum. The current due to the charged particles created in the intermediate state will be written as $J_\mu^{(v)} = (\mathbf{J}^{(v)}, \rho^{(v)})$. Hereunder (i) , (v) will always denote particles existing in vacuum in the initial state and created in the intermediate state, respectively.

In order to perform the calculations in such a way as not to destroy the relativistic covariance, we must determine an integration-domain which is relativistically invariant, as we have stated elsewhere.**

In vacuum polarization, a pair of charged particles with momenta $\mathbf{P} + \frac{\mathbf{l}}{2}$ and $-\mathbf{P} + \frac{\mathbf{l}}{2}$ respectively is created in the intermediate state. Therefore, we need only prescribe the integration-domain to be the interior of a closed surface on which the scalar product of the energy-momentum four-vectors of these two charged particles, $(P \cdot P')$ takes a constant value and subsequently allow this

* II. Umezawa and R. Kawabe; Prog. Theor. Phys. in this issue.
Hereunder referred to as Paper I.

** II. Umezawa and R. Kawabe; Prog. Theor. Phys. in this issue.
Hereunder referred to as Paper II.

domain to become infinitely large. For that purpose, we must transform our variable into v by the following relation:

$$\left\{ \left(E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}} \right)^2 - l^2 \right\} = 2(P, P') + 2\mu^2 = \frac{4\mu^2}{1-v^2} \quad (1)^*$$

The integration-domain for v is $0 \leq v < 1$. We go over from the system of variables $|\mathbf{P}|$, θ , φ (cf., Fig. 1) to v , θ , φ . Then

$$d\rho = \frac{4\mu^2 2E_{\mathbf{P}-\frac{\mathbf{l}}{2}} E_{\mathbf{P}+\frac{\mathbf{l}}{2}}}{\rho \left[\frac{4\mu^2}{1-v^2} + l^2(1-x^2) \right]} \frac{v dv}{(1-v^2)^2} \quad x \equiv \cos \theta \quad (2)$$

$$\rho^0 = \frac{\left\{ \frac{4\mu^2}{1-v^2} + l^2 \right\}}{\left\{ \frac{4\mu^2}{1-v^2} + l^2(1-x^2) \right\}} \frac{\mu^2 v^2}{1-v^2} \quad (3)$$

We write here those formulae in Paper I which we make especial use of in the following.

$$\delta J = \iint \left[\left\{ (f_{11}(l) + f_{12}(l)) \frac{2l}{\square} + (f_2(l) + f_3(l)) \right\} (e_1 \mathbf{J}^{(4)}(X') e_1 + (f_2'(l) + f_3'(l)) (e \mathbf{J}^{(4)}(X')) e + (f_2'(l) + f_3'(l)) J_4^{(4)}(X') \right] e^{i(l, X-X')} dX' dl$$

$$f_{11}(l) = \sum_{\mathbf{P}} \frac{2(E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})}{\{l^2 - (E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})^2\}} g_1(l) \quad (31\cdot I)$$

$$f_{12}(l) = (\mathbf{U} | e^2 H_2^2 | \mathbf{U}) \quad (\mathbf{U} \dots \text{photon}) \quad (45\cdot I)$$

$$f_2(l) = \sum_{\mathbf{P}} \frac{-4l(E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})}{\{l^2 - (E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})^2\}} g_1(l) \quad (32\cdot I)$$

$$f_3(l) = \sum_{\mathbf{P}} \frac{4l(E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})}{\{l^2 - (E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})^2\}} \frac{\square}{\{l^2 - (E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})^2 + \square\}} g_1(l) \quad (33\cdot I)$$

$$f_2'(l) = \sum_{\mathbf{P}} \frac{2(E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})}{\{l^2 - (E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})^2\}} g_2(l) \quad (37\cdot I)$$

* The reason we did not take (P, P') itself as v , but choose complex expression like (1) is, as will later be seen to facilitate comparison with Schwinger's result for an electron.

$$f_1'(l) = \sum_{\mathbf{P}} \frac{-2(E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})}{\{l^2 - (E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})^2\} \{l^2 - (E_{\mathbf{P}-\frac{\mathbf{l}}{2}} + E_{\mathbf{P}+\frac{\mathbf{l}}{2}})^2 + \square\}} g_2(l) \quad (38 \cdot I)$$

$$g_1(l) = \frac{2\pi}{l} \left(\text{vac.} | e_1 \mathbf{J}_1^{(n)} | \mathbf{P} - \frac{\mathbf{l}}{2}, -\mathbf{P} - \frac{\mathbf{l}}{2} \right) \left(\mathbf{P} - \frac{\mathbf{l}}{2}, -\mathbf{P} - \frac{\mathbf{l}}{2} | e_1 \mathbf{J}_1^{(n)} | \text{vac.} \right) \quad (34 \cdot I)$$

$$g_2(l) = i \left(\text{vac.} | \rho^{(n)} | \mathbf{P} - \frac{\mathbf{l}}{2}, -\mathbf{P} - \frac{\mathbf{l}}{2} \right) \left(\mathbf{P} - \frac{\mathbf{l}}{2}, -\mathbf{P} - \frac{\mathbf{l}}{2} | \rho^{(n)} | \text{vac.} \right) \\ = (\text{vac.} | V^{(nn)} | \text{vac.}) \quad (39 \cdot I)$$

$$\square = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial t^2}$$

$V^{(nn)} = e_1 \rho^{(n)} \rho^{(n)} = \frac{\rho^{(n)} \rho^{(n)}}{r}$ is the Coulomb potential of the charged particles appearing in the intermediate state. e_1 is the unit vector lying in the plane determined by \mathbf{J} and \mathbf{l} and perpendicular to the latter.

According to these, we see that the factors $f_{11}(l)$, $f_2(l)$, $f_3(l)$, determining the mass term of the photon due to $eH_1^{(n)}$, the charge renormalization term (c.r) $\delta \mathbf{J}_b$, and the observed current term (o.c) $\delta \mathbf{J}_c$ respectively are given by the following equations. Writing for the result of the integration over θ , φ .

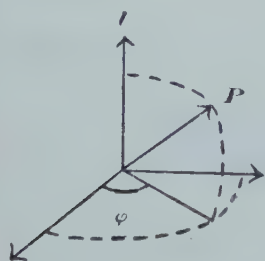


Fig. 1.

$$f_1(l) = \int_0^1 G(v) dv \quad (4)$$

$$f_{11}(l) = \frac{1}{2l} \int_0^1 G(v) \frac{4\mu^2}{1-v^2} dv \quad (5)$$

$$f_3(l) = \frac{\square}{4\mu^2} \int_0^1 G(v) \frac{(1-v^2)}{\left\{ 1 - \frac{\square}{4\mu^2} (1-v^2) \right\}} dv \quad (6)$$

Similary, if we can write

$$f_2'(l) = \int_0^1 G'(v) dv \quad (7)$$

$$f_3'(l) = \frac{\square}{4\mu^2} \int_0^1 G'(v) \frac{(1-v^2)}{\left\{ 1 - \frac{\square}{4\mu^2} (1-v^2) \right\}} dv \quad (8)$$

We next write the interactions between spinor, scalar and vector fields and the electromagnetic field.

i) Interaction between a charged spinor field ψ and an electromagnetic field* $H^{(e)} = cH_1^{(e)} + c^2H_2^{(e)} + V^{(e,t)}$

$$cH_1^{(e)} = -c \int \psi^* \boldsymbol{\alpha} \boldsymbol{A} \psi, \quad c^2H_2^{(e)} = 0, \quad V^{(e)} = \frac{\rho^{(e)} \rho^{(t)}}{r} \quad (9)$$

$$J^{(e)} = (\mathbf{J}^{(e)}, \rho^{(e)}), \quad \mathbf{J}^{(e)} = \psi^* \boldsymbol{\alpha} \psi, \quad \rho^{(e)} = \psi^* \psi \quad (10)$$

$$\mathbf{A} = \sum_{\mathbf{l}, \mu} \sqrt{\frac{2\pi}{l}} \mathbf{e}(\mathbf{l}, \mu) \left\{ e^{i(\mathbf{l}, \mathbf{r})} C_{\mathbf{l}\mu} + e^{-i(\mathbf{l}, \mathbf{r})} C_{\mathbf{l}\mu}^* \right\} \quad (11)$$

μ denotes the direction of polarization.

ii) Interaction between a charged scalar (or pseudoscalar) field U and an electromagnetic field. $H^{(s)} = cH_1^{(s)} + c^2H_2^{(s)} + V^{(s,t)}$

$$cH_1^{(s)} = \frac{i\epsilon}{4\pi} \int \mathbf{A} (U^* \text{grad } U - \text{grad } U \cdot U^*), \quad c^2H_2^{(s)} = \frac{c^2}{4\pi} U^* U, \quad V^{(s,t)} = \frac{\rho^{(s)} \rho^{(t)}}{r} \quad (12)$$

$$U = i \sum_{\mathbf{P}} \sqrt{\frac{2\pi}{E_{\mathbf{P}}}} (C_{\mathbf{P}} - d_{\mathbf{P}}^*) e^{i(\mathbf{P}, \mathbf{r})}, \quad U^* = \sum_{\mathbf{P}} \sqrt{\frac{E_{\mathbf{P}}}{8\pi}} (C_{\mathbf{P}}^* + d_{\mathbf{P}}) e^{-i(\mathbf{P}, \mathbf{r})} \quad (13)$$

U^\dagger is the momentum conjugate to the wave function U of the charged scalar field. Therefore, expanding (12) in a Fourier series,

$$cH_1^{(s)} = - \int \mathbf{A} \mathbf{J}_1^{(s)} = - \frac{i\epsilon}{4\pi} (2\pi)^{3/2} \sum_{\substack{\mathbf{P}, \mathbf{P}', \\ \mathbf{l}, \mu}} \frac{2i(\mathbf{P}', \mathbf{e}(\mathbf{l}, \mu))}{\sqrt{E_{\mathbf{P}} E_{\mathbf{P}'} l}} \left[(C_{\mathbf{P}}^* - d_{\mathbf{P}}) (C_{\mathbf{P}'} - d_{\mathbf{P}'}^*) \right. \\ \left. \times \{ C_{\mathbf{l}\mu}^* \delta(-\mathbf{P} + \mathbf{P}' - \mathbf{l}) + C_{\mathbf{l}\mu} \delta(-\mathbf{P} + \mathbf{P}' + \mathbf{l}) \} \right] \quad (14)$$

$$c^2H_2^{(s)} = - \frac{1}{2} \int \mathbf{A} \mathbf{J}_2^{(s)} = \left(\frac{c^2}{4\pi} \right) (2\pi)^2 \sum_{\substack{\mathbf{P}, \mathbf{P}', \mathbf{l}, \\ \mathbf{l}', \mu, \mu'}} \frac{(C_{\mathbf{P}}^* - d_{\mathbf{P}}) (C_{\mathbf{P}'} - d_{\mathbf{P}'}^*)}{\sqrt{E_{\mathbf{P}} E_{\mathbf{P}'} l l'}} \\ \times \{ C_{\mathbf{l}\mu}^* C_{\mathbf{l}'\mu'}^* \delta(-\mathbf{P} + \mathbf{P}' - \mathbf{l} - \mathbf{l}') + C_{\mathbf{l}\mu}^* C_{\mathbf{l}'\mu'}^* \delta(-\mathbf{P} + \mathbf{P}' - \mathbf{l} + \mathbf{l}') \\ + C_{\mathbf{l}\mu} C_{\mathbf{l}'\mu'}^* \delta(-\mathbf{P} + \mathbf{P}' + \mathbf{l} - \mathbf{l}') + C_{\mathbf{l}\mu} C_{\mathbf{l}'\mu'}^* \delta(-\mathbf{P} + \mathbf{P}' + \mathbf{l} + \mathbf{l}') \} \quad (15)$$

$$J^{(s)} = (\mathbf{J}^{(s)}, \rho^{(s)})$$

$$\mathbf{J}^{(s)} = \mathbf{J}_1^{(s)} + \mathbf{J}_2^{(s)} = - \frac{i\epsilon}{4\pi} (U^* \text{grad } U - \text{grad } U^* \cdot U) - \frac{c^2}{2\pi} \mathbf{A} U^* U \quad (16)$$

$$\rho^{(s)} = -i\epsilon (U^\dagger U - U^\dagger * U^*) = \frac{\epsilon}{2} \sum_{\mathbf{P}, \mathbf{P}'} \left[(C_{\mathbf{P}}^* C_{\mathbf{P}'} - d_{\mathbf{P}} d_{\mathbf{P}'}^*) \left(\sqrt{\frac{E_{\mathbf{P}}}{E_{\mathbf{P}'}}} + \sqrt{\frac{E_{\mathbf{P}'}}{E_{\mathbf{P}}}} \right) \right]$$

* Hereunder we take $\hbar = c = 1$.

We denote the scalar product and vector product of two vectors \mathbf{A}, \mathbf{B} as (\mathbf{A}, \mathbf{B}) and $[\mathbf{A}, \mathbf{B}]$, respectively.

$$-(C_{\mathbf{P}}^* C_{\mathbf{P}'}^* - d_{\mathbf{P}} C_{\mathbf{P}'}) \left(\sqrt{\frac{E_{\mathbf{P}}}{E_{\mathbf{P}'}}} - \sqrt{\frac{E_{\mathbf{P}'}}{E_{\mathbf{P}}}} \right) e^{-i(\mathbf{P}-\mathbf{P}', r)} \quad (17)$$

iii) Interaction between a charged vector (or pseudo-vector) particle $U_{\mu} = (\mathbf{U}, U_0)$ and an electromagnetic field. $H^{(v)} = eH_1^{(v)} + e^2 H_2^{(v)} + V^{(vi)}$

$$eH_1^{(v)} = 4\pi ic (\mathbf{U}^\dagger \mathbf{A} \operatorname{div} \mathbf{U}^{*\dagger} - \mathbf{U}^{*\dagger} \operatorname{div} \mathbf{U}^\dagger) + \frac{i}{4\pi \mu^2} \{ [\mathbf{A} \mathbf{U}^*] \operatorname{Curl} \mathbf{U} - [\mathbf{A} \mathbf{U}] \operatorname{Curl} \mathbf{U}^* \} \quad (18)$$

$$e^2 H_2^{(v)} = 4\pi e^2 (\mathbf{A} \mathbf{U}^\dagger) (\mathbf{A} \mathbf{U}^{*\dagger}) + \frac{e^2}{4\pi \mu^2} [\mathbf{A} \mathbf{U}] [\mathbf{A} \mathbf{U}^*] \quad (19)$$

$$\left. \begin{aligned} \mathbf{U} &= \sum_{\mathbf{P}} \left\{ \frac{i\sqrt{2\pi}}{\sqrt{E_{\mathbf{P}}}} \frac{\mu}{\lambda=1,2} (a_{\mathbf{P},\lambda} - b_{\mathbf{P},\lambda}^*) \mathbf{e}(\mathbf{P}, \lambda) + \sqrt{2\pi E_{\mathbf{P}}} (A_{\mathbf{P}} + B_{\mathbf{P}}^*) \mathbf{e}(\mathbf{P}) \right\} e^{i(\mathbf{P}, r)} \\ \mathbf{U}^\dagger &= \sum_{\mathbf{P}} \left\{ \frac{\sqrt{E_{\mathbf{P}}}}{\sqrt{8\pi\mu}} \frac{\lambda=1,2}{(a_{\mathbf{P},\lambda}^* + b_{\mathbf{P},\lambda}) \mathbf{e}(\mathbf{P}, \lambda) + \frac{i}{\sqrt{8\pi E_{\mathbf{P}}}}} \right. \\ &\quad \left. \times (A_{\mathbf{P}}^* - B_{\mathbf{P}}) \mathbf{e}(\mathbf{P}) \right\} e^{-i(\mathbf{P}, r)} \end{aligned} \right\} \quad (20)$$

\mathbf{U}^\dagger is the momentum conjugate to \mathbf{U} , while $\mathbf{e}(\mathbf{P}, \lambda)$ and $\mathbf{e}(\mathbf{P})$ are the unit vectors in the directions of polarization and propagation, respectively, of the charged vector particle of momentum \mathbf{P} .

$$eH_1^{(v)} = - \int \mathbf{A} \mathbf{J}_1^{(v)} = - \frac{c}{2} \sum_{\mathbf{P}, \mathbf{P}', \mathbf{l}} \sqrt{\frac{2\pi}{l}} \{ q_l \delta(\mathbf{P} - \mathbf{P}' - \mathbf{l}) + q_l^* (\mathbf{P} - \mathbf{P}' + \mathbf{l}) \} \\ \times \left\{ \sum_{\lambda} \frac{iP'}{\mu} \sqrt{\frac{E_{\mathbf{P}}}{E_{\mathbf{P}'}}} (a_{\mathbf{P},\lambda}^* + b_{\mathbf{P},\lambda}) (-A_{\mathbf{P}'} + B_{\mathbf{P}'}^*) (\mathbf{e}(\mathbf{l}, \mu) \mathbf{e}(\mathbf{P}, \lambda)) \right\} \quad (A_1)$$

$$+ \frac{P'}{\sqrt{E_{\mathbf{P}} E_{\mathbf{P}'}}} (-A_{\mathbf{P}}^* + B_{\mathbf{P}}) (-A_{\mathbf{P}'} + B_{\mathbf{P}'}^*) (\mathbf{e}(\mathbf{l}, \mu), \mathbf{e}(\mathbf{P})) \quad (A_2)$$

$$+ \sum_{\lambda, \lambda'} \frac{P'}{\sqrt{E_{\mathbf{P}} E_{\mathbf{P}'}}} (-a_{\mathbf{P},\lambda}^* + b_{\mathbf{P},\lambda}) (-a_{\mathbf{P}',\lambda'} + b_{\mathbf{P}',\lambda'}^*) ([\mathbf{e}(\mathbf{l}, \mu), \mathbf{e}(\mathbf{P}, \lambda)] [\mathbf{e}(\mathbf{P}'), \mathbf{e}(\mathbf{P}', \lambda')]) \quad (A_3)$$

$$+ \sum_{\lambda'} \frac{iP'}{\mu} \sqrt{\frac{E_{\mathbf{P}}}{E_{\mathbf{P}'}}} (A_{\mathbf{P}}^* + B_{\mathbf{P}}) (a_{\mathbf{P}',\lambda'} - b_{\mathbf{P}',\lambda'}^*) ([\mathbf{e}(\mathbf{l}, \mu), \mathbf{e}(\mathbf{P})] [\mathbf{e}(\mathbf{P}'), \mathbf{e}(\mathbf{P}', \lambda')]) \quad (A_4)$$

+ Hermite Conj.

(21)

We denote the four terms in brackets by $(A_1), \dots, (A_4)$.

$$\begin{aligned}
e^2 H_2^{(1)} = & -\frac{1}{2} \int \mathbf{A} \mathbf{J}_2^{(1)} = \sum_{\mathbf{l}\mu} \sum_{\mathbf{l}'\mu'} \sum_{\mathbf{P}, \lambda} \sum_{\mathbf{P}', \lambda'} \frac{\pi}{\sqrt{l l'}} \left[\left\{ \frac{\sqrt{E_{\mathbf{P}} E_{\mathbf{P}'}}}{\mu^2} (a_{\mathbf{P}\lambda}^* + b_{\mathbf{P}\lambda}) (a_{\mathbf{P}'\lambda'} + b_{\mathbf{P}'\lambda}') \right. \right. \\
& \times (e(\mathbf{l}\mu) e(\mathbf{P}, \lambda) (e(\mathbf{l}'\mu') e(\mathbf{P}', \lambda')) - \frac{i}{\mu} \sqrt{\frac{E_{\mathbf{P}}}{E_{\mathbf{P}'}}} (a_{\mathbf{P}\lambda}^* + b_{\mathbf{P}\lambda}) (A_{\mathbf{P}'} - B_{\mathbf{P}'}) \\
& \times (e(\mathbf{l}\mu) e(\mathbf{P}, \lambda)) (e(\mathbf{l}'\mu') e(\mathbf{P}')) + \frac{i}{\mu} \sqrt{\frac{E_{\mathbf{P}'}}{E_{\mathbf{P}}}} (A_{\mathbf{P}}^* - B_{\mathbf{P}}) (a_{\mathbf{P}'\lambda'} + b_{\mathbf{P}'\lambda}') \\
& \times (e(\mathbf{l}\mu) e(\mathbf{P})) (e(\mathbf{l}', \mu') e(\mathbf{P}', \lambda')) + \frac{1}{\sqrt{E_{\mathbf{P}} E_{\mathbf{P}'}}} (A_{\mathbf{P}}^* - B_{\mathbf{P}}) (A_{\mathbf{P}'} - B_{\mathbf{P}'}) \\
& \left. \left. \times (e(\mathbf{l}\mu) e(\mathbf{P})) (e(\mathbf{l}'\mu') e(\mathbf{P}')) \right\} \right] \quad (B_1)
\end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{1}{\sqrt{E_{\mathbf{P}} E_{\mathbf{P}'}}} (a_{\mathbf{P}'\lambda'}^* - b_{\mathbf{P}'\lambda'}) (a_{\mathbf{P}\lambda}^* - b_{\mathbf{P}\lambda}) ([e(\mathbf{l}\mu) e(\mathbf{P}', \lambda')] e(\mathbf{l}', \mu') e(\mathbf{P}, \lambda)) \right\} \\
& + \frac{i}{\mu} \sqrt{\frac{E_{\mathbf{P}}}{E_{\mathbf{P}'}}} (a_{\mathbf{P}'\lambda'}^* - b_{\mathbf{P}'\lambda'}) (A_{\mathbf{P}}^* + B_{\mathbf{P}}) ([e(\mathbf{l}\mu), e(\mathbf{P}'\lambda')] [e(\mathbf{l}', \mu'), e(\mathbf{P})]) \\
& - \frac{i}{\mu} \sqrt{\frac{E_{\mathbf{P}'}}{E_{\mathbf{P}}}} (A_{\mathbf{P}'} + B_{\mathbf{P}'}^*) (a_{\mathbf{P}\lambda}^* - b_{\mathbf{P}\lambda}) ([e(\mathbf{l}\mu) e(\mathbf{P}')] [e(\mathbf{l}', \mu'), e(\mathbf{P}, \lambda)]) \\
& + \frac{\sqrt{E_{\mathbf{P}} E_{\mathbf{P}'}}}{\mu^2} (A_{\mathbf{P}'} + B_{\mathbf{P}'}^*) (A_{\mathbf{P}}^* + B_{\mathbf{P}}) ([e(\mathbf{l}\mu), e(\mathbf{P}')] [e(\mathbf{l}'\mu') e(\mathbf{P})]) \left. \right\} \quad (B_2)
\end{aligned}$$

$$\begin{aligned}
& \times \{ C_{\mathbf{l}\mu}^* C_{\mathbf{l}'\mu'}^* \delta(-\mathbf{P} + \mathbf{P}' - \mathbf{l} - \mathbf{l}') + C_{\mathbf{l}\mu}^* C_{\mathbf{l}'\mu'} \delta(-\mathbf{P} + \mathbf{P}' - \mathbf{l} + \mathbf{l}') \\
& + C_{\mathbf{l}\mu} C_{\mathbf{l}'\mu'}^* \delta(-\mathbf{P} + \mathbf{P}' + \mathbf{l} - \mathbf{l}') + C_{\mathbf{l}\mu} C_{\mathbf{l}'\mu'} \delta(-\mathbf{P} + \mathbf{P}' + \mathbf{l} + \mathbf{l}') \} \quad (22)
\end{aligned}$$

The first and second terms in square brackets are denoted by (B₁), (B₂) respectively.

Current $\mathbf{J}^{(v)} = (\mathbf{J}^{(v)}, \rho^{(v)})$

$$\begin{aligned}
\mathbf{J}^{(v)} = & -\frac{\partial H^{(v)}}{\partial \mathbf{A}} = \mathbf{J}_1^{(v)} + \mathbf{J}_2^{(v)} = -4\pi i c (\mathbf{U}^\dagger \operatorname{div} \mathbf{U}^{* \dagger} - \mathbf{U}^{* \dagger} \operatorname{div} \mathbf{U}^\dagger) - \frac{i c}{4\pi \mu^2} \{ [\mathbf{U}^*, \operatorname{Curl} \mathbf{U}] \\
& - [\mathbf{U}, \operatorname{Curl} \mathbf{U}^*] \} - 4\pi c^2 \{ \mathbf{U}^\dagger (\mathbf{A} \mathbf{U}^{* \dagger}) + (\mathbf{A} \mathbf{U}^\dagger) \mathbf{U}^{* \dagger} \} + \frac{c^2}{4\pi \mu^2} \{ [\mathbf{U}, [\mathbf{A}, \mathbf{U}^*]] \\
& + [\mathbf{A}, \mathbf{U}], \mathbf{U}^* \} \quad (23)
\end{aligned}$$

$$\rho^{(v)} = -i c (\mathbf{U}^\dagger \mathbf{U} - \mathbf{U}^{* \dagger} \mathbf{U}^*) = -i c \sum_{\mathbf{P}, \mathbf{P}'} \left[-\frac{i}{2} \left\{ \sqrt{\frac{E_{\mathbf{P}'}}{E_{\mathbf{P}}}} (a_{\mathbf{P}, \lambda} - b_{\mathbf{P}, \lambda}^*) (a_{\mathbf{P}'\lambda'}^* + b_{\mathbf{P}'\lambda'}) \right. \right.$$

$$\begin{aligned}
 & + \sqrt{\frac{\bar{E}_{\mathbf{P}}}{E_{\mathbf{P}'}}} (a_{\mathbf{P}'\lambda'}^* - b_{\mathbf{P}'\lambda'}) (a_{\mathbf{P}\lambda} + b_{\mathbf{P}\lambda}^*) \{e(\mathbf{P}, \lambda) e(\mathbf{P}', \lambda')\} + \sqrt{\frac{\bar{E}_{\mathbf{P}} \bar{E}_{\mathbf{P}'}}{2\mu}} \\
 & \times \{ (A_{\mathbf{P}} + B_{\mathbf{P}}^*) (a_{\mathbf{P}'\lambda'}^* + b_{\mathbf{P}'\lambda'}) (e(\mathbf{P}) e(\mathbf{P}', \lambda')) - (A_{\mathbf{P}'}^* + B_{\mathbf{P}'}) (a_{\mathbf{P}\lambda} + b_{\mathbf{P}\lambda}^*) \\
 & \times (e(\mathbf{P}') e(\mathbf{P}\lambda')) \} - \frac{\mu}{2\sqrt{E_{\mathbf{P}} E_{\mathbf{P}'}}} \{ (a_{\mathbf{P}\lambda} - b_{\mathbf{P}\lambda}^*) (A_{\mathbf{P}'}^* - B_{\mathbf{P}'}) (e(\mathbf{P}\lambda) e(\mathbf{P}')) \\
 & - (a_{\mathbf{P}'\lambda'}^* - b_{\mathbf{P}'\lambda'}) (A_{\mathbf{P}} - B_{\mathbf{P}}^*) (e(\mathbf{P}') e(\mathbf{P})) \} + \frac{i}{2} \left\{ \sqrt{\frac{\bar{E}_{\mathbf{P}}}{E_{\mathbf{P}'}}} (A_{\mathbf{P}} + B_{\mathbf{P}}^*) \right. \\
 & \times (A_{\mathbf{P}'}^* - B_{\mathbf{P}'}) + \sqrt{\frac{\bar{E}_{\mathbf{P}'}}{E_{\mathbf{P}}}} (A_{\mathbf{P}'}^* + B_{\mathbf{P}'}) (A_{\mathbf{P}} - B_{\mathbf{P}}^*) \left. (e(\mathbf{P}) e(\mathbf{P}')) \right\} \Big] e^{i(\mathbf{P}-\mathbf{P}', r)} \\
 & \quad \quad \quad (24)
 \end{aligned}$$

§ 2. Calculation of the charge renormalization term (c.r.) $f_2(l)$

(i) The (c.r.) due to a charged spinor particle.

From (10), (34.1)

$$\begin{aligned}
 g_1^{(e)}(l) &= \frac{2\pi e^2}{l} (\text{vac.}, \phi^* e_1 a \phi | \mathbf{P} - \frac{l}{2}, -\mathbf{P} - \frac{l}{2}) \\
 & \quad (\mathbf{P} - \frac{l}{2}, -\mathbf{P} - \frac{l}{2} | \phi^* e a \phi | \text{vac.}) \\
 & \quad \quad \quad (25)
 \end{aligned}$$

$$= - \left(\frac{2\pi}{l} \right) \frac{2 \cos^2 \varphi}{2 E_{\mathbf{P}-\frac{l}{2}} \bar{E}_{\mathbf{P}+\frac{l}{2}}} \left[2 P^2 (1-x^2) - \left\{ \left(E_{\mathbf{P}+\frac{l}{2}} + E_{\mathbf{P}-\frac{l}{2}} \right)^2 - l^2 \right\} \right] \quad (26)$$

where $x = \cos \theta$ (cf. Fig. 1)

From (1), (2), (3) (32.1)

$$\begin{aligned}
 f_2^{(e)}(l) &= \frac{e^2 \mu}{2\pi} \int_0^1 v dv \int_{-1}^1 dx \frac{\left(l^2 + \frac{4\mu^2}{1-v^2} \right)}{\left\{ l^2 (1-x^2) + \frac{4\mu^2}{1-v^2} \right\}^{5/2}} \frac{v}{(1-v^2)^{3/2}} \left[\left(\frac{4\mu^2}{1-v^2} + l^2 \right) v^2 (1-x^2) \right. \\
 & \quad \left. - 2l^2 (1-x^2) - \frac{8\mu^2}{1-v^2} \right] = - \frac{e^2}{\pi} \int_0^1 \frac{v^2}{(1-v^2)} \left(1 - \frac{v^2}{3} \right) dv \\
 & \quad \quad \quad (27)
 \end{aligned}$$

The fact that (27) has become an integral depending only on v shows that $f_2(l)$ is a Lorentz invariant quantity.

$$f_2^{(e)}(l) = - \frac{e^2}{3\pi} \log \frac{2(P, P')}{\mu^2} + \frac{5e^2}{9\pi} \quad (P, P') \rightarrow \infty \quad (28)$$

(ii) The (c.r.) due to a charged scalar (or pseudoscalar) field.
From (14). (34.1)

$$g_1^{(s)}(l) = \frac{2\pi}{l} \sum_{\mathbf{P}} \frac{P^2(1-x^2)}{E_{\mathbf{P}-\frac{l}{2}} E_{\mathbf{P}+\frac{l}{2}}} \cos^2 \varphi \quad (29)$$

From (32.1)

$$f_2^{(s)}(l) = -\frac{e^2 \mu}{4\pi} \int_0^1 \tau d\tau \int_{-1}^1 dx \frac{\left\{ l^2 + \frac{4\mu^2}{1-\tau^2} \right\}^2 (1-x^2)}{\left\{ \frac{4\mu^2}{1-\tau^2} + l^2(1-x^2) \right\}^{5/2} (1-\tau^2)^{3/2}} \quad (30)$$

$$= -\frac{e^2}{6\pi} \int_0^1 \frac{\tau^4}{1-\tau^2} d\tau \quad (31)$$

From (1)

$$f_2^{(s)}(l) = -\frac{e^2}{12\pi} \log \frac{2(P, P')}{\mu^2} + \frac{e^2}{18\pi} \quad (P, P') \rightarrow \infty \quad (32)$$

(iii) The (c.r.) due to a charged vector (or pseudovector) field.
According to (21). There are the following combinations as the contribution from $cH_1^{(v)}$ to $f_2^{(v)}(l)$.

$$\begin{aligned} (a) + (b) + (c) + (d) + (e) + (f) + (g) + (h) &= (A_1 A_1^* + A_1^* A_1) + (A_2 A_2^* + A_2^* A_2) \\ &+ (A_3 A_3^* + A_3^* A_3) + (A_4 A_4^* + A_4^* A_4) + (A_1 A_4 + A_1^* A_4^*) + (A_2 A_2 + A_2^* A_2^*) \\ &+ (A_3 A_3 + A_3^* A_3^*) + (A_4 A_1 + A_4^* A_1^*) \end{aligned} \quad (33)$$

$$(a) = (A_1 A_1^* + A_1^* A_1) \propto \sum_{\lambda=1,2} \left(e\left(\mathbf{P} + \frac{\mathbf{l}}{2}, \lambda\right) \right.$$

$$\left. \times e(\mathbf{l}, \mu) \right)^2 = \sin^2 \varphi + \cos^2 \theta'' \cos^2 \varphi$$

$$(b) = (A_2 A_2^* + A_2^* A_2) \propto \left(e\left(\mathbf{P} + \frac{\mathbf{l}}{2}\right), e(\mathbf{l}, \mu) \right)^2$$

$$= \sin^2 \theta'' \cos^2 \varphi$$

$$(c) = (A_3 A_3^* + A_3^* A_3) \propto \sum_{\lambda=1,2} \left(\left[e\left(\mathbf{P} - \frac{\mathbf{l}}{2}\right) e\left(\mathbf{P} - \frac{\mathbf{l}}{2}, \lambda\right) \right] \left[e(\mathbf{l}, \mu) e\left(\mathbf{P} + \frac{\mathbf{l}}{2}, \lambda\right) \right] \right)^2$$

$$= \cos^2 \varphi \sin^2 \theta' + \cos^2 \varphi \sin^2 \theta'' + \sin^2 \varphi \sin^2 (\theta' - \theta'')$$

$$(d) = (A_4 A_4^* + A_4^* A_4) \propto \sum_{\lambda=1,2} \left(\left[e\left(\mathbf{P} - \frac{\mathbf{l}}{2}\right), e\left(\mathbf{P} - \frac{\mathbf{l}}{2}, \lambda\right) \right] \left[e(\mathbf{l}, \mu) e\left(\mathbf{P} + \frac{\mathbf{l}}{2}\right) \right] \right)^2$$

$$= \sin^2 \varphi \sin^2 \left(\frac{\pi}{2} + \theta' - \theta'' \right) + \cos^2 \varphi \cos^2 \theta''$$

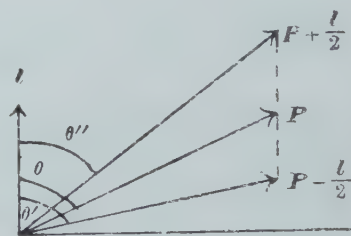


Fig. 2.

$$(e) = (A_1 A_1 + A_1^* A_1^*) \propto \sum_{\lambda=1,2} \left(e\left(\mathbf{P} + \frac{\mathbf{l}}{2}, \lambda\right) e(\mathbf{l}, \mu) \right) \left(\left[e\left(\mathbf{P} + \frac{\mathbf{l}}{2}\right) e\left(\mathbf{P} + \frac{\mathbf{l}}{2}, \lambda\right) \right] \right. \\ \left. \times \left[e\left(\mathbf{P} - \frac{\mathbf{l}}{2}\right) e(\mathbf{l}, \mu) \right] \right) = \cos \theta' \cos \theta'' \cos^2 \varphi + \sin^2 \varphi \sin \left(\frac{\pi}{2} - \theta' + \theta'' \right)$$

$$(f) = (A_2 A_2 + A_2^* A_2^*) \propto \left(e\left(\mathbf{P} + \frac{\mathbf{l}}{2}\right) e(\mathbf{l}, \mu) \right) \left(e\left(\mathbf{P} - \frac{\mathbf{l}}{2}\right) e(\mathbf{l}, \mu) \right) \\ = \sin \theta'' \cos^2 \varphi \cos \left(\frac{\pi}{2} - \theta' \right)$$

$$(g) = (A_3 A_3 + A_3^* A_3^*) \propto \sum_{\lambda=1,2} \left(\left[e\left(\mathbf{P} - \frac{\mathbf{l}}{2}\right) e\left(\mathbf{P} - \frac{\mathbf{l}}{2}, \lambda\right) \right] \left[e\left(\mathbf{P} + \frac{\mathbf{l}}{2}, \lambda\right) e(\mathbf{l}, \mu) \right] \right) \\ \times \left(\left[e\left(\mathbf{P} + \frac{\mathbf{l}}{2}\right) e\left(\mathbf{P} + \frac{\mathbf{l}}{2}, \lambda\right) \right] \left[e\left(\mathbf{P} - \frac{\mathbf{l}}{2}, \lambda\right) e(\mathbf{l}, \mu) \right] \right) = 2 \sin \theta' \sin \theta'' \sin^2 \varphi$$

$$(h) = (A_4 A_4 + A_4^* A_4^*) \propto \sum_{\lambda=1,2} \left(e\left(\mathbf{P} - \frac{\mathbf{l}}{2}\right), e(\mathbf{l}, \mu) \right) \left(\left[e\left(\mathbf{P} - \frac{\mathbf{l}}{2}, \lambda\right) e\left(\mathbf{P} - \frac{\mathbf{l}}{2}\right) \right] \right. \\ \left. \left[e(\mathbf{l}, \mu) e\left(\mathbf{P} + \frac{\mathbf{l}}{2}\right) \right] \right) = \cos \theta' \cos \theta'' \cos^2 \varphi + \sin^2 \varphi \sin \left(\frac{\pi}{2} + \theta' - \theta'' \right)$$

The integrations are for θ , which is related to θ' , θ'' as follows (cf. Fig. 2.)

$$\begin{aligned} \sin \theta' &= \frac{P}{\left| \mathbf{P} - \frac{\mathbf{l}}{2} \right|} \sin \theta & \cos \theta' &= \frac{P \cos \theta - \frac{l}{2}}{\left| \mathbf{P} - \frac{\mathbf{l}}{2} \right|} \\ \sin \theta'' &= \frac{P}{\left| \mathbf{P} + \frac{\mathbf{l}}{2} \right|} \sin \theta & \cos \theta'' &= \frac{P \cos \theta + \frac{l}{2}}{\left| \mathbf{P} + \frac{\mathbf{l}}{2} \right|} \\ \sin (\theta' - \theta) &= \frac{l}{2 \left| \mathbf{P} - \frac{\mathbf{l}}{2} \right|} \sin \theta & \cos (\theta' - \theta) &= \frac{P - \frac{l}{2} \cos \theta}{\left| \mathbf{P} - \frac{\mathbf{l}}{2} \right|} \\ \sin (\theta - \theta'') &= \frac{l}{2 \left| \mathbf{P} + \frac{\mathbf{l}}{2} \right|} \sin \theta & \cos (\theta'' - \theta) &= \frac{P + \frac{l}{2} \cos \theta}{\left| \mathbf{P} + \frac{\mathbf{l}}{2} \right|} \\ \sin (\theta' - \theta'') &= \frac{lP}{\left| \mathbf{P} - \frac{\mathbf{l}}{2} \right| \left| \mathbf{P} + \frac{\mathbf{l}}{2} \right|} \sin \theta & \cos (\theta' - \theta'') &= \frac{P^2 - \frac{l^2}{4}}{\left| \mathbf{P} - \frac{\mathbf{l}}{2} \right| \left| \mathbf{P} + \frac{\mathbf{l}}{2} \right|} \end{aligned} \quad (34)$$

From this, and using (1), coefficients of

$$\left(-\frac{c^2\pi^2}{\mu^2} \frac{\left(\frac{4\mu^2}{1-\tau^2} + l^2 \right)^{1/2} v d\tau dx}{\left\{ \frac{4\mu^2}{1-\tau^2} + l^2(1-x^2) \right\}} \cdot P \right) \text{ are ;}$$

$$\begin{aligned} (a) + (d) \cdots \rightarrow \frac{1}{\mu^2} \left\{ \left| P + \frac{l}{2} \right|^2 \left| P - \frac{l}{2} \right|^2 + 2 \left(Px + \frac{l}{2} \right)^2 \left| P - \frac{l}{2} \right|^2 + \left(P^2 - \frac{l^2}{4} \right)^2 \right\} \\ + \left\{ \left| P - \frac{l}{2} \right|^2 + \frac{2 \left(Px + \frac{l}{2} \right)^2 \left| P - \frac{l}{2} \right|^2 + \left(P^2 - \frac{l^2}{4} \right)^2}{\left| P - \frac{l}{2} \right|^2} \right\} \end{aligned}$$

$$(e) + (h) \cdots \rightarrow -\frac{1}{\mu^2} \left(P^2 - \frac{l^2}{4} \right) \{ 2P^2(1+x^2) - l^2 \} + \frac{1+\tau^2}{1-\tau^2} \{ 2P^2(1+x^2) - l^2 \}$$

$$(b) + (c) \cdots \rightarrow \left\{ 1 + \frac{2 \left| P - \frac{l}{2} \right|^2 + l^2}{\left| P + \frac{l}{2} \right|^2} \right\} P^2(1-x^2)$$

$$(f) + (g) \cdots \rightarrow 3l^2(1-x^2)$$

Finally

$$f_2^{(n)}(l) = (a) + (b) + (c) + (d) + (e) + (f) + (g) + (h) = \text{(I)} + \text{(II)} + \text{(III)}$$

$$\begin{aligned} \text{(I)} = -\frac{c^2\pi^2}{\mu^2} \frac{1}{(2\pi)^3} \int_0^1 d\tau \left(\frac{4\mu^2}{1-\tau^2} + l^2 \right)^2 \frac{\tau^4 \mu^3}{(1-\tau^2)^{2/3}} \left(8 + \frac{2l^2}{\mu^2} + \frac{2(1+\tau^2)}{1-\tau^2} \right) \\ \times \int_{-1}^1 \frac{dx}{\left\{ \frac{4\mu^2}{1-\tau^2} + l^2(1-x^2) \right\}^{5/2}} \end{aligned}$$

$$\begin{aligned} \text{(II)} = \frac{c^2\pi^2}{\mu^2} \frac{1}{(2\pi)^3} \int_0^1 d\tau \left(\frac{4\mu^2}{1-\tau^2} + l^2 \right)^2 \frac{\tau^4 \mu^3}{(1-\tau^2)^{3/2}} \left(4 + \frac{2l^2}{\mu^2} - \frac{2(1+\tau^2)}{1-\tau^2} \right) \\ \times \int_{-1}^1 \frac{x^2 dx}{\left\{ \frac{4\mu^2}{1-\tau^2} + l^2(1-x^2) \right\}^{5/2}} \end{aligned}$$

$$\begin{aligned} \text{(III)} = \frac{c^2\pi^2}{\mu^2} \frac{1}{(2\pi)^3} \int_0^1 d\tau \left(\frac{4\mu^2}{1-\tau^2} + l^2 \right)^2 \frac{\tau^2 \mu}{(1-\tau^2)^{1/2}} \frac{8\tau^2 l^2}{(1-\tau^2)} \\ \times \int_{-1}^1 \frac{dx}{\left\{ \frac{4\mu^2}{1-\tau^2} + l^2(1-x^2) \right\}^{3/2}} \end{aligned}$$

Integration over x ,

$$f_2^{(v)}(l) = -\frac{e^2}{2\pi} \int_0^1 \frac{v^4}{1-v^2} dv - \frac{2e^2}{3\pi} \int_0^1 \frac{v^4}{(1-v^2)^2} dv \quad (35)$$

From (1)

$$f_2^{(v)}(l) = -\frac{e^2}{6\pi} \frac{(P, P')}{\mu^2} + \frac{e^2}{4\pi} \log \frac{2(P, P')}{\mu^2} \quad (P, P') \rightarrow \infty \quad (36)$$

§ 3. Proof of the relations $f_2(l) = f_2'(l)$ and $f_3(l) = f_3'(l)$.

(i) For the case of a charged spinor field.

From (39.1), (11)

$$\begin{aligned} g_2^{(e)}(l) &= \frac{4\pi e^2}{l^2} \left(\text{vac} | \phi^* \psi \left(\mathbf{P} - \frac{\mathbf{l}}{2}, -\mathbf{P} - \frac{\mathbf{l}}{2} \right) \left(\mathbf{P} - \frac{\mathbf{l}}{2}, -\mathbf{P} - \frac{\mathbf{l}}{2} | \phi^* \psi \text{ vac.} \right) \right. \\ &= \sum_{\mathbf{P}} \frac{8\pi e^2}{E_{\mathbf{P}+\frac{\mathbf{l}}{2}} E_{\mathbf{P}-\frac{\mathbf{l}}{2}}} \frac{P^2(1-x^2) + \mu^2}{\left\{ \left(E_{\mathbf{P}+\frac{\mathbf{l}}{2}} + E_{\mathbf{P}-\frac{\mathbf{l}}{2}} \right)^2 - l^2 \right\}} \end{aligned} \quad (37)$$

From (37.1)

$$f_2'^{(e)}(l) = -\frac{e^2}{\pi} \int_0^1 \left(1 - \frac{1}{3} v^2 \right) \frac{v^2 dv}{(1-v^2)^2} \quad (27')$$

Comparing this with (27)

$$f_2'^{(e)}(l) = f_2^{(e)}(l) \quad (38)$$

Consequently, from (6), (8)

$$f_3'^{(e)}(l) = f_3^{(e)}(l) \quad (39)$$

(ii) For the case of a charged scalar (or pseudoscalar) field.

From (39.1), (17)

$$\begin{aligned} g_2^{(s)}(l) &= \sum_{\mathbf{P}} \frac{e^2 \pi}{l^2} \left(\sqrt{\frac{E_{\mathbf{P}+\frac{\mathbf{l}}{2}}}{E_{\mathbf{P}-\frac{\mathbf{l}}{2}}}} - \sqrt{\frac{E_{\mathbf{P}-\frac{\mathbf{l}}{2}}}{E_{\mathbf{P}+\frac{\mathbf{l}}{2}}}} \right), \\ f_2'^{(s)}(l) &= -\frac{e^2}{2\pi} \int_0^1 \frac{\mu^4}{(1-v^2)} \left\{ \frac{4\mu^2}{1-v^2} + l^2 \right\} \frac{dv}{(1-v^2)^{1/2}} \int_{-1}^1 \frac{x^2 dx}{\left\{ \frac{4\mu^2}{1-v^2} + l^2(1-x^2) \right\}^{3/2}} \\ &= -\frac{e^2}{6\pi} \int_0^1 \frac{v^2}{1-v^2} dv \end{aligned} \quad (31)'$$

Comparing, this with (31)

$$f_2'^{(s)}(l) = f_2^{(s)}(l) \quad (40)$$

Consequently from (6), (8):

$$f_3'^{(s)}(l) = f_3^{(s)}(l) \quad (41)$$

(iii) For the case of a charged vector (or pseudovector) field.

From (39.1), (24)

$$\begin{aligned}
 g_2^{(v)}(l) &= \frac{e^2 \pi}{l^2} \sum_{\lambda, \lambda'} \left[\left(\sqrt{\frac{E_{\mathbf{P}+\frac{l}{2}}}{E_{\mathbf{P}-\frac{l}{2}}}} - \sqrt{\frac{E_{\mathbf{P}-\frac{l}{2}}}{E_{\mathbf{P}+\frac{l}{2}}}} \right)^2 \left\{ \left(e\left(\mathbf{P}-\frac{l}{2}, \lambda\right) e\left(\mathbf{P}+\frac{l}{2}, \lambda'\right) \right)^2 \right. \right. \\
 &\quad \left. \left. + \left(e\left(\mathbf{P}-\frac{l}{2}\right) e\left(\mathbf{P}+\frac{l}{2}\right) \right)^2 \right\} + \left(\frac{1}{\mu} \sqrt{\frac{E_{\mathbf{P}+\frac{l}{2}} E_{\mathbf{P}-\frac{l}{2}}}{E_{\mathbf{P}+\frac{l}{2}} E_{\mathbf{P}-\frac{l}{2}}}} - \frac{\mu}{E_{\mathbf{P}+\frac{l}{2}} E_{\mathbf{P}-\frac{l}{2}}} \right)^2 \right. \\
 &\quad \left. \times \left\{ \left(e\left(\mathbf{P}-\frac{l}{2}\right) e\left(\mathbf{P}+\frac{l}{2}, \lambda'\right) \right)^2 + \left(e\left(\mathbf{P}-\frac{l}{2}, \lambda\right) e\left(\mathbf{P}+\frac{l}{2}\right) \right)^2 \right\} \right] \\
 &= \frac{e^2 \pi}{l^2} \left[\left(\frac{E_{\mathbf{P}+\frac{l}{2}}}{E_{\mathbf{P}-\frac{l}{2}}} + \frac{E_{\mathbf{P}-\frac{l}{2}}}{E_{\mathbf{P}+\frac{l}{2}}} - 2 \right) (1 + 2 \cos^2(\theta'' - \theta')) + \frac{1}{2} \left(\frac{1}{\mu^2} E_{\mathbf{P}+\frac{l}{2}} E_{\mathbf{P}-\frac{l}{2}} \right. \right. \\
 &\quad \left. \left. + \frac{\mu^2}{\sqrt{E_{\mathbf{P}+\frac{l}{2}} E_{\mathbf{P}-\frac{l}{2}}}} - 2 \right) \sin^2(\theta'' - \theta') \right] \\
 &= \frac{e^2 (2\pi)}{l^2} \frac{1}{E_{\mathbf{P}-\frac{l}{2}} E_{\mathbf{P}+\frac{l}{2}}} \left[3 \left(\mu^2 + P^2 + \frac{l^2}{4} - E_{\mathbf{P}+\frac{l}{2}} E_{\mathbf{P}-\frac{l}{2}} \right) + \frac{P^2 l^2}{\mu^2} (1 - x^2) \right] \quad (42)
 \end{aligned}$$

From (37.1)

$$\begin{aligned}
 f_2^{(v)}(l) &= -\frac{2e^2}{\pi} \int_0^1 d\tau \frac{\mu^2 \tau^5}{(1-\tau^2)^{5/2}} \left\{ \frac{4\mu^2}{1-\tau^2} + l^2 \right\} \left[\int_{-1}^1 \frac{1}{2\mu^2} \frac{dx}{\left\{ \frac{4\mu^2}{1-\tau^2} + l^2(1-x^2) \right\}^{3/2}} \right. \\
 &\quad \left. + \frac{(1-3\tau^2)}{(1-\tau^2)} \int_{-1}^1 \frac{x^2 dx}{\left\{ \frac{4\mu^2}{1-\tau^2} + l^2(1-x^2) \right\}^{5/2}} \right] \\
 &= -\frac{2e^2}{3\pi} \int_0^1 \frac{\tau^4}{(1-\tau^2)^2} d\tau - \frac{e^2}{2\pi} \int_0^1 \frac{\tau^4}{1-\tau^2} d\tau \quad (35')
 \end{aligned}$$

Comparing this with (35)

$$f_2^{(v)}(l) = f_2^{(v)}(l) \quad (43)$$

Consequently, from (6), (8)

$$f_3^{(v)}(l) = f_3^{(v)}(l) \quad (44)$$

§ 4. Calculation of the Observed Current Term (o. c.)

Using the results for $f_2(l)$ as obtained in § 2, $f_3(l)$ is immediately obtained from (6) :

(i) Charged spinor field.

$$f_3'^{(e)}(l) = f_3^{(e)}(l) = -\frac{e^2 \square}{4\pi\mu^2} \int_0^1 \tau^2 \left(1 - \frac{\tau^2}{3}\right) \frac{d\tau}{\left\{1 - \frac{\square}{4\mu^2} (1 - \tau^2)\right\}} \quad (45)$$

This result is identical with that obtained by J. Schwinger¹⁾ by method of canonical transformation.

When the spatio-temporal variation of the field is small, and consequently it can be expanded in terms of \square ,

$$f_3'^{(e)}(l) = f_3^{(e)}(l) \simeq -\frac{e^2 \square}{15\pi\mu^2} - \frac{e^2 \square^2}{140\pi\mu^5} \dots \quad (45')$$

(ii) Charged scalar (or pseudoscalar) field.

$$f_3'^{(s)}(l) = f_3^{(s)}(l) = -\frac{e^2 \square}{24\pi\mu^2} \int_0^1 \frac{v^4 dv}{\left\{1 - \frac{\square}{4\mu^2} (1 - v^2)\right\}} \quad (46)$$

$$\simeq -\frac{e^2 \square}{120\pi\mu^2} - \frac{e^2 \square^2}{1630\pi\mu^4} \dots \quad (46')$$

(iii) Charged vector (or pseudovector) field.

$$f_3'^{(v)}(l) = f_3^{(v)}(l) = -\frac{e^2 \square}{6\pi\mu^2} \int_0^1 \frac{v^4}{(1 - v^2)} \frac{dv}{\left\{1 - \frac{\square}{4\mu^2} (1 - v^2)\right\}} - \frac{e^2 \square}{8\pi\mu^2} \times \int_0^1 \frac{\tau^4 d\tau}{\left\{1 - \frac{\square}{4\mu^2} (1 - \tau^2)\right\}} \quad (47)$$

$$\simeq -\left\{\frac{e^2}{12\pi\mu^2} \log \frac{2(P, P')}{\mu^2} - \frac{71e^2}{360\pi\mu^2}\right\} \square - \frac{17e^2 \square^2}{1680\pi\mu^5} \dots \quad (47')$$

§ 5. The Mass-term of the Photon.²⁾

The self-energy W_l of the photon l^- is

$$W_l = f_{11}(l) + f_{12}(l) \quad (48)$$

$f_{11}(l)$ can be calculated immediately from (5) on using the result of § 2.

(i) Charged spinor field,

$$\begin{aligned} W_l^{(e)} = f_{11}^{(e)}(l) &= -\frac{e^2}{2\pi l} \int_0^1 \frac{v^2}{(1 - v^2)} \left(1 - \frac{v^2}{3}\right) \frac{4\mu^2}{(1 - v^2)} dv \\ &= -\frac{2e^2}{3\pi l} \left[\frac{1}{2} (P, P') - \mu^2 \right] \quad (P, P') \rightarrow \infty \quad (f_{12}(l) = 0) \end{aligned} \quad (49)$$

(ii) Charged scalar (or pseudoscalar) field.

$$f_{11}^{(s)}(l) = -\frac{e^2}{12\pi l} \int_0^1 \frac{v^4}{(1-v^2)} \frac{4\mu^2}{(1-v^2)} dv \quad (50)$$

Using (45.1), (15) :

$$f_{12}^{(s)}(l) = \frac{e^2}{\pi l} \int_0^\infty \frac{P^2}{\sqrt{P^2 + \mu^2}} dP \quad (51)$$

The domain of P is a sphere. Also, the surface $v = \text{const.}$, when referred to a special system of coordinates, becomes a sphere. By making these two fields coincide with each other, we can resplace the P -domain by a relativistically invariant v -domain. This is effected by the transformation of variables.

$$P^2 \rightarrow \frac{(P, P') - \mu^2}{2} = \frac{\mu^2 v^2}{(1-v^2)} \quad (52)$$

Then (51) becomes

$$f_{12}^{(s)}(l) = \frac{e^2}{\pi l} \int_0^1 \frac{v^2 \mu^2}{(1-v^2)^2} dv \quad (53)$$

Adding (50) and (53), the self-energy of the photon is

$$W_l^{(s)} = \frac{e^2}{3\pi l} \left[\frac{1}{2} (P, P') - \mu^2 \right] \quad (P, P') \rightarrow \infty \quad (54)$$

(iii) Charged vector (or pseudovector) field.

From (45.1), (36)

$$f_{11}^{(v)}(l) = -\frac{e^2}{4\pi l} \int_0^1 \frac{v^4}{(1-v^2)} \frac{4\mu^2}{(1-v^2)} dv - \frac{e^2}{3\pi l} \int_0^1 \frac{v^4}{(1-v^2)^2} \frac{4\mu^2}{(1-v^2)} dv \quad (55)$$

From (22) (45.1), the contribution from B_1 to $f_{12}^{(v)}(l)$ is

$$\frac{2e^2}{3\pi\mu^2 l} \int_0^\infty P^2 \sqrt{P^2 + \mu^2} dP + \frac{e^2}{3\pi l} \int_0^\infty \frac{P^2}{\sqrt{P^2 + \mu^2}} dP \quad (56)$$

while that from B_2 is

$$\frac{4e^2}{3\pi l} \int_0^\infty \frac{P^2 dP}{\sqrt{P^2 + \mu^2}} + \frac{2e^2}{3\pi l \mu^2} \int_0^\infty P^2 \sqrt{P^2 + \mu^2} dP \quad (56')$$

Therefore

$$f_{12}^{(v)}(l) = (56) + (56') = \frac{4e^2}{3\pi l \mu^2} \int_0^\infty P^2 \sqrt{P^2 + \mu^2} dP + \frac{5e^2}{3\pi l} \int_0^\infty \frac{P^2}{\sqrt{P^2 + \mu^2}} dP \quad (57)$$

Making the transformation of variable (52),

$$f_{12}^{(v)}(l) = \frac{5\mu^2 e^2}{3\pi l} \int_0^1 \frac{v^2 dv}{(1-v^2)^2} + \frac{4\mu^2 e^2}{3\pi l} \int_0^1 \frac{v^2 dv}{(1-v^2)^3} \quad (58)$$

Adding (55) and (58), the self-energy of the photon is

$$W_l^{(v)} = \frac{e^2}{\pi l} \left[\frac{1}{2} (P, P') - \mu^2 \right] \quad (P, P') \rightarrow \infty \quad (59)$$

The operator which gives the self-energy W_l of the photon, that is the mass-term of the photon, is

$$\frac{W_l}{2(2\pi)} \mathbf{A}_\perp^2(X) \quad (60)$$

where \mathbf{A}_\perp is the transversal part of the vector potential.

It is clear that (49), (54), (59) all have the transformation property of $\frac{1}{l}$, that is, that of the mass-term.

§ 6. The Induced Current.

Using the above results, the current δJ induced by an arbitrary current $J^{(i)}$ can be calculated from (48·I). On making arrangements, the final result is as follows; irrespectively of the kind of (i),

$$\delta J(X) = \iint \left[W_l \frac{2l}{\square} (\mathbf{e}_1 \mathbf{J}^{(i)}(X')) \mathbf{e}_1 + (f_2(l) + f_3(l)) J^{(i)}(X') \right] e^{i(l, X-X')} dX' dl \quad (61)$$

(i) Vacuum polarization due to a charged spinor field,

$$f_2^{(e)}(l) = -\frac{e^2}{3\pi} \log \frac{2(P, P')}{\mu^2} + \frac{5e^2}{9\pi} \quad (P, P') \rightarrow \infty \quad (28)$$

$$f_3^{(e)}(l) = -\frac{e^2 \square}{4\pi \mu^2} \int_0^1 \frac{v^2 \left(1 - \frac{v^2}{3}\right)}{\left\{1 - \frac{\square}{4\mu^2} (1-v^2)\right\}} dv \quad (45)$$

$$W_l^{(e)} = -\frac{2e^2}{3\pi l} \left[\frac{1}{2} (P, P') - \mu^2 \right] \quad (P, P') \rightarrow \infty \quad (49)$$

(ii) Vacuum polarization due to a charged scalar (or pseudoscalar) field.

$$f_2^{(s)}(l) = -\frac{e^2}{12\pi} \log \frac{2(P, P')}{\mu^2} + \frac{e^2}{18\pi} \quad (P, P') \rightarrow \infty \quad (32)$$

$$f_3^{(s)}(l) = -\frac{e^2 \square}{24\pi \mu^2} \int_0^1 \frac{v^4 dv}{\left\{1 - \frac{\square}{4\mu^2} (1-v^2)\right\}} \simeq -\frac{e^2 \square}{120\pi \mu^2} - \frac{e^2 \square^2}{1630\pi \mu^4} \dots \dots \quad (46)$$

$$W_l^{(s)} = \frac{e^2}{3\pi l} \left[\frac{1}{2} (P, P') - \mu^2 \right] \quad (P, P') \rightarrow \infty \quad (54)$$

(iii) Vacuum polarization due to a charged vector (or pseudovector) field.

$$f_2^{(v)}(l) = -\frac{e^2}{6\pi} \frac{(P, P')}{\mu^2} + \frac{e^2}{4\pi} \log \frac{2(P, P')}{\mu^2} \quad (P, P' \rightarrow \infty) \quad (36)$$

$$f_2^{(v)}(l) = -\frac{e^2 \square}{6\pi \mu^2} \int_0^1 \frac{\tau^4}{(1-\tau^2)} - \frac{d\tau}{\left\{1 - \frac{\square}{4\mu^2} (1-\tau^2)\right\}} - \frac{e^2 \square}{8\pi \mu^2} \int_0^1 \frac{\tau^4 d\tau}{\left\{1 - \frac{\square}{4\mu^2} (1-\tau^2)\right\}} \quad (47)$$

$$= -\left\{ \frac{e^2}{12\pi \mu^2} \log \frac{2(P, P')}{\mu^2} - \frac{71e^2}{360\pi \mu^4} \right\} \square - \frac{17e^2 \square^2}{1680\pi \mu^6} \dots \quad (47')$$

$$W_l^{(v)} = \frac{e^2}{\pi l} \left[\frac{1}{2} (P, P') - \mu^2 \right] \quad (P, P') \rightarrow \infty \quad (59)$$

§ 7. Discussion

Using the above results, we shall investigate the problem of vacuum polarization, considering various charged particles simultaneously.

The first term of (61) has only a component in the \mathbf{e}_1 direction. Consequently, it is convenient to consider it after transforming it as follows. Denoting the transversal part of the electromagnetic vector potential at the position of $f^{(i)}(X)$ by $\mathbf{A}_\perp(X)$,

$$\square \mathbf{A}_\perp(X) = -4\pi (\mathbf{e}_1 \mathbf{J}^{(i)}(X)) \mathbf{e}_1$$

Therefore, the first term $\delta \mathbf{J}_a$ of (61) is

$$\delta \mathbf{J}_a(X) = - \iint 2 \frac{W_l l}{2(2\pi)} \mathbf{A}_\perp(X') e^{i(l, \mathbf{x} - \mathbf{x}')} dX' dl \quad (61a)$$

This current may be thought of as due to the following additional term in the Hamiltonian function.

$$\int \frac{W_l l}{2(2\pi)} \mathbf{A}_\perp^2(X') e^{i(l, \mathbf{x} - \mathbf{x}')} dX' dl \quad (60')$$

Comparing this with (60) it is seen to be the photon mass-term.

Observing (49), (54), (59), we find a simple law to hold for the self-energies of the photon due to various charged particles. That is the magnitudes are respectively proportional to the degree of freedom of the charged particle field, and the sign is positive for a Fermi field and negative for a Bose field. However, the relations of the finite terms are not always simple, because the proper mass μ is not the same, for the various particles. But this is because we have taken the relativistically invariant quantity (P, P') as the common variable for each particle, so that circumstances are extremely different from the customary method of integrating for \mathbf{P} . Though it is possible in a special system of co-

ordinates, to make the τ -space domain coincide with the \mathbf{P} -space sphere (when $l=0$), in this case $(P, P')=2P^2+\mu^2$ so there is a difference by a term relating to mass. But, since it is impossible to make a scalar quantity involving only P and independent of μ , it is totally ambiguous what function of (P, P') to use as the common variable for various particles in extending the customary P -integration into a relativistically invariant method of integration. Consequently, we cannot make satisfactory discussions of the finite terms of the self-energy of photon.

We previously maintained that the self-energy of the photon may be made finite by assuming scalar and spinor particles to exist with a relative abundance of 2:1.²⁾ According to the above results, $W^{(s)}:W^{(v)}:W^{(v)}=-2:1:3$, so that if the number of spinor, scalar (or pseudoscalar) and vector (pseudovector) fields are n , m and l respectively, and the relation

$$-2n+m+3l=0 \quad (62)$$

holds, it is seen that the self-energy of the photon can be made to vanish. But in this case the question of finite terms is ambiguous, as mentioned above, and it appears that, in general, writing $\mu^{(s)}$, $\mu^{(s)}$ and $\mu^{(v)}$ for the masses of the spinor, scalar (or pseudoscalar), vector (or pseudovector) fields respectively, the following relation must hold between them:

$$-2\sum_{i=1}^n (\mu_i^{(s)})^2 + \sum_{i=1}^m (\mu_i^{(s)})^2 + 3\sum_{i=1}^l (\mu_i^{(v)})^2 = 0 \quad (63)$$

However, the fact that the μ 's are involved in $W^{(s)}$, $W^{(s)}$ and $W^{(v)}$ in the same form shows that the self-energy of the photon can be made to vanish by taking a mixed field of the form (62) without requiring the condition (63). That is, we need only take $(P, P')=2\mu^2$ as the common quantity for each particle. Then the relation (62) by itself suffices to annul the self-energy of the photon. In any case, it seems that, at the present stage, the mixed field (62) provides a method of avoiding the difficulty of the self-energy of photons. Considered together with the success to some extent of the C -meson in the case of the self-energy of electron, it appears that, at the present stage, the method of mixed field provides a powerful spring-board for the future development of elementary particle theory.

We next examine the contributions δJ_b , δJ_c from the second term $f_2(x)$, $f_3(x)$ of (61) to δJ .

$$\delta J_b^{(X)} = \int \{f_2(l) f^{(s)}(X') e^{i(l, X-X')} dX' dl \quad (64)$$

$$\delta J_c^{(X)} = \int \{f_3(l) f^{(v)}(X') e^{i(l, X-X')} dX' dl \quad (65)$$

Since δJ_b is, as may be seen from the calculations up to now (28), (32), (36),

Mr. Oda pointed out that this relation is just as that between zero-point energies due to various charged particles. Therefore, (62) is also available to annul zero-point energies. The relation between the self-energy of the photon and the zero point energy will be precisely discussed elsewhere.

merely a constant, it can be amalgamated into $J^{(i)}$. Though its value, of course, depends on the particular type of particle, actually the sum of the contributions from all the charged particles gives ∂J_b , as will be shown. Thus, this has a value independent of the type of particles.³⁾

$$\partial J_b = \sum_{i=1}^n \partial J_{b_i}^{(e)} + \sum_{i=1}^m \partial J_{b_i}^{(s)} + \sum_{i=1}^l \partial J_{b_i}^{(n)} \quad (66)$$

∂J_c cannot be amalgamated into $J^{(i)}$ because it involves the \square , and moreover, it diverges logarithmically for a charged vector particle (as seen from (45)), so that this seems to present a new difficulty to the Tomonaga-Schwinger theory.

Since the charge renormalization terms and the observed current terms have the same signs independent on the spin properties of charged particles. This fact, it is that we pointed out in our paper I by scrutinizing on the general formula of vacuum polarization. The more detailed discussion of this problem will be published elsewhere. In conclusion, we would like to express our sincere thanks to Prof. S. Sakata and Dr. Y. Tanikawa for their interest in this work.

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The Self-energy of a Dirac Particle, and its Relativistic Covariance.

Rokuo KAWABE and Hiroomi UMEZAWA

Institute of Theoretical Physics, Nagoya University.

(Received June 9, 1949)

Hitherto, when the electromagnetic self-energy of an electron in motion was calculated directly by perturbation theory, the result as a whole did not show a relativistic covariance, so convenient prescriptions were applied as the necessity arose. For instance, the fact that a direct calculation of the self-energy $W(\mathbf{P})$ of an electron of momentum \mathbf{P} gives an indeterminate result of the form of a difference between two quadratically diverging terms led Weisskopf¹⁾ to calculate the self-energy $W(0)$ of an electron at rest—in which case the ambiguities are not so manifest, owing to the spherical symmetry of the angular distribution of the momentum of a photon—from which $W(\mathbf{P})$ can be obtained by means of a Lorentz transformation. He also predicted that the result thus obtained would be equivalent to that of a direct calculation of $W(\mathbf{P})$ in which suitable precautions are taken in performing the subtraction between the two diverging integrals.

Similarly, Pais²⁾ calculated $W(\mathbf{P})$ as

$$W(\mathbf{P}) = W(0) \sqrt{1 - \beta^2}$$

Also, Oneda and Ozaki³⁾ recently employed the method of mixing a C-meson⁴⁾ as well as using the super-many-time formalism and contact transformation in showing that the diverging part of an electron in motion has the property of covariance, but this is not a case for the finite term, even when the deviation from spherical symmetry of the angular distribution of the momentum of the virtually emitted photon is taken into account.

However, when the prescription recently proposed by us⁵⁾ is employed the electromagnetic self-energy, and that due to the C-meson of a Dirac particle in motion are both seen to possess just the transformation property of the mass term, that is, the transformation $W(\mathbf{P}) = W(0) \sqrt{1 - \beta^2}$ makes a consequential appearance. Noting that the spherical domain hitherto taken is not a relativistically invariant one, and also that a momentum conservation not satisfying the energy conservation is not a relativistically invariant relation, we take, as the integration domain, the interior of a closed surface $w = \text{const.}$ (where w is a certain four dimensional scalar quantity in the energy-momentum space), while the lower and upper limits are determined by placing the condition that, in a special case (momentum=0), the above domain become a sphere in the momentum space in

order to yield results coinciding with those hitherto obtained. When a Fourier expansion is made, the above precautions must always be taken in order to perform the integrations in a relativistically invariant manner. It is only natural that the relativistic invariance should be destroyed in the course of calculation and the results be devoid of the correct transformation property in both our previous perturbation calculations and those of Ôneda and Ozaki which take the deviations from spherical symmetry into account since they are both defined by the momentum conservation which is not a relativistically invariant relation, and since the limits of integration probably cannot be uniquely determined. Although Ôneda and Ozaki use the same formalism as Schwinger⁽⁶⁾ who obtained results showing the correct transformation property, the above precautions were wanting in their calculations, which were performed resolved into Fourier components.

§ 1. The Electromagnetic Self-Energy of a Dirac Particle

The calculations were performed on positron theory, and, in order to make the relativistic invariance evident, the longitudinal part was not excluded from the outset, but eliminated in the course of calculations, by using the Lorentz condition. That is, writing the scalar and longitudinal parts of the four-potential $A_i = (\mathbf{A}, iA_0)$ (three-dimensional vectors will always be denoted by boldfaced letters in the following) as

$$A_0 = \sum_l \sqrt{\frac{2\pi}{l}} (d_l e^{i\mathbf{l}\cdot\mathbf{r}} + d_l^* e^{-i\mathbf{l}\cdot\mathbf{r}})$$

and

$$A_3 = \sum_l \sqrt{\frac{2\pi}{l}} (c_l e^{i\mathbf{l}\cdot\mathbf{r}} + c_l^* e^{-i\mathbf{l}\cdot\mathbf{r}})$$

respectively, and substituting these in the Lorentz condition

$$(\text{div } \mathbf{A} + A_0)\psi = 0$$

$d\psi$ and $d^*\psi$ can be rewritten as $ic\psi$ and $-ic^*\psi^*$. Taking this into account and constructing the self-energy from the interaction,

$$\Delta E^{(1)} = \frac{e^2}{(2\pi)^3} \int d\mathbf{l} \left\{ \frac{(\mathbf{P} | \sum_{i=1}^4 a_i A_{\mathbf{P}-\mathbf{l}}^+ | \mathbf{P})}{E_{\mathbf{P}} - E_{\mathbf{P}-\mathbf{l}} - l} - \frac{(\mathbf{P} | \sum_{i=1}^4 a_i A_{\mathbf{P}-\mathbf{l}}^- | \mathbf{P})}{-E_{\mathbf{P}} - E_{\mathbf{P}-\mathbf{l}} - l} \right\} \frac{2\pi}{l} \quad (1.1)$$

$$A_{\mathbf{P}-\mathbf{l}}^\pm = (E_{\mathbf{P}-\mathbf{l}} \pm (\boldsymbol{\alpha} \cdot \mathbf{P} - \mathbf{l}) \pm \beta\mu) / 2E_{\mathbf{P}-\mathbf{l}}, \quad a_i = (\boldsymbol{\alpha}, a_4), \quad a_4 = i\alpha_0 = i$$

where \mathbf{P} is the momentum of the Dirac particle (whose proper mass is μ) and \mathbf{l} that of the photon.

Taking the spur and deformining, we have

* Note Added in Proof: The same procedure as we had used, was developed more precisely by several authors. S. T. Ma. Phys. Rev. **75** (1949) 535, F. J. Belinfante. Phys. Rev. **76** (1949), 226.

$$\Delta E^{(1)} = \frac{e^2}{2\pi^2 E_P} \left[\mu^2 \int \frac{1}{(E_{\mathbf{P}-\mathbf{l}} + l)^2 - E_P^2} \frac{E_{\mathbf{P}-\mathbf{l}} + l}{E_{\mathbf{P}-\mathbf{l}}} \frac{d\mathbf{l}}{l} \right. \\ \left. + \frac{1}{2} \int \frac{E_{\mathbf{P}-\mathbf{l}} + l}{l E_{\mathbf{P}-\mathbf{l}}} d\mathbf{l} - \int \frac{d\mathbf{l}}{E_{\mathbf{P}-\mathbf{l}}} \right] \quad (1.2)$$

This equation corresponds to Eq. (7) in Feynman's paper in Phys. Rev. **74**, 1430 (1948). Feynman says that the third term is not "invariant," but actually, according to our prescription,^{b)} this term, too, can be expressed in terms of an energy-momentum space four-dimensional scalar w , and is independent of \mathbf{P} . Consequently, Feynman's argument that its invariance is destroyed by the C-meson (or f -Field) does not hold.

In this case, we take the following quantity as:

$$2(l, P-l) = (E_{\mathbf{P}-\mathbf{l}} + l)^2 - E_P^2 = \mu^2 w \quad (1.3)$$

where $(l, P-l)$ is the scalar product of the energy-momentum four vector of the two particles appearing in the intermediate state; and it can readily be seen that this takes the form given here, for either of the two processes responsible for the self-energy, that is for the virtual state either involving the creation of an electron pair. Then

$$l = \frac{\mu^2 w}{2[\{(1+w)\mu^2 + P^2\}^{1/2} - P_x]} \quad (x \equiv \cos \theta) \\ \frac{d\mathbf{l}}{E_{\mathbf{P}-\mathbf{l}}} = \frac{\mu^2}{2[\{(1+w)\mu^2 + P^2\}^{1/2} - P_x] \{(1+w)\mu^2 + P^2\}^{1/2}}$$

Making the transformation of variables $l \rightarrow w$, and integrating over the angles,^{*)}

$$\Delta E^{(1)} = \frac{e^2 \mu^2}{2\pi E_P} \left[\int_0^\infty \frac{d\tau w}{1+\tau w} + \frac{1}{2} \int_0^\infty \frac{\tau w}{1+\tau w} d\tau w - \frac{1}{2} \int_0^\infty \frac{\tau w^2}{(1+\tau w)^2} d\tau w \right] \\ = \frac{e^2 \mu^2}{E_P} \frac{3\mu}{2\pi} \left[\frac{1}{2} \log(1+w) - \frac{1}{6} \right]_{w \rightarrow \infty} \quad (1.4)$$

This results posses the transformation property of mass, and also coincides with of Schwinger⁶⁾ inclusive of the finite terms.

*) For instance, the third term of (1.2),

$$\int \frac{d\mathbf{l}}{E_{\mathbf{P}-\mathbf{l}}} = \int_0^\infty \int_{-1}^1 \int_0^{2\pi} \frac{l^2 d\mathbf{l} d\chi d\varphi}{\sqrt{\mu^2 + (\mathbf{P}-\mathbf{l})^2}} = \frac{\pi \mu^6}{2} \int_0^\infty \frac{\tau w^2}{\{(1+\tau w)\mu^2 + P^2\}^{1/2}} \int_{-1}^1 \frac{d\chi}{[\{(1+\tau w)\mu^2 + P^2\}^{1/2} - P_x]^3} \\ = \frac{\pi \mu^2}{2} \int_0^\infty \frac{\tau w^2}{(1+\tau w)^2} d\tau w$$

§ 2. The Self-energy of a Dirac particle due to the C-meson.

The self-energy of a Dirac particle due to C-meson is⁴⁾

$$\Delta E^{(2)} = -\frac{f^2}{(2\pi)^2 E_{\mathbf{P}}} \int \frac{(Pl + 2\mu^2)(E_{\mathbf{P}-\mathbf{l}} + \epsilon_{\mathbf{l}}) - E_{\mathbf{P}}^2 \epsilon_{\mathbf{l}}}{(E_{\mathbf{P}-\mathbf{l}} + \epsilon_{\mathbf{P}})^2 - E_{\mathbf{P}}^2} \frac{1}{\epsilon_{\mathbf{l}} E_{\mathbf{P}-\mathbf{l}}} dl \quad (2.1)$$

where \mathbf{l} , $\epsilon_{\mathbf{l}}$ and x are the momentum, energy, and mass of the C-meson, respectively. In this case, we take

$$2(l, P-l) + x^2 = (E_{\mathbf{P}-\mathbf{l}} + \epsilon_{\mathbf{l}})^2 - E_{\mathbf{P}}^2 = w \quad (2.2)$$

as the four-dimensional scalar quantity w in the energy-momentum space, and assume its dimension to the square of mass. Then

$$l = \frac{av \pm b\sqrt{a^2 - 4x^2(b^2 - v^2)}}{2(b^2 - v^2)}$$

Since l takes all positive values, we take the $+$ in the double sign.

$$\epsilon_{\mathbf{l}} = \frac{ab + v\sqrt{a^2 - 4x^2(b^2 - v^2)}}{2(b^2 - v^2)} \quad (2.3)$$

$$\frac{dl}{E_{\mathbf{P}-\mathbf{l}} \epsilon_{\mathbf{l}}} = \frac{dw}{b\sqrt{a^2 - 4x^2(b^2 - v^2)}} \quad (w + x^2 \equiv a, \sqrt{w + E_{\mathbf{P}}^2} \equiv b, Px \equiv v)$$

Using the abbreviations

$$A \equiv \sqrt{s - 4x^2v^2}, \quad s \equiv a^2 - 4x^2b^2, \quad B \equiv b^2 - v^2$$

(21) can be written as

$$\begin{aligned} \Delta E^{(2)} = & -\frac{f^2}{2\pi E_{\mathbf{P}}} \int_{w_0}^{\infty} \frac{dw}{w} \int_{-1}^1 \left\{ \frac{b(4x^2 - a)}{2} + \frac{abw}{2B} + \frac{avA}{2B} \right\} \\ & \times \left\{ \frac{a^2v^2}{4bB^2A} + \frac{bA}{4B^2} + \frac{av}{2B^2} \right\} dx \\ & (w_0 = x^2 + 2x\mu) \end{aligned} \quad (2.4)$$

Since w is a scalar quantity, an integral whose integrand involves only w is an invariant quantity. We now subject the lower limit w_0 of the integral to a Lorentz transformation. When the particle is in motion, this seems to depend on \mathbf{P} . However, since the lower limit is a section of the Lorentz-invariant domain enclosed by the closed surface $w = \text{const.}$ within the domain restricted by the momentum-conservation, it must take the value (a constant) of w on this surface. Furthermore, since it coincides, in a special system of coordinates, with the lower limit of w for an electron at rest, this value is $w_0 = x^2 + 2x\mu$,⁵⁾ which is the value of w for $\mathbf{l} = \mathbf{P} = 0$, and forms the threshold energy of the C-meson. Among the nine terms arising from the multiplication of the two factors of the

integrand of (2.4), the four which are odd functions of $\tau = px$ vanish as the result of the integration for x . On collecting the remaining terms, the integration for x become as follows:

$$\begin{aligned} & -\frac{1}{8P} \{ (4\mu^2 - a) (a^2 + 4\mu^2 P^2) - 8x^2 a \tau v \} L \\ & + \frac{1}{8P} \{ 2a^2 b^2 (4\mu^2 - a) - 3a \tau v (a^2 + 4x^2 b^2) \} \left(-\frac{s}{2a^2 b^2} M + \frac{a^2 + 4x^2 b^2}{2a^2 b^2} L \right) \\ & + \frac{a^2 b^2 \tau v}{2P} \left\{ \frac{s^2}{4a^4 b^2} N + \frac{1}{a^4 b^4} \left(-\frac{3a^4}{8} + x^2 a^2 b^2 - 10x^4 b^4 \right) M \right. \\ & \quad \left. + \frac{1}{a^4 b^4} \left(-\frac{3a^4}{8} + x^2 a^2 b^2 + 6x^4 a^4 \right) L \right\} \end{aligned}$$

$$L = \int \frac{d\tau}{(b^2 - a^2) \sqrt{s + 4x^2 \tau^2}} = \int \frac{du}{b^2 - a^2 u^2} \quad \left(u = \frac{\tau}{\sqrt{s + 4x^2 \tau^2}} \right)$$

$$M = \frac{u}{b^2 - a^2 u^2} = \frac{2P \{ a^2 - 4x^2 (\tau v + \mu^2) \}^{1/2}}{s (\tau v + \mu^2)}$$

$$N = \frac{u}{(b^2 - a^2 u^2)^2} = \frac{2P \{ a^2 - 4x^2 (\tau v + \mu^2) \}^{3/2}}{s^2 (\tau v + \mu^2)^2}$$

Putting these in order, the coefficient of L becomes zero, while terms involving M and N are reduced to

$$\begin{aligned} & \frac{\mu^2 \tau v \sqrt{a^2 - 4x^2 (\tau v + \mu^2)}}{(\tau v + \mu^2) (w + x^2)} - \frac{\{ a^2 - 4x^2 (\tau v + \mu^2) \}^{3/2}}{4 (\tau v + \mu^2) (\tau v + x^2)} \\ & \quad - \frac{w \{ (w + x^2)^2 - 4x^2 (\tau v + \mu^2) \}^{5/2}}{4 (w + x^2) (\tau v + \mu^2)^2} \end{aligned}$$

which do not depend on \mathbf{P} , so that $\Delta E^{(2)}$ can be expressed in terms of τv alone. That is,

$$\begin{aligned} \Delta E^{(2)} &= -\frac{f^2 \mu^2}{2\pi E_{\mathbf{P}}} \int_{w_0}^{\infty} \left\{ \frac{\{ (\tau v + x^2)^2 - 4x^2 (\tau v + \mu^2) \}^{1/2}}{(\tau v + x^2) (\tau v + \mu^2)} - \frac{\{ (\tau v + x^2)^2 - 4x^2 (\tau v + \mu^2) \}^{3/2}}{4 \tau v (\tau v + x^2) (\tau v + \mu^2)^2} \right\} d\tau v \\ &= -\frac{f^2 \mu^2}{2\pi E_{\mathbf{P}}} \left[\frac{3}{4} \int_{w_0}^{\infty} \frac{d\tau v}{\sqrt{W}} + \frac{\mu^2 - x^2}{4\mu^2} - \frac{x^2 (x^2 - 4\mu^2)^2}{4\mu^4} \int_{w_0}^{\infty} \frac{d\tau v}{w \sqrt{W}} \right. \\ & \quad \left. + \frac{x^2 - \mu^2}{4\mu^4} \{ (x^2 + \mu^2)^2 + 2\mu^2 (\mu^2 - 4x^2) \} \int_{w_0}^{\infty} \frac{d\tau v}{(\tau v + \mu^2) \sqrt{W}} \right] \\ & \quad (W = (\tau v + x^2)^2 - 4x^2 (\tau v + \mu^2)) \end{aligned}$$

Carrying out the integration, and rewriting the result in terms of $\delta \equiv \frac{x}{\mu}$,

$$\Delta E^{(2)} = \begin{cases} -\frac{f^2 \mu^2}{2\pi E_P} \left[\frac{3}{4} \log 'w' + \frac{1}{4} \left\{ 1 - \delta^2 + (\delta^4 - 6\delta^2) \log \delta + \delta(\delta^2 - 4) \sqrt{\delta^2 - 4} \right. \right. \\ \quad \left. \left. \times \log \left(\frac{1}{2} (\delta - \sqrt{\delta^2 - 4}) \right) \right\} \right] & (\delta > 2) \\ -\frac{f^2 \mu^2}{2\pi E_P} \left[\frac{3}{4} \log 'w' - \frac{1}{4} \{ 3 + 8 \log 2 \} \right] & (\delta = 2) \\ -\frac{f^2 \mu^2}{2\pi E_P} \left[\frac{3}{4} \log 'w' + \frac{1}{4} \left\{ 1 - \delta^2 + (\delta^4 - 6\delta^2) \log \delta + \delta(\delta^2 - 4) \sqrt{4 - \delta^2} \right. \right. \\ \quad \left. \left. \times \left\{ \sin^{-1} \left(-\frac{\delta}{2} \right) + \frac{\pi}{2} \right\} \right\} \right] & (\delta < 2) \end{cases} \quad (2.5)$$

among the logarithmic diverging terms of which 'w' is obtained by dividing w by μ^2 . This is a dimensionless quantity just as (1.3)**). (2.5) also has the transformation property of mass.

On placing the condition $f^2 = 2c^2$,⁴⁾ we get, from (2.5) and (1.4),

$$\Delta E = \Delta E^{(1)} + \Delta E^{(2)} = \begin{cases} -\frac{c^2 \mu^2}{4\pi E_P} \left[2 - \delta^2 + (\delta^4 - 6\delta^2) \log \delta + \delta(\delta^2 - 4) \sqrt{\delta^2 - 4} \right. \\ \quad \left. \times \log \left\{ \frac{1}{2} (\delta - \sqrt{\delta^2 - 4}) \right\} \right] & (\delta > 2) \\ +\frac{c^2 \mu^2}{4\pi E_P} [2 + 8 \log 2] & (\delta = 2) \\ -\frac{c^2 \mu^2}{4\pi E_P} \left[2 - \delta^2 + (\delta^4 - 6\delta^2) \log \delta + \delta(\delta^2 - 4) \sqrt{4 - \delta^2} \cos^{-1} \frac{\delta}{2} \right] & (\delta < 2) \end{cases} \quad (2.5)$$

When $\delta < 2$, and μ is taken $= M$ (the proton mass), this gives the mass difference between proton and neutron regarded as being two states, differing only in their e - and f -charges, of a single particle.

Since

$$\cos^{-1} \frac{\delta}{2} = \tan^{-1} \frac{\sqrt{4 - \delta^2}}{\delta}$$

this coincides exactly with Eq. (16) of Pais²⁾ paper III, 4. Thus, the C-meson by itself is insufficient in order to account for the proton-neutron mass difference

**))

$$\int_{w_0}^w \frac{d'w}{\sqrt{W}} = \log 2w - \log 2x\mu = \log \frac{2w}{2\mu^2} - \log \frac{2x\mu}{2\mu^2} = \log 'w' - \log \delta$$

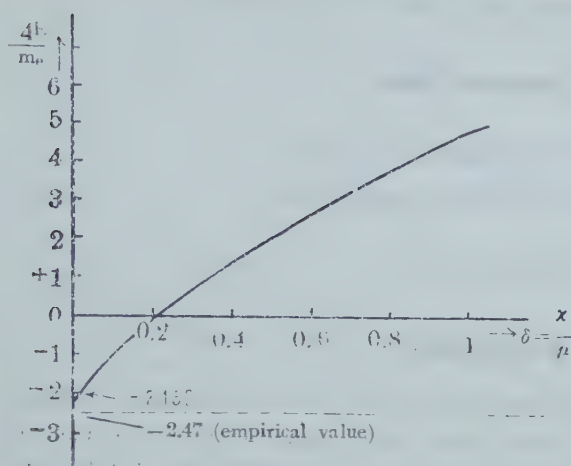


Fig. 1.

As a result, new suggestions might be revealed concerning such problems as the asymmetry between the proton-proton force and proton-neutron force.

In concluding, authors wish to express their sincere gratitude to Prof. Sakata and Dr. Tanikawa for their interest throughout this work.

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(see Fig. 1), and if its mass is taken as ≈ 100 , which was hitherto thought appropriate, the mass difference amounts to only about $-1.4m_p$.

The question of mass-difference seems to be one which must be taken up again considering the effect of the nuclear force meson as well. Consequently, the problem of mirror nuclei, which was hitherto thought to be a "stumbling block" for C-meson theory, must presumably also be re-examined from this new standpoint.

On the Nuclear Stars.

Yoichi FUJIMOTO and Yoshio YAMAGUCHI

Physics Institute, Tokyo University.

(Received June 27, 1949)

§ 1. Introduction.

About ten years ago, it was discovered that the cosmic radiation produced stars in photographic emulsions¹⁾. These stars were interpreted as nuclear disruptions. A theoretical analysis of these phenomena was undertaken by Bagge²⁾ on the basis of the evaporation theory of nuclear physics. But, because his consideration was made referring to the cosmic ray and the energy of the star-producing agent was not known, the result could not be quantitative.

Recently, however, Berkeley 184" cyclotron was built and it has become possible to study various high energy nuclear reactions excited by an agent of definite energy. Thus the stars initiated by high energy deuterons were treated theoretically by Horning and Baumhoff³⁾.

Independently of them, we have been studying the same problem in a somewhat different way, which will be reported in the present paper. The main difference of our consideration from the above cited American authors' is that we take account of the change in binding energy of neutrons and protons when the nucleus loses these particles step by step by the evaporation process. This effect, which was neglected by them, affects considerably the prong number distribution.

In § 2, we will discuss the mean behavior of the evaporation process of excited nuclei, generalizing Bagge's treatment. The fluctuation problem is treated in § 3. § 4 is then devoted to the application of our consideration and comparison with experiments, such as stars produced by π^- -meson capture, stars by high energy deuteron bombardment, nuclear reaction $\text{As}^{75} + 190 \text{ MeV D}$, etc. For this purpose it is necessary to find the statistical distribution of the initial excitation energy. We treat this problem by a simple classical model, which can be used also to predict nuclear transparency.

§ 2. Evaporation Theory.

When a nucleus is excited with the energy X , the probability per second of emitting a proton with the energy ξ , $\xi + d\xi$ is given by the evaporation theory of Weisskopf⁴⁾ as follows,

$$W_p(X, \xi) d\xi = \gamma(\xi - V - E_p) \exp [S_B(X - \xi) - S_A(X)] d\xi, \quad (2)$$

where S_A and S_B are the entropy of the nucleus before and after the evaporation, E_p is the binding energy of a proton, and V the Coulomb barrier height. The kinetic energy of an emitted proton is $\xi - E_p$ in this notation. γ is a certain constant which is given by

$$\gamma = \sigma_0 j m / \pi^2 \hbar^2 \quad (2)$$

where σ_0 is the geometrical cross-section of a nucleus, j the statistical factor, and m the proton mass.

The probability per second $W_n d\xi$ of emitting a neutron with energy ξ , $\xi + d\xi$ is obtained from (1) by replacing E_p by E_n , which is the binding energy of a neutron, and putting the barrier height $V=0$.

Thus the total probability per second of evaporating a proton or a neutron is, respectively,

$$\left. \begin{aligned} \Gamma_p(X) &= \int_{V+E_p}^X W_p(X, \xi) d\xi, \\ \Gamma_n(X) &= \int_{E_n}^X W_n(X, \xi) d\xi. \end{aligned} \right\} \quad (2)$$

If we adopt the Fermi gas model for a nucleus, the entropies can be explicitly written as,

$$\left. \begin{aligned} S_A(X) &\simeq 0.632 (AX)^{1/2}, \\ S_B(X) &\simeq 0.632 [(A-1)X]^{1/2}, \end{aligned} \right\} \quad (3)$$

where X is measured in MeV, and A is the mass number of the initial nucleus. Using (3), we obtain an approximate expression for the ratio of $\Gamma_n(X)$ to $\Gamma_p(X)$.

$$\frac{\Gamma_n(X)}{\Gamma_p(X)} \approx \frac{X - E_n}{X - E_p - V} \exp. [S_n(X - E_n) - S_p(X - E_p - V)], \quad (4)$$

which is valid for not too small X .

Next, we consider the successive emission of nucleons from an excited nucleus. From (1), we see that the mean energies taken away by the evaporated proton and the neutron are roughly given by

$$E_p + V + 2T, \quad E_n + 2T,$$

respectively, where T is the temperature of a nucleus with excitation energy X , which is given by,

$$T = 1 / \left(\frac{dS_A(X)}{dX} \right).$$

In the course of the evaporation process, the nucleus loses its excitation energy gradually and is cooled down. This decrease of the excitation energy X is, on an average, expressed by the following mean cooling law, which is a generalization of Bagge's one²⁾,

$$\left. \begin{aligned} -\frac{\partial X}{\partial p} &= E_p + T + 2T, \\ -\frac{\partial X}{\partial n} &= E_n + 2T. \end{aligned} \right\} \quad (5)$$

Here n and p denote the number of evaporated neutrons and protons.

Now the ratio of differentials dn/dp , which defines the average direction of transmutation of the evaporating nucleus, is determined by (4), i. e.

$$\frac{dn}{dp} = \frac{\Gamma_n(X)}{\Gamma_p(X)} \quad (6)$$

The set of simultaneous equations (5), (6) determine the mean behavior of the evaporation process.

Before going to solve (5) and (6), we must give the explicit expressions for E_p and E_n . These are in the first approximation given by⁵⁾

$$\left. \begin{aligned} E_p &\approx E_b - \beta \left(n - \frac{1+\mu}{1-\mu} p \right), \\ E_n &\approx E_b + \beta \left(n - \frac{1+\mu}{1-\mu} p \right), \end{aligned} \right\} \quad (7)$$

where E_b is the binding energy of a nucleon in a stable nucleus. μ represents the direction of the Heisenberg valley in a nuclear chart, i. e.

$$\mu = \frac{d}{dA} (A - 2Z_A).$$

Z_A corresponds to the most stable charge of a nucleus with mass number A . β in (7) is the constant determining the slope of binding energies of protons and neutrons on the nuclear chart. As we are interested in the stars from heavy nuclei in emulsion, such as Ag or Br, we will take $A=100$. In this case, the constants stated above have the following values:

$$E_b \approx 8 \text{ MeV}, \quad \beta \approx 0.9 \text{ MeV}, \quad \mu \approx 1/4. \quad (8)$$

We have solved the equations (5), (6) with constants (8) by numerical integration, for four different values of X . The solution determines the path in the n - p plane, as is shown in Fig. 1. The end point of a path expresses the mean total number of emitted protons and neutrons.

The concavity of a path is the effect of the change of E_p and E_n . If we neglect this change, the path becomes convex, which is also shown in Fig. 1. This is because, the lower is the temperature of nucleus, the more difficult it is for a proton to go over the barrier. We see that the mean number of emitted protons is too small, when we neglect this change of the binding energies. This number becomes larger in our treatment, because the decrease of the binding energy of a proton in the course of the evaporation process is properly taken into consideration.

Finally we must remark one point. In the above treatment, we have neglected altogether the emission probability of α -particles and other heavier nuclei. This neglect seems to be valid for our case $A=100$, because the potential barrier is here so high that the emission of heavier particles is much reduced.

§ 3. Fluctuation.

As we have calculated the mean behavior of evaporation process, we will consider its fluctuation. From Fig. 1, we see that the number p of protons evaporated from the nucleus is considerably small, so that its fluctuation will be nearly of Poisson type.

To examine the fluctuation of number $a=n+p$ of emitted nucleons (neutrons and protons), we consider the following simple model: a nucleus which consists of neutrons only and the emission of neutrons from this nucleus with constant temperature T . Then the energy distribution of the evaporated neutrons is approximately given by

$$\frac{1}{T^2} e^{-\xi/T} \xi d\xi \quad (9)$$

The probability for a nucleus to evaporate n or more neutrons is obtained by integrating (9) through the energetically allowed region, i. e.

$$\begin{aligned} P_n(X) &= \frac{1}{T^{2n-2}} \int \dots \int \xi_1 \dots \xi_{n-1} \exp \left[- \sum_{j=1}^{n-1} \xi_j / T \right] d\xi_1 \dots d\xi_{n-1} \\ (n-1) E_b &\leq \sum_{j=1}^{n-1} \xi_j \leq X - n E_b \\ &= \Gamma(2n-2, \frac{X-nE_b}{T}) / \Gamma(2n-2), \end{aligned} \quad (10)$$

where X is the excitation energy of a nucleus corresponding to the temperature T .

If we replace n in (10) by $a(=n+p)$, this model represents the actual nuclear evaporation considerably well, because of the following fact. In the actual

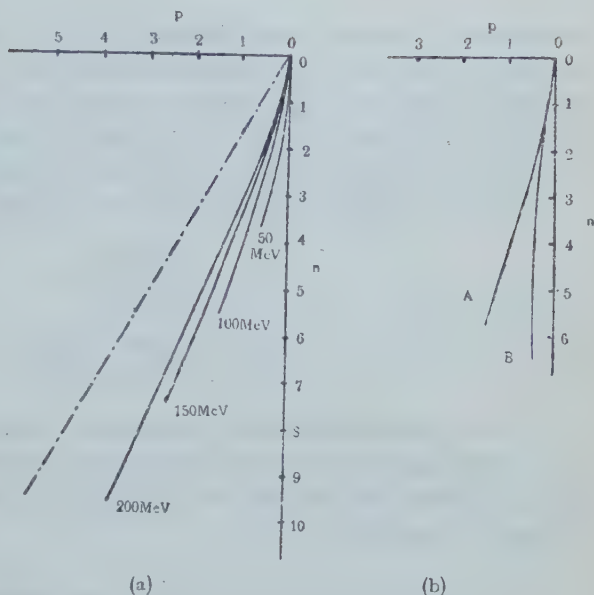


Fig. 1. (a) Solid curves: Solutions of eqs. (5), (6), and (7) for initial excitation energies $X=50, 100, 150$, and 200 Mev. Chain curve: the line $n=1.7p$, which shows the most stable nuclei. (b) Solution of eqs. (5) and (6) for $X=100$ MeV, A: with E_p and E_n given in (7); B: with $E_p = E_n = E_b = 8$ MeV.

evaporation process, as is shown in Fig. 1, more neutrons are emitted in the beginning, and the rate of protons increases gradually later on. So the mean energy carried away by one nucleon is roughly constant throughout the whole process. This corresponds to the constant temperature of the above model.

Approximating (10) by Gaussian distribution, we have the mean number $\langle a \rangle$ of the emitted nucleons and its root mean square deviation Δa as follows:

$$\langle a \rangle = \frac{X+3T}{E_b+2T}, \quad (11)$$

$$\Delta a = \frac{E_b}{T} \cdot \left(\frac{2X-3E_b}{E_b+2T} \right)^{1/2} \quad (12)$$

The error introduced by the assumption of constant temperature can be shown to be small, which follows from the fact that the nuclear matter has a large specific heat, so that the temperature changes only little even when the energy content changes considerably.

§ 4. Applications.

a) Nuclear transparency. In order to apply the theory of § 2, § 3, we must calculate the excitation energy X of a nucleus and its distribution, when it is bombarded by a particle with energy E . For the high incident energy, nuclear matter becomes transparent⁶⁾. Here we will treat this problem in a semi-quantitative way, and more accurate treatment will be done in a subsequent paper.

At high energies ($\gtrsim 100\text{MeV}$) the wave length of a nucleon is so small compared with the nuclear radius R , that we can imagine the classical orbit of an incident particle. Let ρ be the path length of an incident particle in a nucleus whose radius is denoted by R . Then the distribution function of ρ is given by

$$2 \cdot \frac{\rho}{2R} \cdot \frac{2R}{d\rho}. \quad (13)$$

Let λ be the mean free path of an incident nucleon in a nuclear matter, and ϵ be the mean energy transfer per one collision with a nucleon in a nucleus. Using these two quantities, we can calculate the mean energy transfer X to a nucleus when the path length is ρ ,

$$X = \frac{\rho}{\lambda} \epsilon. \quad (14)$$

Therefore the distribution of excitation energy X is, together with (13) and (14)

$$2 \left(\frac{\lambda}{2R} \right)^2 \frac{X dX}{\epsilon^2}, \quad \left(0 \leq X \leq \frac{2R}{\lambda} \epsilon \right). \quad (15)$$

When the incident particle is a deuteron, the corresponding distribution function of X for a large nucleus ($R \gg d = \text{radius of a deuteron}$) is roughly given by⁷⁾

$$2 \left(\frac{\lambda}{4K} \right)^2 \frac{XZY}{\epsilon^2}, \quad (0 \leq X \leq \frac{4K}{\lambda} \epsilon). \quad (16)$$

(22) and (23) can be applied only when the incident energy E is so high that a nucleus becomes completely transparent. If the energy of an incident nucleon E is $< \frac{2R}{\lambda} \epsilon$, a nucleus becomes partially opaque, and some modifications must be done in the above formula.

From the experiments of high energy nucleon-nucleon collision, λ is known to be proportional to E and ϵ will be expected proportional to E . Thus λ/ϵ is approximately independent of the incident energy E . According to the calculation of Serber⁶⁾, its numerical value is ($\lambda \approx 4 \times 10^{-13}$ cm, $\epsilon \approx 25$ MeV)

$$\lambda/\epsilon \approx 1.6 \times 10^{-14} \text{ cm MeV}^{-1}.$$

b) High energy reaction $\text{As}^{75} + 190 \text{ MeV D}$. In Fig. 2 the experimental relative yields of the products in this reaction⁸⁾ are given. In the same figure, we have reproduced the paths given in Fig. 1. These calculated paths agree fairly well with the maximal line of the yields and our evaporation theory seems to be able to explain the main character of this high energy reaction. The products, which correspond to more emitted neutrons and protons than the end point of the path for $X=200$ MeV, must be due to the fluctuations. These fluctuations were already calculated in § 3, and the results agree approximately

with the experimental ones.

c) Stars initiated by the high energy deuterons. The stars initiated by bombardment of 35, 90, 130, and 190 MeV deuterons were observed at Berkeley⁹⁾, and their prong number distributions are shown in Fig. 3. Applying our calculations in § 2, § 4 a), we obtain the average prong number distribution of the stars, which originate from the heavy nuclei in emulsion. These results are also shown by broken lines in Fig. 3. We see that the discrepancy between the calculated and the observed ones is more pronounced as the energy of the incident deuterons becomes lower.

In this connection, we remark several points:

1) In the above estimation, we have taken account of protons of all energies. If we consider the insensitivity of emulsions to high energy (< 15 MeV) protons, the calculated prong number will be considerably reduced. So it will be hopeless that we explain all of these stars

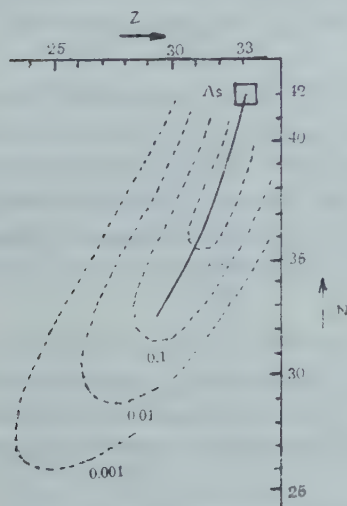


Fig. 2. $^{75}_{33}\text{As} + 190 \text{ MeV D}$. Dotted curves represent the relative "equi-yield" curves, obtained from the experiment⁸⁾. The solid curve is the solution of (5), (6), and (7) for $X=200$ MeV.

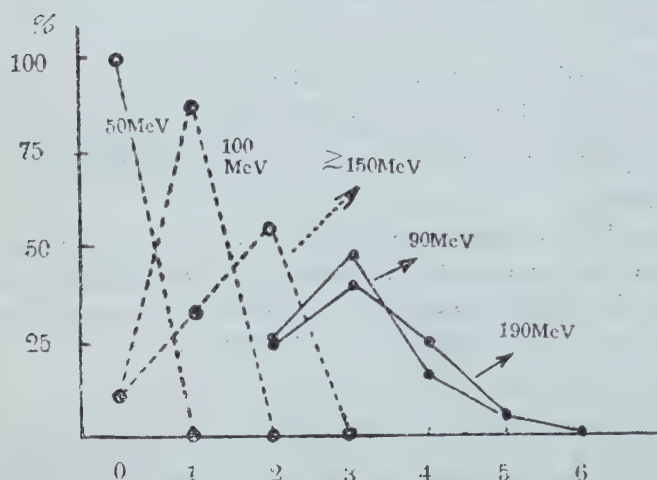


Fig. 3. Percentage of stars of given number of prongs. Solid curves: experimental¹⁰⁾. Dotted curves: theoretical (the stars of heavy elements in emulsion, bombarded by deuterons with the energies indicated in the figure).

The theoretical and experimental values of the percentage differ in normalization. If here, we do not take account of this difference as well as the effect of fluctuation. On these points, see ref. 16).

low energy nuclear collisions. We suppose that almost all the stars observed at Berkeley may be of such a kind.

III) This consideration seems to be supported by two experimental facts. First, the observed angular distribution of star prongs can not be explained by the evaporation process of heavy nuclei^{3) 9)}. Second, the observed prong number distribution in emulsion by deuteron bombardment and the one in Wilson chamber (filled with alcohol and water vapour) by 100 MeV neutron bombardment¹⁰⁾ show almost the same character. The latter concerns only the stars from light nuclei.

d) Stars by π^- -meson capture.^{*)} The capture probability of a stopped π^- -meson for various nuclei in chemical compounds is approximately proportional to the atomic number Z and the density¹¹⁾. If a π^- -meson is stopped in emulsion, the capture probabilities for light nuclei (C, N, and O) and heavy nuclei (Ag and Br) are ~ 0.2 and 0.8 , respectively.

First we will consider the stars from heavy nuclei, assuming that:

- i) The probability of π^- -meson capture is equal for all protons in the nucleus.
- ii) When a proton captures a π^- -meson, it changes into a neutron and receives the kinetic energy which is equal to the rest energy of a π^- -meson. The initial direction of motion of this π^- -meson-captured proton is isotropic.

Then the distribution of path length ρ of the π^- -captured proton in a nucleus is

using the evaporation theory.

II) Besides the heavy nuclei, the emulsion contains light nuclei, such as C, N, and O. As the evaporation theory can not be applied to these light nuclei, we have heretofore disregarded the stars from them. If we assume the α -particle model for them, we may imagine that these light nuclei will easily be split up into several α -particles and nucleons by nuclear collisions. Thus a light nucleus will more easily produce a star than a heavy one, especially by

$$\frac{3}{2} \left\{ 1 - \left(\frac{\rho}{2R} \right)^2 \right\} \frac{d\rho}{2R}. \quad (17)$$

From (17) we have the distribution of energy transfer X to the nucleus

$$\frac{3}{2} \left\{ 1 - \left(\frac{\lambda}{2R} \frac{X}{\epsilon} \right)^2 \right\} \frac{\lambda}{2R} \frac{dX}{\epsilon}, \quad (0 \leq X \leq \frac{2R}{\lambda} \epsilon). \quad (18)$$

Using (18) and the calculation in § 2, we obtain the prong number distribution of stars originating from π^- -captured heavy nuclei.

Next we treat the stars from light nuclei. The evaporation theory can not be applied to this case. But we may assume, referring to the consideration in § 4 c), that the stars of π^- -captured light nuclei will not be very different from the stars of light nuclei bombarded with high energy particles. Thus we may use the number of prongs borrowed from the results of Gardner's experiment.⁹⁾

The results of this analysis is shown in Table I comparing with experimental data¹²⁾. Agreement between the calculated and experimental values is not satisfactory. However, it may be expected to reach more satisfactory results, if we would take account of the effect of fluctuations.

Table I.

Number of prongs		0	1	2	3	4	5
Theoretical (in %)	Heavy	45	35	0	—	—	—
	Light	—	—	5	10	5	—
	Sum	45	35	5	10	5	—
Experimental (in %)		27.0	23.4	24.0	14.8	8.7	1.9

e) Taketani and Sasaki¹³⁾ have discussed the μ^- -capture in detail. According to their results, a light neutral particle carries away the major part of the liberated

energy, and the excitation of the nucleus is only about 10 MeV. So, it will be rare for the nucleus to emit a proton, and this was confirmed by many experiments.

f) Proton-decay¹⁴⁾. After the evaporation process, the binding energy of a proton E_p in the residual nucleus becomes considerably smaller than that of a neutron E_n (see Fig. 4), so that a proton can be emitted through penetration of the potential bar-

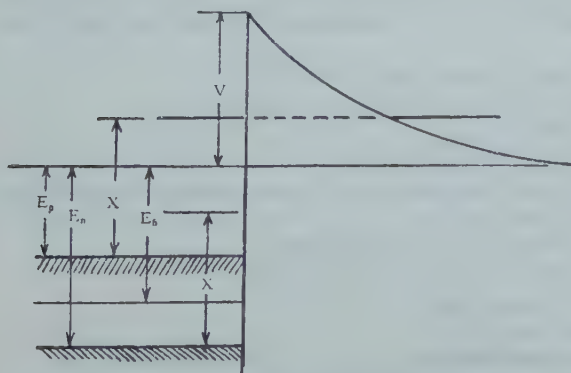


Fig. 4. The graphical representation of the state of a nucleus, when the evaporation process has finished.

rier, while a neutron can not escape from the nucleus energetically. Thus we may expect that a proton decay of a final nucleus takes place, and a slow proton in a star with energy smaller than the barrier height will be observed. In fact, such protons were observed by Perkins¹⁵⁾.

§ 5. Conclusion.

We have calculated the stars and the related nuclear phenomena using the statistical model of the nucleus, i. e. Serber model and Weisskopf's evaporation theory. The results seem to agree with experiments. Accordingly we think that the above picture will be valid for general high energy nuclear phenomena¹⁶⁾.

When the incident energy becomes higher than the threshold for meson production, the meson will play an important role and our model will become invalid. For example, the very large stars caused by cosmic radiation¹⁷⁾ do not seem to be explained by our theory. Also the stars by π^- -meson capture¹⁸⁾ would be out of the scope of our consideration. Such problems, which will have close relation with the penetrating showers in cosmic radiation, must be examined taking account of the mesonic effects.

Acknowledgement: We wish to express our sincere thanks to Prof. Tomonaga, Prof. Taketani and Mr. Hayakawa for their kind interest in this work.

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Addendum, see p. 581.

Applicability of Pauli's Regulator to the γ -Decay of Neutrettos*

H. FUKUDA and Y. MIYAMOTO, T. MIYAZIMA and S. TOMONAGA
Physics Institutes, Tōkyo University and Tōkyo Bunrika Daigaku

and

S. ŌNEDA, S. OZAKI and S. SASAKI
Physics Institute, Tōhoku University

(Received July 17, 1949)

§ 1. Possible origin of ambiguity involved in the field theory.

Although the present formulation of the field theory seems to be "proved" to satisfy the requirement of invariance with respect to the Lorentz and gauge transformations, puzzling facts are encountered in many problems that the results are non-invariant, such as the non-vanishing of the photon self-energy,⁽¹⁾ the non-gauge-invariance of the matrix elements for the γ -decay of neutrettos,⁽²⁾ or for the radiative decay of heavy mesons. Detailed examination,⁽³⁾ however, shows that, in most of these cases, results depend on the method of calculation, so that the conclusions are not unique, but quite ambiguous.

Where on earth are involved the origin of these ambiguities in the formalism of the present field theory? To our opinion, the reason for this seems to lie in the fact that the consistent formulation of the relativistic commutation relations and of the generalized Schrödinger equation can be attained only by using a singular function $\Delta(x)$.

In the first place, Δ plays a role of Green's function which expresses a wave field $A(x')$ by its initial value given on a space-like surface σ by

$$A(x') = \int_{\sigma} d\sigma_{\mu} A(x) \frac{\partial}{\partial x_{\mu}} \Delta(x-x'). \quad (1)$$

$\Delta(x)$ is the solution of the wave equation that vanishes in the space-like region. From the condition that $A(x')$ must approach to $A(x)$ when x' approaches to a point x on σ , one must have

$$\lim_{\sigma \rightarrow \sigma_0} \int_{\sigma} d\sigma_{\mu} \frac{\partial \Delta(x)}{\partial x_{\mu}} = 1, \quad (2)$$

*Though this study was, at first, made independently by the Tōkyo and Tōhoku groups, the results obtained were found to be almost identical when discussed together, and it was then approved to publish the paper jointly.

where σ_0 is an arbitrary space-like surface passing through the origin. This property of Δ would be inconsistent if it were finite at the origin. As Δ is an invariant, $\frac{\partial \Delta}{\partial x_\mu}$ should be a vector. However, on an arbitrary space-like plane through the origin its space-like components are zero and the time-like component has the value $i\delta^3(\mathbf{x})$. Thus, the value of $\frac{\partial \Delta}{\partial x_\mu}$ at the origin has no meaning, because it has only the time-like component in *any* reference system.

However, this singular behavior of the Green's function is not an essential difficulty of the field theory, but is common with any classical wave field. The difficulty gets worse by the fact that the same singular function $\Delta(x)$ also plays an important role in expressing relativistic commutation relations between field variables. The field quantities have thus infinite degrees of freedom, and fluctuation of various quantities becomes infinite. This is expressed by the more singular function $\Delta^{(0)}(x)$ that expresses the vacuum expectation value of a bilinear combination of the field quantity with its adjoint. The singularity of Δ is entangled with that of $\Delta^{(0)}$, and thus the essential divergence and ambiguity of the present theory result from this situation. In fact, the most integrals of products of several Δ and $\Delta^{(0)}$ become divergent. At times some integrals apparently converge, but they have ordinarily a form of $\infty - \infty$ or $\infty \times 0$. Thus the divergence affects even the finite terms and makes them ambiguous.

Since the consistency of the fundamental equations and the commutation relations could be formally proved, it is expected that reasonable results would be obtained by some careful procedure choosing suitable expressions out of the ambiguous ones. In fact, it is shown that we can obtain in this way gauge invariant results for the photon self-energy and the matrix elements for γ -decay of neutrettos. Pauli's regulator⁽⁴⁾ seems to provide us an automatic procedure obtaining the reasonable results. If a set of conditions for the regulator were found in order to regulate reasonably all possible ambiguities it would become a powerful method filling up the gaps involved in the field theory.

At first sight, it would seem desirable to regulate Δ or $\Delta^{(0)}$ itself from the beginning, but replacement of Δ by some regular function may destroy correspondence of the interaction representation to the Heisenberg-Pauli theory. The field quantity will then no longer satisfy the wave equation, and, as a result of it, the current expression $i\bar{\varphi}\gamma_\mu\varphi$ will lose its meaning, because it no longer satisfies the continuity relation. The regularization must, therefore, be applied to the resulting matrix elements for individual processes.

In this paper we want to examine applicability of the regulator in the case of γ -decay of neutrettos.

§ 2. γ -decay of neutrettos.

The general formulation and some solutions of the γ -decay problem were

given by Fukuda and Miyamoto.⁽²⁾ Let us briefly summarize their results. The generalized Schrödinger equation describing the system of the neutretto $\Psi(x)$, the proton $\varphi(x)$ and the electromagnetic field, $A_\mu(x)$ is given by

$$i \frac{\partial \Psi}{\partial \sigma} = (H_i + H_f) \Psi, \quad (3)$$

where $H_i = -i \bar{\varphi} \gamma_\mu \varphi A_\mu$ represents the interaction between the proton and the electromagnetic field, and H_f that between the neutretto and the proton. The latter is given by

$$H_f = f \bar{\varphi} \varphi I^* \quad (4)$$

for the scalar neutretto with the scalar interaction, and by

$$H_f = if \bar{\varphi} \gamma_\mu \varphi \partial_\mu I^* + 2\pi f^2 (\bar{\varphi} \gamma_\mu \varphi)^2 \quad (5)$$

and

$$H_f = if \bar{\varphi} \gamma_\mu \varphi V \quad (6)$$

for the pseudoscalar neutretto with the pseudovector and pseudoscalar couplings respectively. In (5) γ_N means the normal component of γ on the surface σ .

The matrix element responsible to the γ -decay of the neutretto can be obtained by the following canonical transformation $\Psi = U_c \Phi$ with U_c determined by

$$i \frac{\partial U_c}{\partial \sigma} = H_i U_c \quad (7)$$

The original equation (3) is transformed into

$$i \frac{\partial \Phi}{\partial \sigma} = U_c^{-1} H_f U_c \Phi = \langle H_f \rangle \Phi \quad (8)$$

The matrix element is obtained from this by evaluating the vacuum expectation value of $\langle \bar{\varphi} L \varphi \rangle$ with respect to the proton field, where $L = 1$, γ_μ and γ_5 for (4), (5) and (6) respectively.

Now the gauge invariance of $\langle \bar{\varphi} L \varphi \rangle$ and the identity

$$\partial_\mu \langle \bar{\varphi} \gamma_\mu \varphi \rangle = -2m \langle \bar{\varphi} \gamma \varphi \rangle \quad (9)$$

(m : proton mass)

are evident without any proof, because $\langle \bar{\varphi} L \varphi \rangle$ is regarded as the Heisenberg representation of $\bar{\varphi} L \varphi$ in the presence of electromagnetic field. But the direct proof shall be given here in order to make sure. For the transformation

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \partial_\mu A, & \square^2 A &= 0, \\ \varphi &\rightarrow \varphi, & \bar{\varphi} &\rightarrow \bar{\varphi}, \end{aligned} \quad (10)$$

the equation (7) for U_c is invariant if U_c is transformed according to

$$U_c \rightarrow e^{i\alpha} U_c$$

$$\gamma(\sigma) = -ic \int_{\sigma} A(x') d\sigma_{\mu}' \bar{\varphi}(x') \gamma_{\mu} \varphi(x'). \quad (11)$$

The equation (7) is then transformed into

$$e^{i\tau} i \frac{\partial}{\partial \sigma} e^{-i\tau} U_e = e^{i\tau} (H_e - ic \bar{\varphi} \gamma_{\mu} \varphi \partial_{\mu} A) e^{-i\tau}. \quad (12)$$

Now, as in the case of consistency proof of the generalized Schrödinger equation and the auxiliary condition, the commutability of γ with $\bar{\varphi} L \varphi$ taken on the surface σ plays an important role for the following proof. The commutability is "proved" as follows:

$$[\gamma(\sigma), \bar{\varphi} L \varphi] = -ic \int_{\sigma} A(x') d\sigma_{\mu}' (\bar{\varphi}' \gamma_{\mu} S(x' - x) L \varphi - \bar{\varphi} L S(x - x') \gamma_{\mu} \varphi')$$

which, in the reference system whose time axis coincides with the normal of σ at the point x , and using the property that $\frac{\partial A}{\partial x_{\mu}}$ has only the time-like component $i\delta^3(x)$ on the tangential plane of σ at x , becomes

$$[\gamma(\sigma), \bar{\varphi} L \varphi] = -c \int A(x') d^3 x \delta^3(x' - x) (\bar{\varphi}' L \varphi - \bar{\varphi} L \varphi') = 0 \quad (13)$$

The conclusion (13) is true only under the assumption that, in the first place, the relation (2) holds when σ is equated with σ_0 from the beginning, and in the second place, $\bar{\varphi}' L \varphi$ approaches $\bar{\varphi} L \varphi$ uniformly when x' approaches x . As we remarked in the preceding paragraph, the last assumption is quite uncertain, because, for example, the vacuum expectation value of $\bar{\varphi}' L \varphi$ has just the singularity of $S^{(1)}(x' - x)$ and (13) is indeterminate in the strict sense.

Nevertheless, admitting (13) for the moment, we shall show that (12) is identical with (7). This is easily seen from the relation:

$$e^{i\tau} i \frac{\partial}{\partial \sigma} e^{-i\tau} U_e = i \frac{\partial U_e}{\partial \sigma} - ic \bar{\varphi} \gamma_{\mu} \varphi \cdot \partial_{\mu} A \cdot U_e.$$

The invariance of $\langle \bar{\varphi} L \varphi \rangle$ with respect to the gauge transformation is also trivial.

The proof of the identity (9) runs similarly. First we notice that $\langle \bar{\varphi} L \varphi \rangle$ is a point function independent from the form of σ , and we suppose that σ is a space-like plane. $\partial_{\mu} \langle \bar{\varphi} \gamma_5 \gamma_{\mu} \varphi \rangle = \partial_{\mu} (U_e^{-1} \bar{\varphi} \gamma_5 \gamma_{\mu} \varphi U_e)$ is divided into the differentiation of $\bar{\varphi} \gamma_5 \gamma_{\mu} \varphi$ and that of U_e^{-1} and U_e . The former gives $-2m \langle \bar{\varphi} \gamma_5 \varphi \rangle$. In the latter case, the differentiation in the space-like direction is zero because U_e is unchanged for the displacement in the plane. For the time-like differentiation we utilize the differential equation for U_e and U_e^{-1} :

$$i\partial_4 U_e = \int d^3 x' H_e(x') \cdot U_e \quad \text{and} \quad -i\partial_4 U_e^{-1} = U_e^{-1} \cdot \int d^3 x' H_e(x').$$

Consequently we have

$$i < [\int d^3x' H_e(x'), \bar{\varphi} \gamma_5 \gamma_4 \varphi] >$$

for the latter, which vanishes similarly.

Now, according to the calculation of Fukuda and Miyamoto⁽²⁾, the term $< \bar{\varphi} L \varphi >_2$ of order ϵ^2 in $< \bar{\varphi} L \varphi >$ becomes

$$f < \bar{\varphi} \varphi >_2 = \frac{fm}{8\pi^2} \epsilon^2 \int_1^1 d\tau \int_1^\infty \frac{du}{u} \left[\frac{1}{u^2} A_\mu^2 - \frac{1}{m^2} \left(1 - \frac{1-\tau^2}{u^2} \right) [u^2]^{-1} \frac{1}{2} F_{\mu\nu}^2 \right], \quad (14)$$

$$\begin{aligned} f < \bar{\varphi} \gamma_5 \gamma_\mu \varphi >_2 = & \frac{f}{8\pi^2} \epsilon^2 \int_1^1 d\tau \int_1^\infty \frac{du}{u} \left[\frac{1}{u^3} \sum A_\nu F_{\sigma\tau} \right. \\ & \left. + \frac{1}{m^2} \frac{1-\tau^2}{u^2} [u^2]^{-1} \sum_{\lambda \text{ cycle}} F_{\lambda\nu} \partial_\lambda F_{\sigma\tau} \right], \end{aligned} \quad (15)$$

$$\begin{aligned} f < \bar{\varphi} \gamma_5 \varphi >_2 = & \frac{f}{4\pi^2 m} \epsilon^2 \int_{-1}^1 d\tau \int_1^\infty \frac{du}{u} \left[\frac{1}{u^2} + \frac{1-\tau^2}{4m^2 u^2} \square^2 [u^2]^{-1} \right] \\ & \times (F_{23} F_{14} + F_{31} F_{24} + F_{12} F_{34}) \end{aligned} \quad (16)$$

respectively, where $\gamma_5 \gamma_\mu = \gamma_5 \gamma_\mu \gamma_\sigma \gamma_\tau$ and $(\nu \sigma \tau)$ assumes (324), (134), (214), (123) as μ runs from 1 to 4, and $[u^2]$ is written for $u^2 - \frac{1-\tau^2}{4m^2} \square^2$. That the non-gauge-invariant terms A_μ^2 and $\sum A_\nu F_{\sigma\tau}$ still survive in (14) and (15) is the serious contradiction that $< \bar{\varphi} L \varphi >_2$ should, in general, be gauge invariant. Besides, the identity (9) is not satisfied. In fact, we have from (15)

$$\partial_\mu < \bar{\varphi} \gamma_5 \gamma_\mu \varphi >_2 = - \frac{1}{2\pi^2} \epsilon^2 \int_{-1}^1 d\tau \int_1^\infty \frac{du}{u} \left[\frac{1}{2u^3} + \frac{1-\tau^2}{4m^2 u^2} \square^2 [u^2]^{-1} \right] (F_{23} F_{14} + \dots), \quad (17)$$

the first term of which differs from that of $-2m < \bar{\varphi} \gamma_5 \varphi >_2$, as $\int_1^\infty du/u^3 = 1/2$ and $\int_1^\infty du/2u^4 = 1/6$. It is worthwhile to note that the gauge invariance and identity hold for all terms of higher order in expansion of \square^2/m^2 .

The results from (14) to (16) were calculated by applying Schwinger's method of integration. It can, however, be shown that the coefficients of A^2 and $\sum A F$ in (14) and (15) are only conditionally convergent, and these ambiguous terms are hoped to be able to be dropped by Pauli's regulator. In Schwinger's method of integration, there seems to be no ambiguity of the first term in the bracket of (16), but Schwinger's method itself is quite a special one and we arrive at a different result when different representation are used for A and A^0 . This situation is clearly seen in the fact that the value of $D^{(1)}$ at the origin is infinite in the Fourier representation but becomes zero in the Schwinger one.

§ 3. Discussions of the results.

First of all, we think it appropriate to drop $\sum AF$ term in (15) for the following reasons:

(1) It might be, at first sight, urged that the presence of the non-gauge-invariant term $\sum AF$ does not destroy the gauge invariance of the real processes,⁽⁵⁾ because $\sum A_\nu F_{\sigma\tau} \partial_\mu V$ can be written as $\partial_\mu (\sum A_\nu F_{\sigma\tau} V) + 2V(F_{23}F_{14} + \dots)$, and the last term is clearly gauge invariant while the first term has no contribution to the real processes on account of conservation of energy and momentum. We suppose, however, the gauge invariance of $\langle \bar{\varphi} \gamma_\mu \varphi \rangle_2$ itself should be required because the presence of the non-gauge-invariant term $\sum AF$ in $\langle \bar{\varphi} \gamma_\mu \varphi \rangle_2$ constitutes a difficulty in the case of pseudovector neutretto U_μ interacting with the nucleon by pseudovector coupling. In this case the corresponding matrix element has the form $\langle \bar{\varphi} \gamma_\mu \varphi \rangle U_\mu$ and the existence of the problematic term in $\langle \bar{\varphi} \gamma_\mu \varphi \rangle_2$ gives rise to $\sum U_\mu A_\nu F_{\sigma\tau}$ which is no longer gauge invariant even for the real processes. Indeed, the life time evaluated from it is infinite in the case of the neutretto at rest, but not so for moving one, whence the life time of the pseudovector neutretto does not transform correctly under Lorentz transformation if one retains this term.

(2) As the coefficient of $\sum AF$ is independent from the mass m , this term is dropped off by the regulator. That is to say, if we suppose that there are auxiliary Fermi particles of mass m_i , interacting with the neutretto by the coupling constant f_i which satisfy the condition

$$\sum_i f_i = 0, \quad (18)$$

then $\sum AF$ terms disappear, and the remaining terms retain their original value by making m_i infinitely large.

Next, we consider the case of pseudoscalar coupling. All terms contained in (16) are Lorentz and gauge invariant, so that we have no reason to drop the first term in the bracket. But, if we drop the $\sum AF$ term in the case of pseudovector coupling, this first term must also be dropped in order to preserve the equivalence between the pseudoscalar and pseudovector couplings.

In this situation we have two alternatives: either to abandon the equivalence or to admit to drop the first term even if it seems convergent and invariant. We are not sure which of these alternatives should be taken, but we remember that the equivalence has been "proved" and the latter alternative does not seem altogether impossible. In order that the pseudovector coupling $if_{pv} \langle \bar{\varphi} \gamma_\mu \varphi \rangle \partial_\mu V$ is equivalent to the pseudoscalar one $if_{ps} \langle \bar{\varphi} \gamma_5 \varphi \rangle V$, we must have the relation

$$\frac{1}{2m} f_{ps} = f_{pv} \quad (19)$$

If we use the regulator in the case of pseudoscalar coupling after we put (19)

into (16), we find that the first term $F_{23}F_{14} + \dots$ in the bracket of (16) really disappears by the condition similar to (18):

$$\sum_i (f_{pv})_i = 0. \quad (20)$$

This procedure can be interpreted either as mixing auxiliary fields with different masses maintaining Pauli's condition (18) but, on the other hand, assuming the coupling constants to depend on the masses, $f_{ps} \propto m$, or as mixing the fields with the condition

$$\sum_i \frac{1}{m} (f_{ps})_i = 0, \quad (21)$$

which differs from Pauli's one.

Similar situation is encountered in the case of scalar interaction (14). In this case there would be no reason to retain the non-gauge-invariant term A^2 because this term can be shown only conditionally convergent. Pauli's condition is, however, not sufficient to drop this term, and a more strict condition

$$\sum_i m_i (f_{sc})_i = 0 \quad (22)$$

must be required. As there is in this case no identity such as (9), we have no reason to consider the coupling constant f_{sc} to be inversely proportional to the mass. (22) must be taken as a new condition for the regulator.

There are two alternatives interpreting the regulator method, either as a mixed fields theory or as a formal procedure. If we take the first point of view, the regulator must satisfy the conditions

$$\sum_i f_i = 0, \quad \sum_i m_i f_i = 0 \quad \text{and} \quad \sum_i \frac{1}{m_i} f_i = 0. \quad (23)$$

If these conditions were to be universally applied, the third condition would bring a serious change of the life time of the scalar neutretto, because then the first term of expansion in \square^2/m^2 of the second term of (14) would also be dropped.*

If we take the second point of view and require that the regulator must always be applied only to the function of even power in m , we must first separate a factor of an odd power of m (m and $1/m$ in the cases of (14) and (16) respectively) and afterwards apply the regulator to the remaining terms. In this case the second term of (14) is unchanged, in contrast to the first point of view. But then there remains ambiguity in separating an odd power of m , because it is also possible to separate m^3 in the case of (14).

In the case of vector neutretto, two- γ decay is forbidden, and the matrix element for three- γ decay turns out to

*If the third condition be true, the anomalous magnetic moment of electron will vanish, which contradicts with the experimental fact.

$$\langle \bar{\varphi} \gamma_{\mu} \varphi \rangle_0 = -\frac{1}{24\pi^2} \epsilon^3 A^2 A_{\mu} + \dots,$$

the first term of which can be made to vanish by the regulator.

We are thus inclined to believe that the present theory does not teach us which of various alternatives we should take. We have hoped that Pauli's regulator could give decision at this point, but it seems that there remains still some ambiguity how to use the regulator. In any case, we think that our problem will present a severe test to the regulation method and in this situation we think it desirable that some experiment which could detect the γ -decay of neutretto will answer this problem and provide us some clue to the correct future theory.

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On the Reaction of Radiation Field

Muneo SASAKI and Ryoji SUZUKI

Physical Institute, Tokyo Bunrika Daigaku

(Received July 26, 1949)

§ I. Introduction

To the excellent experiment on the hydrogen fine structure levels observed by Lamb and Retherford¹⁾, Bethe²⁾ has given the interpretation that the displacement of spectrum are attributed to the interaction between the electron and the radiation field. As is well known, the current field theory hitherto has serious difficulties that it gives always diverging results, when one would calculate the radiative mass of a particle, or proceed to the higher order approximation of interaction energy. In the case of the level shift of hydrogen atom, this difficulty appears too, giving rise to diverging results. Bethe, however, has pointed out the fact that the difference between thus calculated interaction energy and "properly evaluated" self-energy of a free electron should be finite and give the observed level shift. Although his preliminary calculations, made in non-relativistic approximation, was not good enough to verify that the theory would give actually converging results, it was expected that relativistic treatment would yield satisfactory consequence. Thereafter, Tomonaga, Miyamoto, and Fukada³⁾, Lamb and Kroll⁴⁾ have made calculations relativistically for the Dirac electron and verified that one obtains really finite value of the level shifts.

The present paper deals with the similar calculation, assuming a scalar particle to be a first approximation of Dirac electron. For this case, Dyson⁵⁾ has already given the calculation, and our result agrees with his excepting small constant terms.

§ II. Subtraction Method and Self-Energy

We are going to investigate the system in which electron (or scalar particle) interacts with outer electromagnetic field $\hat{A}_\alpha(x)$ and quantized radiation field $A_\alpha(x)$. The field equation for this system is then given by:

$$\left(\hat{H}(x) + H(x) + \frac{1}{i} \frac{\partial}{\partial c} \right) \Psi[c] = 0^{(6)} \quad (1)$$

with

$$H(x) = ic(A_\alpha \hat{j}_\alpha) + c^2 \psi^* \psi (\partial_\alpha \hat{N}_\alpha + N_\alpha \hat{N}_\alpha) A_\alpha A_\alpha = H_1(x) + H_2(x) \quad (2)$$

$$J_a(x) = \phi^* \frac{\partial \phi}{\partial x_a} - \frac{\partial \phi^*}{\partial x_a} \phi, \quad (3)$$

and a similar expression for $\hat{H}(x)$. Here $\phi(x)$ denotes the quantity for scalar particle; N_a normal vector to the surface C .

According to the Tomonaga-Schwinger theory the self-energy part of the interaction is interpreted to be the correction to the kinetic mass and must be subtracted initially in order to avoid the divergency resulting from the interaction between particle and radiation field. It is indicated that if we confine ourselves to the order of ϵ^2 , which is sufficient for the consideration of level shifts, this part of Hamiltonian, H_{mass} , is obtained from

$$-\frac{i}{2} \left[H(x), \int_{C_2} H(x') dx' \right]. \quad (4)$$

Here we have neglected the term which comes from $H_2(x)$ in (2), as it gives no contribution to the present problem.

Now the field quantities $\phi(x)$ and $A_a(x)$ satisfy the following commutation relations:

$$\begin{aligned} [A_a(x), A_b(x')] &= -i\delta_{ab}D(x-x'), \\ [\phi(x), \phi^*(x')] &= -i\Delta(x-x'), \end{aligned} \quad (5)$$

where $D(x)$ and $\Delta(x)$ are Pauli's D-functions belonging to radiation field and matter field, resp. To separate out the self-energy operator, we decompose $\phi(x)$ and $A_a(x)$ into two parts:

$$\phi(x) = \hat{\xi}(x) + \eta^*(x), \quad \phi^*(x) = \hat{\xi}^*(x) + \eta(x), \quad A_a(x) = a_a(x) + a_a^*(x) \quad (6)$$

Then the commutation rules for $\hat{\xi}(x)$, $\eta(x)$ and $a_a(x)$ are

$$\begin{aligned} [\hat{\xi}(x), \hat{\xi}^*(x')] &= i\overset{+}{\Delta}(x-x'), \quad [\eta(x), \eta^*(x')] = -i\overset{-}{\Delta}(x'-x) \\ [a_a(x), a_a^*(x')] &= i\overset{+}{D}(x-x') \end{aligned} \quad (7)$$

with

$$\overset{+}{\Delta}(x) = -\overset{-}{\Delta}(-x) = \frac{1}{2(2\pi)^3} \int \frac{e^{iPx}}{P_4} d^3p \quad (8)$$

and similar expression for $D(x)$. With these quantities and by introducing the differential operator $P_a = \frac{1}{i} \frac{\partial}{\partial x_a}$ which operates backward and $P_a^* = -\frac{1}{i} \frac{\partial}{\partial x_a}$ which operates forward, we can write the interaction Hamiltonian as

$$H(x) = ie \{ \hat{\xi}^*(x) T\phi(x) - \phi(x) T\eta(x) \}, \quad (9)$$

where

$$T = i(A_a(x)P_a + P_a^*A_a(x)).$$

Here we shall pass over to the special representation; i. e., at first we expand $\hat{\xi}(x)$ and $\eta(x)$ into Fourier series as

$$\hat{\xi}(x) = \frac{1}{2(2\pi)^{3/2}} \int \hat{\xi}(p) \frac{e^{iPx}}{P_4} d^3p, \quad \eta(x) = \frac{1}{2(2\pi)^{3/2}} \int \eta(p) \frac{e^{iPx}}{P_4} d^3p. \quad (10)$$

Then there hold between the Fourier components, $\hat{\xi}(p)$ and $\eta(p)$, the following commutation rules:

$$[\hat{\xi}(p), \hat{\xi}^*(p')] = [\eta(p), \eta^*(p')] = iP_4 \delta(p - p'), \quad (11)$$

in order that the relations (7) are established. Therefore, if we take here the representation, where $\hat{\xi}^*(p)$ and $\eta^*(p)$ are diagonal, so $\hat{\xi}(p)$ and $\eta(p)$ are considered as differential operators, $-\frac{P_4}{i} \frac{\partial}{\partial \hat{\xi}^*(p)}$, $-\frac{P_4}{i} \frac{\partial}{\partial \eta^*(p)}$ and (6) is written

$$\begin{aligned} \psi^*(x) &= \frac{1}{2(2\pi)^{3/2}} \int \left\{ \hat{\xi}^*(p) e^{-iPx} - \frac{P_4}{i} \frac{\partial}{\partial \eta^*(p)} e^{iPx} \right\} d^3p, \\ \psi(x) &= \frac{1}{2(2\pi)^{3/2}} \int \left\{ -\frac{P_4}{i} \frac{\partial}{\partial \hat{\xi}^*(p)} e^{iPx} + \eta^*(p) e^{-iPx} \right\} d^3p, \end{aligned}$$

or more symbolically as

$$\begin{aligned} \psi^*(x) &= \hat{\xi}^*(x) - \frac{1}{i} \frac{\partial}{\partial \eta^*(x)}, \\ \psi(x) &= -\frac{1}{i} \frac{\partial}{\partial \hat{\xi}^*(x)} + \eta^*(x), \end{aligned} \quad (12)$$

and, as the consequence, we have the relations:

$$\begin{aligned} \hat{\xi}^*(x') \psi(x) &= \hat{\xi}^*(x') \psi(x) \\ \psi(x) \hat{\xi}^*(x') &= \left\{ -\frac{1}{i} \frac{\partial}{\partial \hat{\xi}^*(x')} + \eta^*(x) \right\} \hat{\xi}^*(x') = -\frac{1}{i} \hat{\xi}^*(x') - \frac{\partial}{\partial \hat{\xi}^*(x')} \\ &\quad + i\mathcal{A}(x-x') + \hat{\xi}^*(x') \eta^*(x) = \hat{\xi}^*(x) \psi(x) + i\mathcal{A}(x-x'), \text{ etc.} \end{aligned} \quad (13)$$

The similar relations hold also for radiation field quantities $A_\alpha(x)$. These facts indicate, as Tomonaga has suggested, that we have only to interchange the positions of the field quantities in the products so as to come all those ones with asterisks forward. Thus, by substituting (9) into (4) and using the relations (13), we obtain the the self-energy operator, if only such terms which contain merely two ψ 's are collected:

$$\begin{aligned} -\frac{i}{2} [H(x), \int^C H(x')] dx' &= -\frac{e^2}{4} \int [(\psi_\alpha^*(x) \psi_\alpha(x') - \psi_\alpha^*(x') \psi_\alpha(x)) f_\alpha(x-x') + \text{Comp. Conj.}] dx' \\ &\quad + (\psi_\alpha^*(x) \psi_\alpha(x') - \psi_\alpha^*(x') \psi_\alpha(x)) f_\alpha(x-x') + \text{Comp. Conj.} \end{aligned} \quad (14)$$

with

$$\begin{aligned} g(x) &= \mathcal{A}(x) D^{(1)}(x) + \mathcal{A}^{(1)}(x) D(x), \\ f_\alpha(x) &= \frac{\partial \mathcal{A}(x)}{\partial x_\alpha} D^{(1)}(x) + \frac{\partial \mathcal{A}^{(1)}(x)}{\partial x_\alpha} D(x). \end{aligned}$$

The expression for self-energy (14) is subtracted from the original Hamiltonian, $H(x)-H_{\text{mass}}$ and the solution of the Schrödinger equation with thus modified energy operator is given in the form

$$\Psi=U(C)\Psi, \tag{15}$$

where $U(C)$ is a unitary operator, satisfying the relation :

$$i\frac{\delta U}{\delta C}=(H(x)-H_{\text{mass}}(x))U. \tag{16}$$

Equation (16) is easily solved by transforming it into an integral equation of the term :

$$U=1+(-i)\int^C(H(x')-H_{\text{mass}})U'dx' \tag{17}$$

and by successive substitution of (17) into itself:

$$U=1+(-i)\int^{C_x}H(x')dx'+(-i)^2\int^{C_x}dx'\int^{C_x}dx''H(x'')H(x')-(-i)\int^{C_x}H_{\text{mass}}(x')dx'. \dots\dots\dots(18)$$

§ III. Level Shift of Hydrogen Atom

With the help of the above expression for U , the interaction energy of the outer field $\overset{\circ}{A}_\alpha(x)$ and scalar particle due to $\overset{\circ}{H}(x)$ on the presence of the radiation field is obtained by calculating the expectation value :

$$<\overset{\circ}{H}>_{AV}=(\Psi U^{-1}\overset{\circ}{H}U\Psi) \tag{19}$$

where we assume the initial state vector Ψ to correspond to the levels of hydrogen atom. Here, if we write the matrix $U^{-1}HU$ in (19) separately, it becomes as follows :

$$U^{-1}\overset{\circ}{H}U=\overset{\circ}{H}(x)-i[\overset{\circ}{H}(x),\int^{C_x}H(x')dx']-\frac{1}{2}[[\overset{\circ}{H}(x),\int^{C_x}H(x')dx'],\int^{C_x}H(x'')dx'']-\left[\overset{\circ}{H}(x),\int^{C_x}\left\{\frac{1}{2}[H(x'),\int^{C_{x'}}H(x'')dx'']-H_{\text{mass}}(x')\right\}dx'\right] \tag{20}$$

The first term in (20) is considered to be the first order levels of the hydrogen, the second the Bremsung due to the outer field, while the third and the fourth terms are supposed to contain the radiative corrections to the first term. If we choose the suitable terms which correspond to the corrections, by the procedure mentioned above, i. e., by collecting the terms which have only two ϕ 's after rearrangements of the products, we have 96 terms which are written with the exception of the common factor, $-\frac{i}{4}e^3\overset{\circ}{A}_\alpha$ as follows :

$$\begin{aligned}
\delta H \sim & \int^{C_x} \int^{C_x} \left[\phi^* \phi'' \left[D(x' - x'') \left\{ \frac{\partial \Delta(x - x')}{\partial x_a} \frac{\partial \Delta^{(1)}(x' - x'')}{\partial x'_a} - \frac{\partial^2 \Delta(x - x')}{\partial x_a \partial x'_a} \Delta^{(1)}(x' - x'') \right\} \right. \right. \\
& + D(x' - x'') \left\{ \frac{\partial \Delta(x - x')}{\partial x_a} \frac{\partial \Delta(x' - x'')}{\partial x'_a} - \frac{\partial^2 \Delta(x - x')}{\partial x_a \partial x'_a} \Delta(x' - x'') \right\} - \text{comp. conj.} \\
& + \phi^* \phi'' \left[D(x' - x'') \left\{ \frac{\partial^2 \Delta(x - x')}{\partial x_a \partial x'_a} \frac{\partial \Delta^{(1)}(x' - x'')}{\partial x''_a} - \frac{\partial \Delta(x - x')}{\partial x_a} \frac{\partial^2 \Delta^{(2)}(x' - x'')}{\partial x'_a \partial x''_a} \right\} \right. \\
& + D(x' - x'') \left\{ \frac{\partial^2 \Delta(x - x')}{\partial x_a \partial x'_a} \frac{\partial \Delta(x' - x'')}{\partial x''_a} - \frac{\partial \Delta(x - x')}{\partial x_a} \frac{\partial^2 \Delta(x' - x'')}{\partial x'_a \partial x''_a} \right\} - \text{comp. conj.} \\
& + \phi^*_a \phi''_3 \left[D(x' - x'') \left\{ \frac{\partial \Delta(x - x')}{\partial x'_3} \Delta(x' - x'') - \Delta(x - x') \frac{\partial \Delta^{(1)}(x' - x'')}{\partial x'_3} \right\} \right. \\
& + D^{(1)}(x' - x'') \left\{ \frac{\partial \Delta(x - x')}{\partial x'_3} \Delta(x' - x'') - \Delta(x - x') \frac{\partial \Delta(x' - x'')}{\partial x'_3} \right\} - \text{comp. conj.} \\
& + \phi^*_a \phi'' \left[D(x' - x'') \left\{ \Delta(x - x') \frac{\partial^2 \Delta^{(1)}(x' - x'')}{\partial x'_3 \partial x''_3} - \frac{\partial \Delta(x - x')}{\partial x'_3} \frac{\partial \Delta^{(1)}(x' - x'')}{\partial x''_3} \right\} \right. \\
& + D^{(1)}(x' - x'') \left\{ \Delta(x - x') \frac{\partial^2 \Delta(x' - x'')}{\partial x'_3 \partial x''_3} - \frac{\partial \Delta(x - x')}{\partial x'_3} \frac{\partial \Delta(x' - x'')}{\partial x''_3} \right\} - \text{comp. conj.} \\
& + \phi^* \phi'' \left[D(x' - x'') \left\{ \frac{\partial \Delta^{(1)}(x - x'')}{\partial x'_a} \frac{\partial \Delta(x - x')}{\partial x'_3} - \Delta^{(1)}(x - x'') \frac{\partial^2 \Delta(x - x')}{\partial x_a \partial x'_3} \right\} \right. \\
& + D^{(1)}(x' - x'') \left\{ \frac{\partial \Delta(x - x'')}{\partial x_a} \frac{\partial \Delta(x - x')}{\partial x'_3} - \Delta(x - x'') \frac{\partial^2 \Delta(x - x')}{\partial x_a \partial x'_3} \right\} - \text{comp. conj.} \\
& + \phi^*_i \phi''_3 \left[D(x' - x'') \left\{ \Delta^{(1)}(x - x') \frac{\partial \Delta(x - x')}{\partial x_a} - \frac{\partial \Delta^{(1)}(x - x'')}{\partial x_a} \Delta(x - x') \right\} \right. \\
& + D^{(1)}(x' - x'') \left\{ \Delta(x - x'') \frac{\partial \Delta(x - x')}{\partial x_a} - \frac{\partial \Delta(x - x'')}{\partial x_a} \Delta(x - x') \right\} - \text{comp. conj.} \\
& + \phi^* \phi'' \left[D(x' - x'') \left\{ \frac{\partial \Delta^{(1)}(x - x'')}{\partial x''_3} \frac{\partial^2 \Delta(x - x')}{\partial x_a \partial x'_3} - \frac{\partial^2 \Delta^{(1)}(x - x'')}{\partial x_a \partial x''_3} \frac{\partial \Delta(x - x')}{\partial x'_3} \right\} \right. \\
& + D^{(1)}(x' - x'') \left\{ \frac{\partial \Delta(x - x'')}{\partial x''_3} \frac{\partial^2 \Delta(x - x')}{\partial x_a \partial x'_3} - \frac{\partial^2 \Delta(x - x'')}{\partial x_a \partial x''_3} \frac{\partial \Delta(x - x')}{\partial x'_3} \right\} - \text{comp. conj.} \\
& + \phi^*_i \phi'' \left[D(x' - x'') \left\{ \frac{\partial^2 \Delta^{(1)}(x - x'')}{\partial x_a \partial x''_3} \Delta(x - x') - \frac{\partial \Delta^{(1)}(x - x'')}{\partial x''_3} \frac{\partial \Delta(x - x')}{\partial x_a} \right\} \right. \\
& + D^{(1)}(x' - x'') \left\{ \frac{\partial^2 \Delta(x - x'')}{\partial x_a \partial x''_3} \Delta(x - x') - \frac{\partial \Delta(x - x'')}{\partial x''_3} \frac{\partial \Delta(x - x')}{\partial x_a} \right\} \\
& - \text{comp. conj.} \left. \right] dx' dx'' + \int^{C_x} \int^{C_x'} \left[\phi''^* \phi'_3 D(x' - x'') \right. \\
& \times \left[\left\{ \Delta(x - x') \frac{\partial^2 \Delta^{(1)}(x - x'')}{\partial x_a \partial x''_3} - \frac{\partial \Delta(x - x')}{\partial x_a} \frac{\partial \Delta^{(1)}(x - x'')}{\partial x''_3} \right\} \right.
\end{aligned}$$

$$\begin{aligned}
& + \left\{ J^{(1)}(x-x') \frac{\partial^2 J(x-x')}{\partial x_a \partial x'_a} - \frac{\partial J^{(1)}(x-x')}{\partial x_a} \frac{\partial J(x-x'')}{\partial x'_a} \right\}] - \text{comp. conj.} \\
& + \psi''^* \psi' D(x'-x'') \left[\left\{ \frac{\partial J(x-x')}{\partial x_a} \frac{\partial J^{(1)}(x-x'')}{\partial x'_a} - J(x-x') \frac{\partial J^{(1)}(x-x'')}{\partial x_a} \right\} \right. \\
& + \left. \left\{ \frac{\partial J^{(1)}(x-x')}{\partial x_a} \frac{\partial J(x-x'')}{\partial x'_a} - J^{(1)}(x-x') \frac{\partial J(x-x'')}{\partial x_a} \right\} \right] - \text{comp. conj.} \\
& + \psi''^* \psi' D(x'-x'') \left[\left\{ \frac{\partial^2 J^{(1)}(x-x')}{\partial x_a \partial x'_a} \frac{\partial J^{(1)}(x-x'')}{\partial x'_a} - \frac{\partial J(x-x')}{\partial x'_a} \frac{\partial^2 J^{(1)}(x-x'')}{\partial x_a \partial x'_a} \right\} \right. \\
& + \left. \left\{ \frac{\partial J^{(1)}(x-x')}{\partial x_a \partial x'_a} \frac{\partial J(x-x'')}{\partial x'_a} - \frac{\partial J^{(1)}(x-x')}{\partial x'_a} \frac{\partial J(x-x'')}{\partial x_a \partial x'_a} \right\} \right] - \text{comp. conj.} \\
& + \psi''^* \psi' D(x'-x'') \left[\left\{ \frac{\partial J(x-x')}{\partial x'_a} \frac{\partial J^{(1)}(x-x'')}{\partial x_a} - \frac{\partial^2 J(x-x')}{\partial x_a \partial x'_a} J^{(1)}(x-x'') \right\} \right. \\
& + \left. \left\{ \frac{\partial J^{(1)}(x-x')}{\partial x'_a} \frac{\partial J(x-x'')}{\partial x_a} - \frac{\partial^2 J(x-x')}{\partial x_a \partial x'_a} J(x-x'') \right\} \right] - \text{comp. conj.}] dx' dx''
\end{aligned}$$

The integration with respect to x' and x'' is elementary but considerably complicated and as the final result we have, putting $A_a = (0, 0, 0, iV)$

$$\delta H = -\frac{e^3}{3\pi x^2} \left[\text{"log"} + \frac{5}{48} - \log 2 \right] (J^{(1)})^2 \psi^2 \quad (21)$$

where "log" denotes the term obtained by Bethe

$$\text{"log"} = \log \frac{x}{< E_n - E_m >_{AV}} = 7.7169 - 0.0293. \quad (22)$$

As the energy δH is given in the unit of x^2 , we must multiply it with $\frac{2\pi}{x}$, to obtain the expression in the unit of frequency. Substituting $-e/r$ for V , we have the level shift of 2S state,

$$LS = < \delta H >_{AV} = \frac{a^2}{3\pi} \text{Ry} \left[\text{"log"} + \frac{5}{48} - \log 2 \right], \quad (23)$$

where Ry is Rydberg constant and its numerical value is given by

$$\frac{a}{3\pi} \text{Ry} = 135.580 \text{ MC/sec.}$$

Putting the value for Ry and "log" into (23), we have for the LS of 2S-state

$$LS = 967 \text{ MC/sec.}$$

Here the polarization effect due to the outer field was omitted. This term is calculated by Uehling to be -3 MC/sec and accordingly lowers the above value somewhat.

§IV. Conclusion

The above calculations are to be compared with the case for Dirac electron. It is well known that the scalar particle shows stronger divergency than the Dirac electron. In reality, there appear linearly diverging terms in the course of calculations, while in Dirac electron case, only logarithmically diverging terms. Nevertheless, they cancel out each other, when they are put together, only convergent terms remain.

The numerical value for the level shift is considerably smaller than that experimentally obtained. Dirac electron gives the value: LS (Dirac el.) = 984 MC/sec which differs with ours only several Mc. However, it is added thereof spin-coupling energy, which amounts to 68 MC/sec and the total level shift becomes 1052 MC/sec, agreeing with the observation. As the scalar particle has no spin and consequently lacks the corresponding term, there occurs no further increase of the interaction energy. The spin effect is already susceptible in the first order levels for hydrogen and thus it seems no use to expect the close analogy between scalar particle and Dirac electron.

From the interaction energy (21), the radiative correction for elastic scattering cross section is also calculated as:

$$\delta\sigma_{sc} \sim \frac{d\Omega}{q^2} \lim_{\varepsilon \rightarrow 0} \left\{ \log \frac{2\varepsilon}{x} - \frac{5}{48} \right\}, \quad (24)$$

where $q = |\mathbf{P}_0 - \mathbf{P}|$, \mathbf{P}_0 , \mathbf{P} being initial and final momenta of electron, resp. The cross section for the Bremsung is the same with the Dirac electron case in the approximation above made:

$$\delta\sigma_{\text{Brems}} \sim \frac{d\Omega}{q^2} \lim_{\varepsilon \rightarrow 0} \left\{ \log \frac{x}{\varepsilon} + \text{const.} \right\}. \quad (25)$$

These two cross sections have so-called infra-red difficulties, i. e., as q tends to zero, both expressions diverge to infinite values. However, the sum of the two cancels out the diverging terms, as Bloch and Nordsieck have suggested, and there remain only convergent terms. Therefore we can see that there occurs no infra-red difficulties also in our case.

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On the Gauge-Difficulty in the γ -Decay of the Neutral Scalar Meson.

Daisuke IRÔ and Tatsuaki MIYAZIMA.

Physics Institutes, Tokyo Bunrika Daigaku.

(Received August 1, 1949)

§ 1. Introduction

Contrary to the former predictions, the convergence of the transition probability for the γ -decay of the neutral meson is demonstrated by Miyamoto and Fukuda's beautiful calculations.¹⁾ But, as they pointed out, their results are, in spite of the gauge-covariance proof of the fundamental theory, not gauge covariant, and do not satisfy a certain identity, which must hold about them. As these difficulties are closely related to the underlying difficulties of the present quantum field theory, many investigators devoted their attention to these difficulties.

Recently, W. Pauli proposed a new attempt to avoid the ambiguities of the field theory by a skillful limiting process, which is referred as "Regulator method".²⁾ Of course, from this point of view the above mentioned gauge-difficulties can be treated, but it will be discussed elsewhere in this journal.

In addition to these "Formalistic" procedure, there may be another "Realistic" method, such as C-meson³⁾ (or f -field)⁴⁾ theory for the electron self energy problem, or the Umezawa-Taketani's (et. al.) theory⁵⁾ for the vacuum polarization problem, which assumes various fields in the intermediate states, to compensate the divergence of the photon self energy. In reality, the equivalent result with the regulator method is obtained assuming the another proton pair having coupling constant of reversed sign.

The aim of the present paper is to treat the gauge-difficulties of the γ -decay process of the neutral meson from the realistic point of view. If we calculate the transition probability for the process, in which a neutral scalar meson decays into two photons through a virtual pair of a proton and an anti-proton in the intermediate states, a non-gauge-covariant term will appear in the resulting formula. But if we assume the co-existence of the field of other type, and consider the same processes through both virtual pairs, we can compensate away the terms that give rise to the gauge- and identity-difficulties. Among many possibilities, we chose a charged scalar meson (Assumption of "Bose Proton") as a intermediate states, and took just the same direction of calculation as the Fermi-proton case of Miyamoto and Fukuda. But on applying the Schwinger formalism to the meson field, a special care must be required, as we deal with a normal depending interaction

Hamiltonian density. Details of this point will be discussed in the following section.

§ 2. Assumption of the scalar proton.

In this section, we assume a Bose particle, which is interacting with the neutral scalar meson, described by the charged scalar field $\phi(x)$

$$[\phi(x), \phi^*(x')] = i\Delta(x-x') \quad (1)$$

and calculate the effective matrix element for the γ -decay of neutral meson through this hypothetical field. The interaction of this *proton* field with the neutral scalar meson $U(x)$, and with the electromagnetic field are described by the following interaction Hamiltonians respectively,⁶⁾

$$H_g = g\phi^*\phi U, \quad (2.1)$$

$$H_1 = ie\left(\phi^* \frac{\partial \phi}{\partial x_\lambda} - \frac{\partial \phi^*}{\partial x_\lambda} \phi\right) A_\lambda = ie\phi^* J_\lambda \phi A_\lambda, \quad (2.2)$$

$$H_2 = e^2 \phi^* \phi (A_\lambda^2 + (n_\lambda A_\lambda)^2), \quad (2.3)$$

where J_λ is a kind of differential operator, defined by the following relation.

$$f(x) J_\lambda g(x) = f(x) \frac{\partial g(x)}{\partial x_\lambda} - \frac{\partial f(x)}{\partial x_\lambda} g(x) \quad (3)$$

The effective matrix element for our process is given by

$$\langle H_g \rangle_2 = i \int_{-\infty}^{\infty} \langle [H_2(x'), H_g(x)] \rangle dx' - \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} \langle [H_1(x'') [H_1(x') H_g(x)]] \rangle. \quad (4)$$

The integral of the first term in the right-hand side of (4) has no accurate meaning in itself, as its integrand H_2 is defined at a point on some space-like surface, and contains the normal of the surface. In order to give the precise meaning to this integral, we assign to each world point a space-like surface. Then the normal of the surface is uniquely defined at each point. That the total expression is independent of the assignment of the surfaces, can easily be proved; this also becomes clear in the course of calculation. The expectation value of the bracket expressions can be obtained by ordinary procedure

$$-\langle [H_1(x''), [H_1(x'), H_g(x)]] \rangle = -e^2 g A_\mu' A_\nu'' U \{ \Delta_1(x''-x) J_\nu'' \Delta(x''-x') J_\mu' \Delta(x'-x) + \Delta(x'-x) J_\mu' \Delta_1(x''-x') J_\nu'' \Delta(x''-x) \}, \quad (5.1)$$

$$i \langle [H_2(x'), H_g(x)] \rangle = -e^2 g U \{ A_\lambda'^2 + (n_\lambda' A_\lambda')^2 \} \Delta_1(x'-x) \Delta(x'-x). \quad (5.2)$$

Inserting these into (4), we must perform the integration. In order to make use of Schwinger's integration method, it is necessary to extend the upper limit of the integration to infinite future using a sign functions $\varepsilon(\sigma, \sigma')$'s. But in our case, special care must be taken about the derivatives of the invariant delta-function.

To the first derivatives we can multiply the sign function under the differentiation sign, but in the case of a second derivatives, we cannot multiply so simply. For instance, we must reduce it into first derivatives by means of integration by parts, and then multiply the sign function under the differential symbol, that is

$$\begin{aligned}
 & \int_{-\infty}^{+\infty} dx'' e^2 g A_\nu'' A_\mu' U \mathcal{J}_1(x''-x) \mathcal{J}(x'-x) \frac{\partial^2 \mathcal{J}(x''-x')}{\partial x_\nu'' \partial x_\mu'} \\
 &= e^2 g A_\mu' \mathcal{J}(x'-x) U \left\{ \int_{-\infty}^{+\infty} dx'' \frac{\partial}{\partial x_\nu''} \left(A_\nu'' \mathcal{J}_1(x''-x) \frac{\partial \mathcal{J}(x''-x')}{\partial x_\mu'} \right) \right. \\
 &\quad \left. - \int_{-\infty}^{+\infty} dx'' \frac{\partial (A_\nu'' \mathcal{J}_1(x''-x))}{\partial x_\nu''} \frac{\partial \mathcal{J}(x''-x')}{\partial x_\mu'} \right\} \\
 &= e^2 g A_\mu' \mathcal{J}(x'-x) U \left\{ - \int_{-\infty}^{+\infty} dx'' A_\nu'' A_\nu' \mathcal{J}_1(x''-x) \frac{\partial \mathcal{J}(x''-x')}{\partial x_\mu'} \right. \\
 &\quad \left. + \int_{-\infty}^{+\infty} dx'' \frac{\partial (A_\nu'' \mathcal{J}_1(x''-x))}{\partial x_\nu''} \frac{\partial \mathcal{J}(x''-x')}{\partial x_\mu'} \right\}. \quad (6)
 \end{aligned}$$

This is the only term that contains second derivative in the expression (5). The surface integral of this expression is just the same one as the normal dependent term in the second order interaction Hamiltonian, except its sign. These two terms cancel each other, and our expression contains no more such normal dependent terms. This result is directly obtained, if we proceed according to following rules. I) The sign function can be multiplied to the second derivative of invariant delta-function directly under its differentiation symbol, namely $\epsilon \partial_\nu \partial_\mu \mathcal{J} = -2 \partial_\nu \partial_\mu \mathcal{J}$. II) The normal dependent term can be omitted. By the same procedure as in the case of Fermi proton, we obtain the following formula

$$\begin{aligned}
 & - \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dx'' \langle [H', [H', H_g]] \rangle \rightarrow -e^2 g U \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx' dx'' A_\nu'' A_\mu' \left\{ \mathcal{J}_1(x''-x) f_\nu'' \mathcal{J}(x''-x') \right. \\
 &\quad \left. \times f_\mu' \bar{\mathcal{J}}(x'-x) + \frac{1}{2} \bar{\mathcal{J}}(x'-x) f_\mu' \mathcal{J}_1(x''-x') f_\nu'' \mathcal{J}(x'-x) \right\}, \quad (7.1)
 \end{aligned}$$

$$i \int_{-\infty}^{+\infty} \langle [H_2', H_g] \rangle dx'' \rightarrow -e^2 g U \int_{-\infty}^{+\infty} dx' A_\lambda'' \mathcal{J}_1(x'-x) \mathcal{J}(x'-x). \quad (7.2)$$

By means of integration by parts and supplementary condition, the more simplification is possible,

$$\begin{aligned}
 \langle H_g \rangle_2 &= 2e^2 g U \int_{-\infty}^{+\infty} dx' dx'' A_\nu'' A_\mu' \left\{ \bar{\mathcal{J}}(x''-x) \frac{\partial \mathcal{J}_1(x'-x)}{\partial x_\mu'} \frac{\partial \bar{\mathcal{J}}(x''-x')}{\partial x_\nu''} \right. \\
 &\quad \left. + \bar{\mathcal{J}}(x''-x') \frac{\partial \bar{\mathcal{J}}(x'-x)}{\partial x_\mu'} \frac{\partial \mathcal{J}_1(x''-x)}{\partial x_\nu''} + \mathcal{J}_1(x''-x) \frac{\partial \bar{\mathcal{J}}(x'-x)}{\partial x_\mu'} \frac{\partial \bar{\mathcal{J}}(x''-x')}{\partial x_\nu''} \right\} - \\
 &\quad - e^2 g U \int_{-\infty}^{+\infty} dx' (A_\lambda'')^2 \mathcal{J}_1(x'-x) \bar{\mathcal{J}}(x'-x) \quad (8)
 \end{aligned}$$

These integrations can be evaluated by the usual Schwinger-method. The final result is given by

$$\langle H_g \rangle_2 = \frac{-1}{8\pi^2} g' c^2 \left\{ \int_{-1}^1 d\tau \int_1^\infty \frac{dn}{n^3} A^2 + \int_{-1}^1 d\tau (1 - \tau^2) \int_1^\infty \frac{dn}{n^3} \left[\frac{1}{n^2 - \frac{1 - \tau^2}{4x^2} \square^2} - \frac{1}{2} F_{\mu\nu} F_{\mu\nu} \right] U \right\}. \quad (9)$$

§ 3. Conclusion

According to Miyamoto and Fukuda, the matrix element of the Fermi proton case is given by

$$\langle H_g \rangle_2 = \frac{1}{8\pi^2} g' c^2 \left\{ \int_{-1}^1 d\tau \int_1^\infty \frac{dn}{n^2} A^2 + \int_{-1}^1 d\tau \int_1^\infty \frac{dn}{n^3} \left[\frac{n^2 - 1 + \tau^2}{1 - \tau^2} - \frac{1}{4x^2 \square^2} - \frac{1}{2} F_{\mu\nu} F_{\mu\nu} \right] U \right\}. \quad (10)$$

Comparing this with our result, we can see that if we choose the coupling constants and masses as

$$g = g' \quad x = x' \quad (11)$$

the non-gauge-covariant term $\sim A^2$ drops, and remains only the gauge covariant term

$$\langle H_g \rangle_2 = \frac{1}{8\pi^2} g' c^2 \int_{-1}^1 d\tau \int_1^\infty \frac{dn}{n} \left[\frac{1}{n^2 - \frac{1 - \tau^2}{4x^2} \square^2} - \frac{1}{2} F_{\mu\nu} F_{\mu\nu} \right] U \quad (12)$$

In conclusion, we should like to express our hearty gratitude to Prof. S. Tomonaga for his invaluable guidance. We are also indebted to Mr. Miyamoto and Mr. Fukuda who kindly showed us the manuscript before publication.

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Cosmic-Ray Underground. II.

Satio HAYAKAWA and Sin-itiro TOMONAGA

Meteorological Research Institute and Tokyo Bunrika Daigaku

(Received August 9, 1949)

§ 1. Introduction

In a previous paper^{1,2)} we discussed the intensity-depth relation of cosmic-ray underground on the basis of the electromagnetic interactions with matter of mu-mesons penetrating into the earth. In order to make sure of the validity of the assumption that soft component and showers existing in the deep places mainly produced by the electromagnetic interaction of mu-mesons, we now calculate the number of electrons and the frequency of showers accompanied by a mu-meson, taking into account the knock-on, the radiation and the pair-creation processes by a Dirac particle. In comparing the theoretical results thus obtained with experiments, there have been unfortunately too many experiments which are inconsistent with each other, so that we are puzzled which of the experiments we may adopt. Such inconsistency suggests that the result is largely affected by the experimental conditions, and we must refer to some experiment which appears us to be the most clear-cut one. We think that the experiment carried out by Wilson and Hughes³⁾ is here most reliable. As for the other experiments, which give interest but somewhat curious results^{4,5)}, we shall discuss them in separate papers.

§ 2. The number of electrons accompanied by a meson.

In what follows, we take the critical energy for the cascade electron in earth as

$$\epsilon_j = 6.0 \times 10^7 \text{ eV}, \quad (2.1)$$

and the radiation length

$$X_0 = 27 \text{ g cm}^{-2}. \quad (2.2)$$

Other notations have the same meaning as in I, unless we particularly mention.

The electrons produced by mesons are considered to be in equilibrium with agent mesons, since the formers have far shorter range than the latters. Therefore we may apply the method of the electron track to this problem.

The integral electron track length initiated by the energy ϵ in g cm^{-2} is

$$s(\epsilon) = (\epsilon/\epsilon_j) X_0 \quad (2.3)$$

in the approximation B mentioned by Rossi and Greisen.⁽⁶⁾ Denoting the probability that the meson with energy E produces the soft rays with energy between ϵ and $\epsilon + d\epsilon$ per $g\text{ cm}^{-2}$ as $Q(E, \epsilon)d\epsilon$, the number of electrons which are in equilibrium with the meson with energy E is given by

$$M(E) = \int_{\epsilon_j}^{\infty} Q(E, \epsilon) z(\epsilon) d\epsilon, \quad (2.4)$$

where we neglect the contribution from the produced soft rays with energy lower than ϵ_j because of their too short range.

$Q(E, \epsilon)$ are given for the three processes, collision, radiation and pair creation, as follows.

$$Q_{\text{coll}}(E, \epsilon) d\epsilon = \pi Z r_0^2 \frac{N}{A} 2mc^2 \epsilon_j \left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon E} + \frac{1}{2E^2} \right] d\epsilon, \quad (2.5)$$

$$Q_{\text{rad}}(E, \epsilon) d\epsilon = 4\alpha Z^2 r_0^2 \left(\frac{m}{\mu} \right)^2 \frac{N}{A} \left[\ln \left(\frac{12E}{5Z^{1/3}\mu c^2} \right) - \frac{1}{3} \right] \frac{d\epsilon}{\epsilon}, \quad (2.6)$$

and $Q_{\text{pair}}(E, \epsilon) d\epsilon$ is given in I (3a) to (3d), where denoted by dQ for four energy ranges.

Substituting these $Q(E, \epsilon)$ into (2.4) we obtain the number of electrons for respective processes.

$$\begin{aligned} M_{\text{coll}}(E) &= \pi Z r_0^2 \frac{N}{A} \frac{2mc^2}{\epsilon_j} X_0 \left[\ln \left(\frac{E}{\epsilon_j} \right) - \frac{3}{4} + \frac{\epsilon_j}{E} - \frac{\epsilon_j^2}{4E^2} \right], \\ &\approx 4.8 \times 10^{-2} \left[\ln \left(\frac{E}{\epsilon_j} \right) - \frac{3}{4} + \frac{\epsilon_j}{E} - \frac{\epsilon_j^2}{4E^2} \right], \end{aligned} \quad (2.7)$$

$$\begin{aligned} M_{\text{rad}}(E) &= 4\alpha Z^2 r_0^2 \left(\frac{m}{\mu} \right)^2 \frac{N}{A} X_0 \left[\ln \left(\frac{12E}{5Z^{1/3}\mu c^2} \right) - \frac{1}{3} \right] \frac{E}{\epsilon_j}, \\ &\approx 4.0 \times 10^{-6} [\ln(E/\epsilon_j) - 0.16] (E/\epsilon_j), \end{aligned} \quad (2.8)$$

$$\begin{aligned} M_{\text{pair}}(E) &= \frac{8}{\pi} (\alpha Z)^2 r_0^2 \frac{m}{\mu} \frac{N}{A} X_0 \left[\frac{E}{\epsilon_j} \left\{ \frac{16}{9} \ln \eta + 1 - \frac{\eta \mu c^2}{2E} \right\} \right. \\ &\quad \left. - \frac{7}{9} \ln \eta \left\{ \ln \left(\frac{E}{\epsilon_j} \right) + \ln \left(\frac{m}{\mu} \right) + 1 \right\} \right], \\ &\approx 3.6 \times 10^{-5} \{ (E/\epsilon_j) - 50.7 - 3.4 \ln(E/\epsilon_j) \}. \end{aligned} \quad (2.9)$$

In (2.9) we neglect the terms with factor $(\eta/2\epsilon)^2$, because we are interest only in high energy region.

Comparing these results, we see that M_{rad} and M_{pair} contribute to a small fraction of total electrons up to $E/\epsilon_j \approx 10^4$, e.g. about 6×10^{11} eV. But if we properly take into account the energy spectrum of mesons underground, the contri-

butions of the latter two processes amounts only at most 10% of the collision process in the observed depth.

The energy spectrum of mesons at each depth $f_x(E)dE$ is derived from that at sea level $f(E)$, by considering the energy loss down to the depth under consideration:

$$f(E)dE = f(E_x + E)dE, \quad (2.10)$$

where $f(E)$ is taken from the analysis of experiments in *I* and E_x is given in *I* (12).

Thus we obtain the number of soft rays at each depth accompanied by mesons:

$$N(x) = \int_0^\infty M(E) f_x(E) dE, \quad (2.11)$$

The results of numerical integration are given in Fig. 1 for these three processes and their sum.

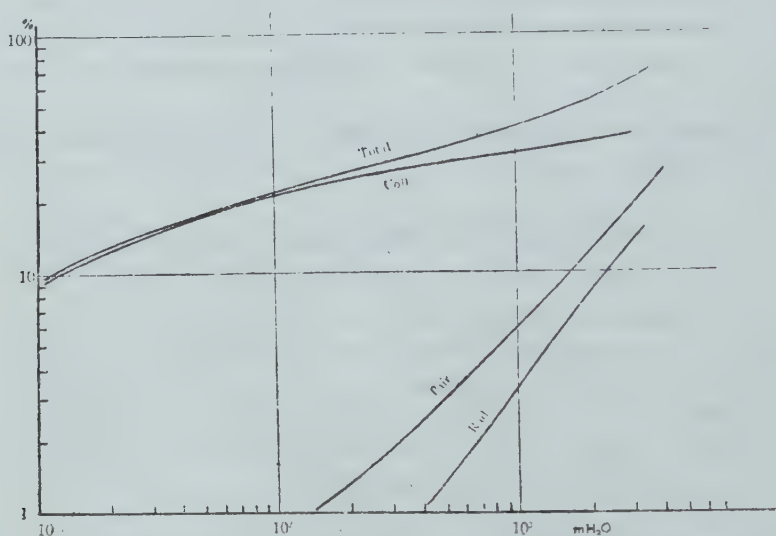


Fig. 1. The fractional number of electrons accompanied by mesons as the function of depth.

§ 3. The frequency of showers accompanied by a meson.

The electrons accompanied by a meson are also appeared as showers, provided that the initial energy is sufficiently high. Such showers reveal themselves as the showers observed in underground experiments without the matter nearby. These are considered to be also in equilibrium with the agent meson and we observe their average behaviour, so that the fluctuation in shower size need not be taken into consideration. This makes the treatment of the problem very simple.

The number of showers of size larger than n accompanied by a meson with energy E is given by

$$S(E) = \int_{\varepsilon_0}^E Q(E, \varepsilon) L(\varepsilon, n) d\varepsilon. \quad (3.1)$$

$L(\varepsilon, n)$ means the length in which the shower with initiating energy ε has electrons more than n . The production of soft rays in this length is responsible for the shower under consideration.

The functional form of $L(\varepsilon, n)$ is approximately obtained from the cascade function $\Pi(\varepsilon, 0)$ in the article of Rossi and Greisen⁶:

$$L(\varepsilon, n) = \{a \ln(\varepsilon/\varepsilon_j) - \beta\} X_0, \quad (3.2)$$

where

$$a = 3.5, \quad \beta = 3.8 + 3.0 \ln n. \quad (3.3)$$

The lower limit of the integral (3.1), ε_0 , means the energy at which $L(\varepsilon, n) = 0$:

$$L(\varepsilon_0, n) = 0. \quad (3.4)$$

Substituting Q into (3.1), we obtain $S(E)$ for the three processes. By the collision process

$$S_{\text{coll}}(E) = \pi r_0^2 Z \frac{N}{A} X_0 2mc^2 a \left[\frac{1}{\varepsilon_0} - \frac{1}{2} \frac{\ln^2(E/\varepsilon_j)}{E} + \left(\frac{\beta}{a} - \frac{1}{2} \right) \times \frac{\ln(E/\varepsilon_j)}{E} - \left\{ \frac{1}{2} \left(\frac{\beta}{a} \right)^2 - \frac{1}{2} \frac{\beta}{a} + \frac{3}{2} \right\} \frac{1}{E} + \frac{1}{2} \frac{\varepsilon_0}{E^2} \right]. \quad (3.5)$$

The most predominant term $1/\varepsilon_0$ does not depend on E , but shows a strong dependence on n . In our problem the second and third terms give a considerable contribution to the result as will be seen later. The radiation process gives

$$S_{\text{rad}}(E) = 4aZ^2 r_0^2 \left(\frac{m}{\mu} \right)^2 \frac{N}{A} X_0 \left[\ln \left(\frac{12E}{5Z^{1/3} \mu c^2} \right) - \frac{1}{3} \right] \times \left[\frac{a}{2} \ln^2 \left(\frac{E}{\varepsilon_j} \right) - \beta \ln \left(\frac{E}{\varepsilon_j} \right) + \frac{1}{2} \frac{\beta^2}{a} \right]. \quad (3.6)$$

The contribution of this process is negligibly small because of the small cross-section. $S(E)$ for pair creation is very complicated, but neglecting the terms with a factor $(\eta/4\xi)^2$,

$$S_{\text{pair}}(E) = \frac{8}{\pi} (aZ)^2 r_0^2 \left(\frac{m}{\mu} \right)^2 \frac{N}{A} X_0 \left[\frac{7}{9} \ln \eta \left\{ \frac{a}{6} \ln^3 \left(\frac{\xi mc^2}{\varepsilon_j} \right) - \frac{\beta}{2} \ln^2 \left(\frac{\xi mc^2}{\varepsilon_j} \right) + \frac{\beta^2}{2a} \ln \left(\frac{\xi mc^2}{\varepsilon_j} \right) - \frac{\beta^3}{6a^2} \right\} + \frac{a}{4} (\ln \eta + 1) \left\{ \ln \left(\frac{\xi mc^2}{\varepsilon_j} \right) + 1 \right\} \right]. \quad (3.7)$$

The main contribution to (3.7) comes from $\xi mc^2 < \varepsilon < 2\eta mc^2$ (c.f. I (3)). If \ln^3

term exceeds the ln^2 term in (3.7), one expects the rapid increase of shower rate. This occurs at about 35 Bev for $n=1$ and 550 Bev for $n=3$. Our previous work²⁾ corresponds to the former case but here we mainly lean on the latter case because of the closeness to the experiments we refer. The predominance of the ln^3 term brings about the predominance of pair creation process. Such a remark is of importance for the interpretation of experimental results.

Thus we can obtain the shower frequency at the depth x as

$$T(x, n) = \int_0^\infty S(E) f_e(E) dE. \quad (3.8)$$

The result of numerical integration for $n=3$ is shown in Fig. 2, compared with experiments.

The showers are often observed under lead shield. In this case we have only to change the values of Z , A , N_0 , ϵ_j and γ . This leads to the fact that S_{pair} becomes effective from the lower energy and smooths out the n -dependence.

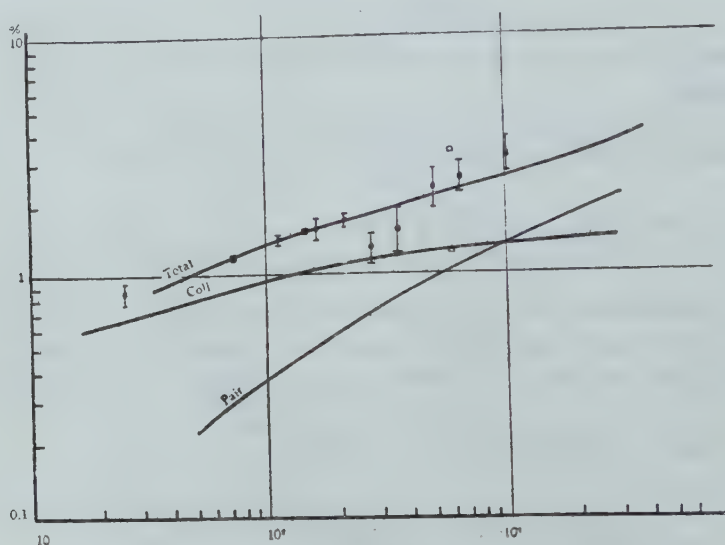


Fig. 2. The fractional frequency of showers accompanied by a meson as the function of depth.

§ 4. The comparison with experiments.

In comparing our theory with experiments, we must notice that the experimental results as to secondary electrons and showers are largely dependent on the experimental conditions. This situation has obstructed the development of this problem. Such remarks are also emphasized by Wilson and Hughes³⁾ and they endeavoured to get the definite results as far as possible.

From the cloud chamber observation at 71m water depth, there is a remark-

able negative excess in soft rays which are considered to be knock-on electrons. The relative number of secondaries accompanied by mesons are at about 13 %, while our theory gives about 20 %. This agreement is satisfactory in view of the roughness of both theory and experiment. Although this rate will slowly increase for greater depth, the additional part of secondary electrons consists in higher energy ones which are likely to be emitted forward. It may follow that the behaviour of absorption does not change at 71 *m* and 657 *m* water depth.

The shower frequency can not be compared with experimental one in absolute value because the effective producing area above the apparatus is different according to experimental condition. Thus we shall only concern the relative value.

The shower frequency *versus* vertical intensity is measured by Wilson⁷⁾ by means of a vertical or horizontal four fold counter train. The result is represented in Fig. 2, normalizing at 112 *m* water depth. The approximate agreement shows that our interpretation is not far from reality.

The experiment of Wilson and Hughes is also plotted in Fig. 2, normalizing at 71 *m* water for four fold coincidence.

From the above analysis we may conclude that cosmic-rays penetrating down to great depth are mu-mesons which suffer the normal electromagnetic interaction with matter. In the extent treated here no extraordinary behaviour seems to exist.

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On the Nature of Cosmic Ray Bursts.

Yoichi FUJIMOTO and Satio HAYAKAWA

*Department of Physics, Tokyo University
and Meteorological Research Institute*

(Received August 9, 1949)

§ 1. Introduction and Summary.

In order to study the interaction between the electromagnetic field and a charged particle in high energy region, cosmic ray bursts, as well as underground phenomena, give many useful data. But for this purpose some care must be taken in the use of the term "bursts," which means, as is well known, the occurrence of momentary vast ionization in an ionization chamber irrespective of its cause, and thus consists of the complexity of different phenomena. Recent investigations on this problem have facilitated us to get a rather clear picture for the nature of bursts, owing to both the deeper analysis of experimental data and the use of fast ionization chambers. But a more satisfactory and quantitative understanding should be needed.

The bursts of smaller size in a thin-walled ionization chamber may be mainly attributed to heavy particles produced in the wall and the gas, thus they have close relations with stars observed in photographic plates and Wilson chambers. Usually both of them seem to be caused by the same agent, i.e. nucleons in the energy range 100~1000 MeV. As will be shown later, we can understand the general behavior of these phenomena by a simplified classification of nucleon component, and on this basis some experimental data are analyzed. (§ 2)

The bursts of larger size in a thin-walled chamber are mainly caused by extensive air showers. This is accepted by the comparisons between the experiments worked out by counters and chambers. (§ 3)

The bursts of larger size under thick absorbers have been interpreted as cascade showers produced by penetrating particles. They are of special interest, since they concern closely the validity of the quantum electrodynamics in high energy region and have been discussed by Christy and Kusaka in this view. We have, in the present paper, calculated their size-frequency curves at sea level with revised constants referring to the underground phenomena. The result assures the current notion, that their agents are mainly μ -mesons with spin 0 or 1/2. (§ 4)

In spite of this success, the large altitude variation of the shielded bursts can never be understood by assuming that bursts are produced solely by μ -mesons.

The bursts at higher altitudes may be caused by nucleons, which produce photons by charge exchange or by other causes. By this interpretation we can explain not only the altitude dependence, but also the absorption behind thicker layer and the distribution of falling directions. (§ 5) Thus we may conclude that the interaction of charged particles with the electromagnetic field in the energy region under consideration, has quite the same nature as that in the lower energy and the validity of the quantum electrodynamics is guaranteed up to about 10^{12} eV.

§ 2. Bursts produced by nuclear disintegrations.

Bursts with smaller size in a thin-walled ionization chamber have often been misinterpreted as air showers. Now these bursts are known as mainly due to heavily ionizing particles. This was indicated by Euler¹⁾ and one of us (S.H.)²⁾ only qualitatively, but the later development allows us more detailed study, especially by making use of the elaborate experiment of Bridge et al³⁾. In what follows, we give a detailed and revised analysis of our preliminary reports on this problem⁴⁾.

Both stars and small bursts are considered to be caused by nucleons with energy one to several hundred MeV. This sort of nucleons, which we call B-component, are produced by the nuclear collisions set up by more energetic nucleons (perhaps more than 1 BeV), A-component. Such nuclear collisions are observed as penetrating showers, the products of which consist of mesons, fast and slow nucleons and eventually electronic component. Among these, the fast nucleons are classified in B-component. Slow nucleons with energy smaller than several ten MeV appeared in penetrating showers and stars are termed as C-component. In this section we will discuss mainly the behaviour of B-component because it may be responsible for stars or bursts.

As is shown by Wilson chamber pictures, the most parts of B-component are neutrons. We will first treat only the B-neutrons, and later show that the B-protons are really rare. We see, on the other hand, from Wilson chamber pictures that A-component consists of nearly equal number of neutrons and protons. But we may neglect the ionization loss in A-component, because only a small fraction of its energy is lost by ionization in an absorption mean free path. Assuming the same absorption coefficient, $1/l = 1/125 \text{ g cm}^{-2}$ in air, for both A- and B-component, and the conservation of the direction by A-B transmutation, we get the diffusion equations as follows,

$$-\frac{dA(x)}{dx} = \frac{1}{l} A(x), \quad (2.1a)$$

$$-\frac{dB(x)}{dx} = \frac{1}{l} B(x) - \frac{2\nu}{l} A(x). \quad (2.1b)$$

In (2.1b) the factor 2 means that about two nuclear collisions occur in l , what

is suggested by the fact that the collision mean free path is about a half of l , i.e. 65 g cm^{-2} in air, if we assume that the cross-section for the nuclear collision is given by the geometrical cross-section of the nucleus. ν in (2.1b) represents the average number of B-neutrons produced at a collision. Solving these equations with the initial condition $A(0)=a$, $B(0)=0$,

$$A(x) = a \exp(-x/l), \quad (2.2a)$$

$$B(x) = 2\nu a(x/l) \exp(-x/l). \quad (2.2b)$$

Since observed stars and bursts are caused by B-component falling from all directions, we must integrate (2.2b) over a half hemisphere.

$$\bar{B}(x) = \int B(x/\cos \theta) d\Omega = 2\pi \cdot 2\nu a(x/l) [-E_4(-x/l)]. \quad (2.3)$$

This should represent the altitude variation of the burst frequency and actually shows a fairly good agreement with experimental one as is seen in Fig. 1. For $x \geq 200 \text{ g cm}^{-2}$, (2.3) can be approximated as $b \exp(-x/132)$.

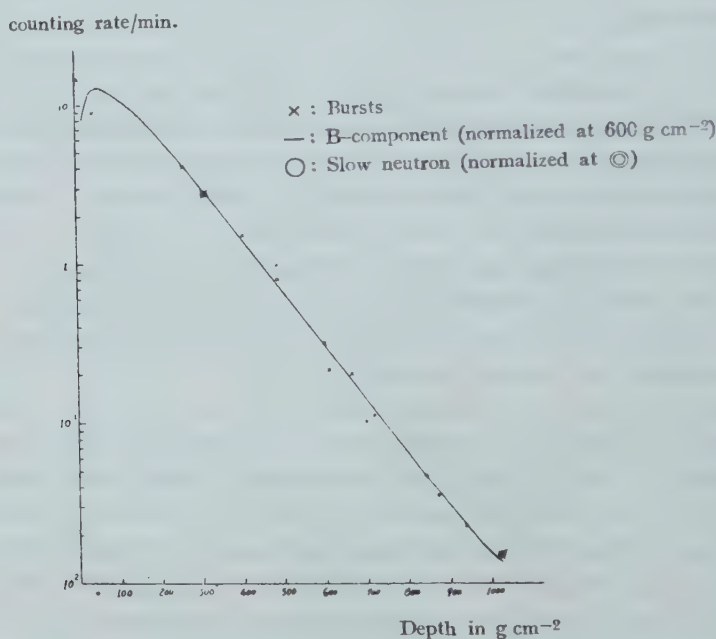


Fig. 1. Altitude variation of burst frequency.

In the above consideration we have considered that B-component consists of neutrons only. This is practically the case as shown by the following estimation. The mean energy of protons in B-component is assumed as 200 MeV, corresponding to the mean range 20 g cm^{-2} in air. As this is shorter than the mean free path of A-component, B-protons should be in equilibrium with A-component. So the intensity of B-protons can be estimated as

$$(20 \times 2/125) \nu a \exp(-x/l). \quad (2.4)$$

This coefficient amounts about only 2.5 % of that of (2.3). Thus we can neglect the protons in B-component and explain why stars are mainly produced by neutrons.

The absolute intensity of the B-component can be determined at the standard altitude, 600 g cm⁻². In this altitude the frequency of stars in photographic plates are about 1/emulsion 1 cc, hr, as estimated by Rossi⁵⁾. Considering that about 63 % (37 %) of the stars are produced in heavy (light) nuclei⁶⁾, and taking into account of the density (3.8) and the constitution ratio of heavy and light nuclei in the emulsion (3.2:0.6), the frequency of stars are given by, respectively,

$$0.37/0.6 \doteq 0.6/\text{g.hr. for light nuclei,} \quad (2.5a)$$

$$0.63/3.2 \doteq 0.2/\text{g.hr. for heavy nuclei.} \quad (2.5b)$$

Multiplying these figures by the collision mean free path, we get the intensity of the star-agents

$$0.6 \times 65 \text{ cm}^{-2} \text{ hr}^{-1} = 1.1 \times 10^{-2} \text{ cm}^{-2} \text{ sec}^{-1} \quad \text{for light nuclei,} \quad (2.6a)$$

$$0.2 \times 130 \text{ cm}^{-2} \text{ hr}^{-1} = 0.72 \times 10^{-2} \text{ cm}^{-2} \text{ sec}^{-1} \quad \text{for heavy nuclei.} \quad (2.6b)$$

This should be read as the intensity of B-component. The slight difference between (2.6a) and (2.6b) will be due to the ambiguous definition of B-component in its energy region. Tentatively we will adopt (2.6a) and equate to (2.3), then we have

$$B = 0.8 \times 10^{-2} \text{ cm}^{-2} \text{ sec}^{-1}, \quad (2.7)$$

On the other hand

$$A = 2 \times 10^{-3} \text{ cm}^{-2} \text{ sec}^{-1}, \quad (2.8)$$

at the same altitude⁵⁾. Therefore, we can determine ν as

$$\nu = 0.42. \quad (2.9)$$

Including protons, the number of B-nucleons produced in a penetrating shower is about 0.8. This should be the minimum estimation, for we have taken the collision mean free path as $l/2$. The maximum estimation, which is obtained by taking the collision mean free path equal to l , is the 4-times of this figure. These estimations are consistent with the Wilson chamber evidence.

To compare the absolute intensity estimated above with the burst frequency, we must know the frequency of stars in the chamber wall (brass). This is from (2.5a) or (2.5b), by considering the geometrical cross-section of nuclei,

$$0.36 \text{ or } 0.24 \text{ g}^{-1} \text{ hr}^{-1}. \quad (2.10)$$

The emitted protons of such a star have an average energy about 10 MeV, which corresponds to the range about 0.1 g cm⁻² brass. Therefore, the frequency of bursts

caused by such stars is estimated as $31 \sim 47 \text{ hr}^{-1}$, multiplying the whole area of the cylinder used by Bridge et al⁹⁾. On the other hand, the observed frequency of bursts (size $\geq 5.8 \text{ MeV}$) at Mt. Evans (610 g cm^{-2}) is 36 hr^{-1} . Both figures, $31 \sim 47 \text{ hr}^{-1}$ and 36 hr^{-1} , can be said to be in fair agreement with each other, in view of the ambiguities in the effective area of the chamber and the estimation of the mean energy of emitted proton.

Above considerations lead us to the conclusion that the stars in the photographic plates and the small bursts in the thin-walled and low pressure ionization chamber are the same in nature. But this conclusion is limited by our approximate classification of nucleons, that is, B-nucleons are not distinguished from each other in their energy. More comprehensive analysis taking the energy spectrum into consideration will be given in a separate paper.

Lastly, we will remark an extraordinary experimental results of Kingshill and Lewis⁷⁾. The wall of their chamber is so thin that protons with down to 10 MeV can penetrate the wall and give rise considerable ionization. These protons are produced not only in air, as was suggested previously²⁾, but also in nearby material. Therefore, the result may be largely dependent on the circumstances, so that we can not treat their data as representing the ordinary cosmic ray phenomena.

§ 3. Bursts produced by extensive air showers.

About 85 % of the bursts with large size in a thin-walled chamber are, as shown by Lapp⁸⁾, coincident with extensive air showers. But it seems that there is discrepancy in the density-frequency distribution between the one measured by ionization chambers and the other by counters²⁾. Such an appearance may be revised by following considerations.

On the one hand, the absolute frequency of extensive air showers at sea level must, following a recent experiment⁹⁾, be greater by the factor $2.0/1.2$ than that previously adopted. On the other hand, the size of the bursts measured by Lapp must be reduced by the factor $1/1.4$, taking account for the precise value of the specific ionization as was kindly suggested by Greisen¹⁰⁾.

Further the burst frequency must be multiplied by 0.85, coincident rate with the extensive air showers, and also be multiplied by 0.6, which means the extrapolation to zero thickness of the wall. Such remarks enable us to compare the two experimental results in Fig. 2, though the regions of density measured are somewhat different. The extrapolation of counter's data agrees fairly well with chamber's one, except for the low density region, where one must consider the effect of heavy particles frequently accompanied with extensive air showers.

§ 4. Bursts produced by μ -mesons.

If one shields the ionization chamber with the absorber, the frequency of bursts increases with increasing thickness of the absorber and reaches the maximum

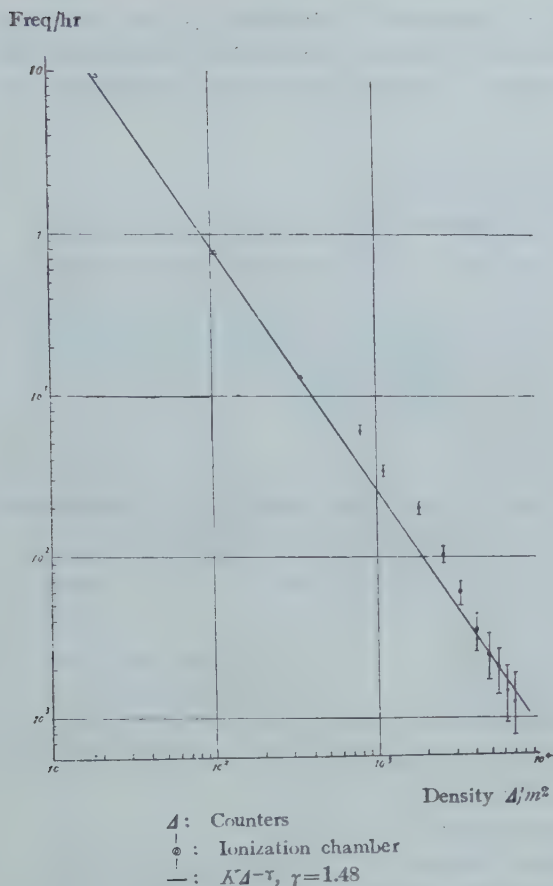


Fig. 2. Density-frequency relation of extensive air showers by the counter coincidence and the ionization chamber.

and then decreases with adding the further absorber^{8) 11)}. This is interpreted as the cascade transition effect of the electron shower which pre-exists in air¹¹⁾.

However this transition curve does not fall as expected from the cascade theory, but reveals the nearly constant frequency behind 20 radiation length or more. The nature of such bursts was discussed by Oppenheimer *et al*¹²⁾ and a quantitative calculation was carried out by Christy and Kusaka¹³⁾. Their analysis was epoch-making on the point that the result based on the quantum field theory of mesons could firstly give the quantitative explanation of the cosmic ray phenomena and could determine the spin of the meson. Although their analysis may be so perfect that no addition is needed, we have recalculated the same problem, because later experiments have established some physical constants which was left somewhat ambiguous in those days, and the further experimental data are now available.

Following the current notion concerning mesons, the bursts under consideration are caused by the electronic component produced by μ -mesons, which are considered to have the mass 217 times the electron mass and the spin 1/2. As for the production process of the electronic component, the knock-on electrons are not so effective as the radiation in the burst size under consideration, as was already shown by Christy and Kusaka. The pair creation is also negligible because of the rapid fall of the energy spectrum of μ -mesons, though this process is most effective in the shower under the great depth of the earth. The hypothetical existence of C-mesons might be another cause of the shower which would give the comparable frequency of showers with that of photons, if the mass of a C-meson were smaller than that of a μ -meson. But even if the C-meson really exists, the mass may be so large that the appreciable effect can not be seen in this phenomenon¹⁴⁾. So we have only to consider the normal radiation process of a spin 1/2 particle.

Firstly, we determine the energy spectrum of μ -mesons falling on the spherical ionization chamber, considering the angular distribution. Assuming the differential spectrum of the vertical incident as $dE/(E+a)^3$, the whole directional intensity is given by

$$2\pi \int d \cos \theta / (E+a/\cos \theta)^3 \equiv D(E)/E^3 \\ \approx (2\pi/E^3) [1 + (3a/E) \ln(a/E) + (5/2)(a/E)]. \quad (4.1)$$

Since a is taken as 2 BeV, the whole directional intensity can be obtained simply by multiplying 2π in higher energy regions, where the spectrum deviates appreciably from dE/E^3 . The accurate spectrum of vertical incident $f(E)$ is given by the intensity-depth relation underground¹⁵⁾. Multiplying it by $D(E)$ given in (4.1), we get the energy spectrum which should be used in what follows.

$$\mu(E) = f(E) \cdot D(E). \quad (4.2)$$

This is represented in Fig. 3, together with the spectrum adopted by Christy and Kusaka. It is noticeable that ours is as large as about twice of theirs.

The probability per g cm^{-2} for the μ -meson with energy E to radiate a photon with energy between ϵ and $\epsilon + d\epsilon$ is

$$R(E, \epsilon) d\epsilon = (N/A) 4\pi Z^2 r_0^2 (m/\mu)^2 \\ \times [\ln(12E/5Z^{1/2} \mu c^2) - 1/3] d\epsilon/\epsilon. \quad (4.3)$$

The notations used are current manner and need not be explained.

The radiated photon makes a cascade shower and brings about electrons in the ionization chamber. The probability that the burst with size $\geq n$ is brought about by the initiating photon energy ϵ is given by Christy and Kusaka¹³⁾ as follows,

$$F(\epsilon, n) = 13.5 X_0 (\epsilon/15\beta n)^{1/2} \exp(-15\beta n/\epsilon). \quad (4.4)$$

In this formula the values of $13.5X_0$ and 15β contain some inevitable ambiguity concerning the effect of fluctuations in a cascade shower and the fact that the meson spectrum is not a pure power law. Here we take the same value as the previous one, in order to facilitate the comparison. So

$$X_0 = 6.7 \text{ g cm}^{-2}, \quad \beta = 16 \text{ MeV}, \quad (4.5)$$

following Lapp⁸⁾.

The probability for a μ -meson with energy E to produce a burst with size $\geq n$ is obtained from

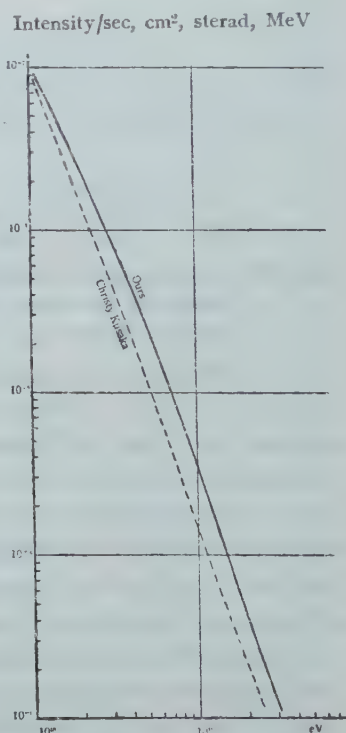


Fig. 3. Differential energy spectrum of μ mesons at sea level.

(4.3) and (4.4)

$$\begin{aligned}
 S(E, n) &= \int_0^E d\epsilon R(E, \epsilon) P(\epsilon, n) \\
 &= \frac{NV}{A} \cdot 4aZ^2r_0^2(m/\mu)^2 \left[\ln\left(\frac{12E}{5Z^{1/3}\mu c^2}\right) - \frac{1}{3} \right] \times 13.5X_0 \\
 &\quad \times \left[\left(\frac{E}{15\beta n}\right)^{1/2} \exp\left(-\frac{15\beta n}{E}\right) - \sqrt{\pi} \left\{ 1 - \Phi\left(\sqrt{\frac{15\beta n}{E}}\right) \right\} \right], \quad (4.6)
 \end{aligned}$$

where $\Phi(x)$ means the error function.

Now the size-frequency distribution of the bursts is given by

$$H(n) = \int_0^\infty dE \mu(E) S(E, n). \quad (4.7)$$

We have integrated (4.7) numerically.

The result is represented in Fig. 4, in which a comparison with the result of Christy and Kusaka and with the experimental data of Schein and Gill, and Lapp is given. The size obtained by the latter authors is reduced by the factor 1/1.4 following the notice of Greisen¹⁰⁾ that the specific ionization taken by them was

—: Ours
 - - - : Christy-Kusaka
 ○ : Schein and Gill
 × : Lapp
 Freq./cm² sec.

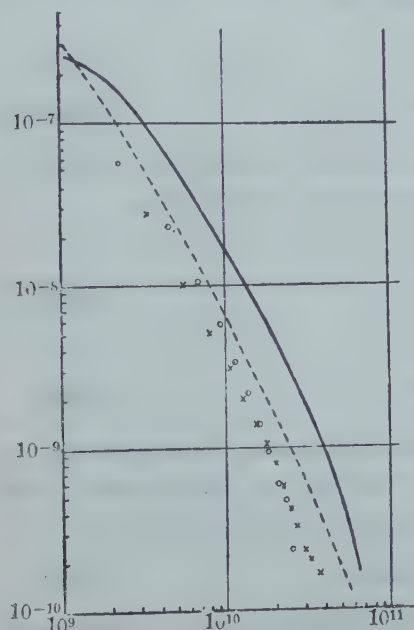


Fig. 4. Size-frequency relation of bursts at sea level.

too small. But we do not adopt his another notice that the path length of the electron in the gas of the chamber is longer than the straight one because of the multiple scattering. This is because the mean energy of shower electrons may not be so small as affected by the multiple scattering, for the iron wall with the thickness 1.25 cm seems to cut out considerably the low energy electrons emerging from the lead absorber, and moreover if we obey the latter notice the discrepancy between the theory and the experiment becomes greater. The difference between Christy and Kusaka's and ours is mainly due to the different energy spectrum of μ -mesons as is seen in Fig. 3. We see that our result gives higher frequency than the experimental one, though our meson spectrum is taken more carefully than Christy and Kusaka's. This seems perhaps to be due to the fact that Christy and Kusaka's formula (4.4) may favour to the larger size. Such a possibility may be plausible, since the formula (4.4) is based on Furry's model and Furry's fluctuation is now considered to be larger than the real one¹⁶⁾. Kusaka¹⁷⁾ has, however, pointed out

that the fluctuation plays no essential role in their calculation. To see this situation, we have to calculate the burst frequency neglecting the fluctuations.

§ 5. Bursts produced by nucleons.

In spite of the brilliant success of Christy and Kusaka to explain the burst at sea level, no satisfactory explanation has been given for the altitude dependence of the burst frequency under thick absorber. Anyone has only suggested that these bursts at the higher altitudes might be caused by the strongly absorbable agents, but could not manifest their definite nature and their interactions with matter. Recently Bridge and Rossi¹¹⁾ have concluded from their ingenuous experiment that these may be caused by nucleons, although the mechanism of production of the electronic component was left unknown. As for this point, one of us (S.H.) has proposed the photon production by the charge exchange of the nucleon¹⁸⁾ and reported some preliminary calculations¹⁹⁾. This process should be compared with the radiation of the μ -meson. Here we do not consider the effect of π -mesons because of their negligible contributions.

The intensity of nucleons is determined from the absolute intensity of the primary rays²⁰⁾ assuming the exponential absorption in the atmosphere, the coefficient of which is $1/125 \text{ g cm}^{-2}$. The integration through all directions is necessary for only μ -mesons, whereas the nucleons come mostly from the vertical direction because of their high absorbability. Thus we get the relative intensity of both components at the atmospheric depth 700 g cm^{-2} .

$$\text{Nucleons: } N(E)dE = 0.048 dE/E^{\tau+1} \quad (5.1a)$$

$$\mu\text{-mesons: } \mu(E)dE = (B/(E+B))dE/E^{\tau+1} \quad (5.1b)$$

where $B \approx 3.4 \times 10^{11} \text{ eV}$ concerns the life of π - μ decay¹⁵⁾. Each of these components produces a photon in lead with the cross-section, respectively,

the charge exchange of nucleons:

$$6.5 \times 10^{-26} k \ln(E/Mc^2) d\epsilon/\epsilon \text{ cm}^2, \quad (5.2a)$$

the radiation of μ -mesons:

$$3.3 \times 10^{-28} \ln(E/\mu^0) d\epsilon/\epsilon \text{ cm}^2. \quad (5.2b)$$

It must be noticed that the both cross-sections show the same ϵ -dependence, so that we can compare both contributions only in their factor. Multiplying (5.1) by (5.2), the burst frequencies can be roughly determined for $E \ll B$,

$$(\text{by nucleons})/(\text{by } \mu\text{-mesons}) = 4.8 k \quad (5.3)$$

The experimental evidences, though not accurately consistent with each other, show that the ratio is nearly $5^{(8)(11)(21)(22)}$. Therefore, if the unknown factor k is of the

Freq. ratio to sea level

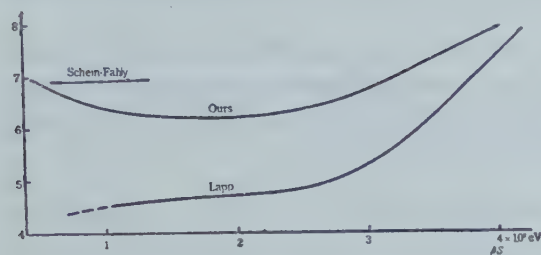


Fig. 5. Altitude variation of burst frequency under thick absorber.

700 g cm^{-2} to at sea level is plotted as the function of their size in Fig. 5, and one sees the approximate agreement with experiments.

The competing process for producing the electronic component by a nucleon is the decay of neutral mesons produced in a nuclear collision. It may be doubted that this process plays a main role in the electronic radiation accompanying a nuclear collision¹⁸. But we should not overlook the possibility that a neutral meson decays considerably quickly so that it contributes to the burst production. In the case of the burst production by the nucleon, the contribution of such a decay process should be at most the same order as that of the charge exchange, as is seen from the above analysis. On the basis of this fact, we can estimate the shorter limit of the life of a neutral meson as follows.

In order to facilitate the comparison of the decay process with the charge exchange process, we assume the energy spectrum of produced mesons as dE/E , which is considered to be not far from reality²⁰. The energy E is imparted to two photons or an electron and a positron, but one has only to consider the total energy converted into soft rays as far as the average behaviour of the shower is concerned. Then we compare the constant factors of the production cross-sections of both processes above, since these have the same energy dependence $d\epsilon/\epsilon$.

Both processes under consideration occur by the geometrical cross-sectional area of the collided nucleus following the current concept. Therefore, we concern the factor which have to be multiplied to this area. In the collision with an air nucleus the fractional energy converted into mesons is estimated as about $1/3$ ²⁰. In the case of the lead nucleus, we must multiply this figure by the ratio of $A^{1/3}$, $(207/14.4)^{1/3} = 2.44$, provided that all nucleons transversed by an incident nucleon are effective for the meson production. Further, we multiply $1/3$, which means the fraction of neutral mesons. $(1/3) \times 2.44 \times (1/3) = 0.27$ represents the fractional energy converted to soft rays by this process. Transforming this figure into the frequency, we get $(0.27)^{1.8} = 0.096$. On the other hand, the corresponding factor for the charge exchange process is

$$kA^{1/3}(2/137\pi) \cdot \ln(E/Mc^2) \approx 0.065k \quad \text{for } E=10^{10} \text{ eV}.$$

This gives about $1.5/k$ times more frequency for the neutral meson process than

order of unity, the radiation by charge exchange can well account for the good part of the bursts. The possible contribution from the decay of neutral mesons produced in the nuclear collision shall be discussed below.

The size-frequency relations are obtained by the same procedure as in § 4. The ratio of frequencies at

the charge exchange one. In the following discussion, we shall tentatively take this factor as 2, considering $k \lesssim 1$.

In order that the neutral meson process should not exceed the charge exchange process, about 1/2 of neutral mesons should disintegrate in the shield. In the case of lead shield we may put the effective length $l \approx 5$ cm, that the neutral meson flies before the decay. Then its life is estimated as

$$\tau \gtrsim 2(\mu c^2/E) \cdot (l/c) \approx 2 \times 10^{-11} \text{ sec.} \quad (5.7)$$

for the energy of the neutral meson $E \approx 2 \times 10^9 \text{ eV}$, considering that the number of produced mesons in a collision may be 5 to 6.

This life is consistent with the theoretical results for the two-gamma-decay of the pseudo-scalar meson, 2.2×10^{-11} sec for the mass $286 m_\pi$, obtained by Fukuda and Miyamoto²³⁾. The more precise estimation of the life will be possible by the comparison of lead and iron shields, if more accurate calculation for the electromagnetic process will be possible.

The second evidence for our interpretation that the burst is produced by nucleons is the absorption length of the agents. This is about 430 g cm^{-2} for lead and 320 g cm^{-2} for iron^{22), 10)}. If the agent were μ -mesons, this value should be about 3600 g cm^{-2} ²⁴⁾. Such a large discrepancy manifestly rules out the assumption that μ -mesons are the agents. The ratio of the absorption length for lead to the one for iron may be consistent with the ratio of geometrical cross-sections of each nucleus. Their absolute values may be affected by the transparency of the nucleus²⁵⁾, or the cascade-like multiple process of the nuclear interactions²⁶⁾.

The third evidence for our interpretation is the directional distribution of the burst agents, which is much steeper than that expected for μ -mesons²¹⁾. The distribution law $\cos^{5-6}\theta$ is nearly equal to that for extensive air showers²⁷⁾, which suggests the common origin, i.e. nucleon component, for both phenomena.

We should like to express our sincerest thanks to Professor Greisen for his oversea remarks and interest. Thanks should also be offered to Professor Tomonaga and Mr. Yamaguchi for their stimulating discussions.

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Note added in proof

We must correct our description about the contribution from the decay of neutral mesons by following reasons.

The fractional energy converted into mesons by a nuclear collision must be reduced by a factor about 1/2, because our previous estimation is based on the work in which $\pi-\mu$ decay is not properly treated²⁸⁾. Then the reducing factor must be corrected as about 0.03 instead of 0.096 and the decay of neutral mesons brings about less frequency of bursts than the charge exchange process, provided $k=1$. Therefore, our estimation of the life is meaningless.

Now the γ -decay of neutral mesons seems us to be established experimentally. Our first argument, mentioned in reference (18), is disproved by the experiment of Vernov²⁹⁾, in which not only the shower is almost always accompanied by a penetrating ray in the stratosphere, but also the shower particles are electronic. The third argument is also disproved by Fretter, who obtains more frequent mixed showers than ever found and shows they are reasonably interpreted as the decay of neutral mesons.³⁰⁾ The most reliable evidence may be Berkeley experiment, which gives one credit to the γ -decay of neutral mesons³¹⁾. Furthermore the life of neutral meson is, against our presumption, found to be shorter than 10^{-13} sec. from the analysis of R-star³²⁾.

Accordingly, we must consider the γ -decay as the established source of bursts and the charge exchange process will contribute to them at most by the same order as the γ -decay. We may therefore put k as about 1/2 or smaller within our rough estimation. Further, in the γ -decay problem²³⁾ we may save gauge invariant terms and give up the equivalence theorem of pseudoscalar meson.

On the Convergence of the Perturbation Method. I.

TOSIO KATO

Department of Physics, Tokyo University.

(Received August 12, 1949)

Introduction.

The perturbation method is one of the most important methods of approximation in quantum mechanics as well as in some fields of classical mechanics. But the question of its convergence has not yet been fully discussed from the systematic point of view.

Many of the discussions hitherto given on the subject are based on plausibility considerations and draw no decisive conclusion. Among a few works of mathematical character, we should mention those of Wilson¹⁾ and Rellich²⁾. The papers of Wilson are rather unsystematic and mainly concerned with bounded operators³⁾, which restricts the fields of its application. Rellich's study is the most complete one in the mathematical sense, and treats the case of the so-called regular operators in which the formal series of perturbation are proved to be convergent for sufficiently small value of the parameter. His results are applicable to many problems, especially in classical mechanics, but are still restricted considerably in application, for it is rather usual that the perturbation method is valid only in the sense of asymptotic expansion but not in the sense of power expansion, and in such cases the perturbation cannot be regular in Rellich's sense.

On the basis of the variational principle in a generalized sense, the writer developed a theory of the perturbation method regarded as an asymptotic expansion, which is much wider in scope of application than Rellich's regular perturbation.

Before entering into this subject, however, it is worth while first to treat the regular perturbation, for the formal part of the perturbation method is completely determined in this case and, moreover, it has itself an important application in quantum mechanics of atoms as we shall show below. But since the original method of Rellich is somewhat complicated and abstruse, we will give in this paper an improved and much simplified treatment of the regular perturbation based on the use of resolvents⁴⁾ and contour integrals. Moreover, our method allows us to give explicit formulas representing eigen-values and eigen-vectors as far as any order of the perturbation. Also the estimation of the convergence radii is much improved. We restrict ourselves, however, to a brief outline of the theory together with some illustrative examples and refer the readers to another paper⁵⁾ of the

writer regarding more detailed and rigorous treatment of the problem. More general case (asymptotic expansion) and the perturbation containing the time will be discussed in subsequent papers.

PART I. Regular Perturbation.

§ 1. Resolvents.

Let A be a linear operator in Hilbert space \mathfrak{H} . If there is a constant c such that $\|Af\| \leq c\|f\|$, A is said to be bounded; the lower limit of c is called the bound or norm of A and denoted by $\|A\|$. Almost every operator appearing in quantum mechanics is unbounded and this is the origin of all sorts of difficulties.

Bounded operators can be regarded as defined everywhere in $\mathfrak{H}^{(9)}$, while this is not the case with unbounded operators. Consequently, addition and multiplication can be defined without restriction for bounded operators. Infinite series $\sum A_n = A$ can also be defined for bounded A_n ; it is e.g. uniformly convergent if $\|A_1 + \dots + A_n - A\| \rightarrow 0$ as $n \rightarrow \infty$. Similarly, the regularity or analyticity of a bounded operator $A(l)$ depending on a real or complex parameter l can be defined.

Let H be a (in general not bounded) self-adjoint⁽⁷⁾ operator, i.e. an Hermitian operator for which the eigen-value problem is completely solvable. Then it is known⁽⁸⁾ that the resolvent⁽⁴⁾

$$R(l) = (H - l)^{-1} \quad (1)$$

exists and is a bounded operator provided that the complex number l does not belong to the spectrum of H (the latter is confined to the real axis!). Further $R(l)$ is a regular (analytic) function of l , and its singular points compose exactly the spectrum of H .

To derive important properties of $R(l)$, we assume for the sake of simplicity that the spectrum of H consists of purely discrete eigen-values $\dots < \lambda_{-1} < \lambda_0 < \lambda_1 < \dots$. Then we can write⁽⁹⁾

$$H = \sum_n \lambda_n E_n, \quad (2)$$

where E_n is the projection operator⁽¹⁰⁾ belonging to the eigen-value λ_n and has the property

$$E_m E_n = \delta_{mn} E_n, \quad \sum_n E_n = 1. \quad (3)$$

Also we have

$$R(l) = \sum_n (\lambda_n - l)^{-1} E_n. \quad (4)$$

The following relations can easily be derived from (4) and (3):

$$\left(\frac{d}{dl} \right)^k R(l) = k! [R(l)]^{k+1}, \quad (k = 1, 2, \dots), \quad (5)$$

$$\|R(l)\| = [\Delta(l)]^{-1}, \quad (6)$$

where $\Delta(l)$ is the distance of l from the spectrum of H . In particular $\Im(l)$ denotes the imaginary part of l

$$\|R(l)\| \leq |\Im(l)|^{-1}. \quad (7)$$

Further we have, if l is purely imaginary,

$$\|H \cdot R(l)\| \leq 1. \quad (8)$$

If C is a closed curve in the complex plane which does not pass through any point of the spectrum of H , we have by (4)

$$-(2\pi i)^{-1} \oint_C R(l) dl = \sum_n E_n, \quad (9)$$

where summation is to be taken for such n that λ_n is inside of C .

(4) can also be written as follows:

$$R(l) = (\lambda_0 - l)^{-1} E_0 + S_0(l), \quad (10)$$

where

$$S_0(l) = \sum_{n \neq 0} (\lambda_n - l)^{-1} E_n \quad (11)$$

is regular at $l = \lambda_0$. We shall call $S_0(l)$ the *reduced resolvent* of H for the eigen-value λ_0 . It can easily be shown that $S_0(l)$ satisfies (5) and hence that

$$S_0(l) = \sum_{k=0}^{\infty} (l - \lambda_0)^k [S_0(\lambda_0)]^{k+1}. \quad (12)$$

§ 2. General Theory of Regular Perturbation.

Let H_x be a self-adjoint operator depending on a real parameter x . Then its resolvent $R_x(l) = (H_x - l)^{-1}$ is also a function of x . H_x is called a regular function²⁾ of x if $R_x(l)$ is regular in the neighborhood of $x=0$ for *some* fixed l . Then it is shown^{2), 5)} that the same is true for *every* l not belonging to the spectrum of H_0 provided that x is sufficiently small. It should be noted that *direct* definition of the regularity of H_x is impossible since H_x is not assumed to be bounded.

In the following we assume H_x to be a regular function of x . Let λ_0 be an isolated eigen-value of H_0 with finite multiplicity m , and let E_0 be the projection operator on the corresponding eigen-space. Then we can draw a closed curve C in the complex plane containing λ_0 in its interior but containing inside or on it no other point of the spectrum of H_0 . It can be shown⁵⁾ that C does not pass through the spectrum of H_x too if x is sufficiently small. We can therefore define

$$E_x = -(2\pi i)^{-1} \oint_C R_x(l) dl. \quad (13)$$

According to (9), E_x is the sum of projections belonging to those eigen-values of H_x lying inside C .

By what we stated above, $R_x(l)$ is regular in x for every l on C provided x is small. Hence it follows from (13) that E_x is also regular for small x . In particular $\dim E_x$ must be a continuous function of x . But as it is an integer, we have

$$\dim E_x = \text{const.} = \dim E_0 = m. \quad (14)$$

It means that the total multiplicity of eigen-values of H_x inside C is just equal to m and independent of x .

Thus we have succeeded in separating the m -dimensional subspace E_x which is regular in x and tends to E_0 for $x \rightarrow 0$. Our problem is therefore essentially reduced to the problem in a finite dimensional Euclidian space, and we can establish the following facts without difficulty³⁾. The eigen-values under consideration consist of s distinct ones λ_{xj} ($j=1, \dots, s$), s being independent of x and $\leq m$. Denoting by E_{xj} the projection on the eigen-space belonging to λ_{xj} , we have

$$E_{xj}E_{xk} = \delta_{jk}E_{xj}, \quad \sum_{j=1}^s E_{xj} = E_x, \quad (15)$$

$$\sum_{j=1}^s m_j = m, \quad m_j = \dim E_{xj}, \quad (16)$$

m_j being constant. Further λ_{xj} and E_{xj} are regular in x and developable into power series of x with non vanishing convergence radii.

Thus the perturbation method is justified to its full extent in the case of regular perturbation.

It should be noted that our results are valid so long as λ_0 is an *isolated* eigen-value of H_0 with *finite* multiplicity. The nature of the rest of the spectrum has no influence on the convergence of the result.

The above argument is also valid when we consider finite number of such isolated eigen-values simultaneously.

Hitherto our consideration has been restricted to small value of x . But as the same argument applies to a small neighborhood of any value of x in the regularity region of H_x we can apply the process of analytic continuation to eigen-values and eigen-spaces of H_x . It follows that they are analytic functions of x , and this is true even when crossing takes place among them, so long as they do not come in contact with continuous spectrum, for such a crossing only means an incidental degeneracy of eigen-values which does not affect our argument.

In this paper we considered exclusively projection operators E_{xj} on eigen-spaces because they have the advantage of being uniquely determined whereas eigen-vectors are devoid of this property (especially when degeneracy occurs). But we can, if necessary, easily obtain m_j independent eigen-vectors of H_x belonging to the eigen-value λ_{xj} by putting

$$\varphi_{\alpha j p} = E_{\alpha j} \varphi_p \quad (p=1, \dots, m_j),$$

where φ_p may be nearly arbitrary. Since $E_{\alpha j}$ is regular, these eigen-vectors are also regular and can be ortho-normalized conserving the regularity.

§ 3 Formal Series.

Once the regularity of the eigen-values and eigen-spaces (or eigen-vectors) is established, it is evident that they are formally represented by the series obtained by the usual procedure. Here we shall show that the same formulas can be obtained in a more compact form by our method of contour integral.

Let H_α be given formally by

$$H_\alpha = H_0 + \alpha V, \quad (17)$$

where H_0 is the unperturbed operator and αV is the perturbation. For the present we proceed quite formally, postponing the examination of the regularity of H_α to later stage. We have

$$\begin{aligned} R_\alpha(l) &= (H_\alpha - l)^{-1} = (H_0 - l + \alpha V)^{-1} \\ &= [1 + \alpha l^{-1} \cdot R(l) \cdot (H_0 - l)]^{-1} \\ &= R_0(l) [1 + \alpha V \cdot R_0(l)]^{-1} \\ &= R_0(l) \sum_{n=0}^{\infty} (-1)^n \alpha^n [l^{-1} \cdot R_0(l)]^n. \end{aligned} \quad (18)$$

Substituting into (13), we obtain

$$E_\alpha = E_0 + \sum_{n=1}^{\infty} \alpha^n A^{(n)}, \quad (19)$$

where

$$A^{(n)} = (-1)^{n-1} (2\pi i)^{-1} \oint_C R_0(l) V R_0(l) V \dots V R_0(l) dl, \quad (20)$$

l^{-1} appearing n times in the integrand.

To evaluate the integral of (20), we note that

$$R_0(l) = (\lambda_0 - l)^{-1} E_0 + S_0(l), \quad (21)$$

where $S_0(l)$ is the reduced resolvent of H_0 and regular at $l = \lambda_0$ (see (10), (11)). Substituting (21) into (20) and noting (12), we can easily calculate (20). Writing $S_0(\lambda_0) = S$, we obtain

$$A^{(n)} = (-1)^{n-1} \sum_{(k_1 + \dots + k_{n+1} = n)} S^{k_1} V S^{k_2} V \dots V S^{k_{n+1}}, \quad (22)$$

where summation should be taken over all combinations of $k_j \geq 0$ such that $k_1 + \dots + k_{n+1} = n$, and S^0 should be replaced by $-E_0$. In particular

$$\begin{aligned}
A^{(1)} &= -E_0 V S - S V E_0, \\
A^{(2)} &= E_0 V S V S + S V E_0 V S + S V S V E_0 \\
&\quad - E_0 V E_0 V S^2 - E_0 V S^2 V E_0 - S^2 V E_0 V E_0.
\end{aligned} \tag{23}$$

On the other hand we have by (13), noting $H_\kappa R_\kappa(l) = 1 + l R_\kappa(l)$,

$$\begin{aligned}
(H_\kappa - \lambda_0) E_\kappa &= - (2\pi i)^{-1} \oint_C (H_\kappa - \lambda_0) R_\kappa(l) dl \\
&= - (2\pi i)^{-1} \oint_C (l - \lambda_0) R_\kappa(l) dl.
\end{aligned} \tag{24}$$

Hence we obtain in the same way as above

$$(H_\kappa - \lambda_0) E_\kappa = \sum_{n=1}^{\infty} \kappa^n B^{(n)}, \tag{25}$$

where

$$\begin{aligned}
B^{(n)} &= (-1)^{n-1} (2\pi i)^{-1} \oint_C (l - \lambda_0) R_0(l) V R_0(l) V \dots V R_0(l) dl \\
&= (-1)^{n-1} \sum_{(k_1 + \dots + k_{n+1} = n-1)} S^{k_1} V S^{k_2} V \dots V S^{k_{n+1}}.
\end{aligned} \tag{26}$$

If the splitting of the eigen-value does not occur ($s=1$), we have (writing λ_κ for $\lambda_{\kappa 1}$) $H_\kappa E_\kappa = \lambda_\kappa E_\kappa$ and $\text{Tr}(H_\kappa E_\kappa) = \lambda_\kappa \text{Tr}(E_\kappa) = m \lambda_\kappa$. Thus we have, by taking the trace of (25),

$$\lambda_\kappa = \lambda_0 + m^{-1} \sum_{n=1}^{\infty} \kappa^n \text{Tr}(B^{(n)}), \tag{27}$$

where the first several coefficients are easily calculated to be

$$\left. \begin{aligned}
\text{Tr}(B^{(1)}) &= \text{Tr}(V E_0), \\
\text{Tr}(B^{(2)}) &= -\text{Tr}(V S V E_0), \\
\text{Tr}(B^{(3)}) &= \text{Tr}(V S V S V E_0) - \text{Tr}(V S^2 V E_0 V E_0).
\end{aligned} \right\} \tag{28}$$

If in particular the eigen-value λ_0 is non-degenerate ($m=1$), the traces above are calculated by introducing the normalized eigen-vector φ_0 of H_0 belonging to λ_0 . Noting $E_0 = P_{[\varphi_0]}$ we have

$$\begin{aligned}
\lambda_\kappa &= \lambda_0 + (V \varphi_0, \varphi_0) \kappa - (S V \varphi_0, V \varphi_0) \kappa^2 + \frac{1}{2} (V S V \varphi_0, S V \varphi_0) \\
&\quad - (V \varphi_0, \varphi_0) \cdot (S V \varphi_0, S V \varphi_0) \kappa^3 + \dots
\end{aligned} \tag{29}$$

As is easily seen, these results are in accordance with the formulas derived in the usual manner. Our method can be extended to the general case where splitting of the eigen-value may take place at any power of κ .

§ 4. Criteria for Regularity. Applications.

Since the definition of the regularity of H_x given in § 2 was indirect, it is desirable to have sufficient conditions for the given H_x to be regular. One of the most important cases is as follows¹¹⁾. Let H_x be as in (17) and let H_0 be (essentially¹²⁾) self-adjoint. If there are two constants a, b such that

$$\|Vf\| \leq a \|H_0 f\| + b \|f\|, \quad (30)$$

then H_x is regular at least for $|x| < a^{-1}$.

To show this put $f = R_0(l)g$ in (30). Then

$$\|VR_0(l)g\| \leq a \|H_0 R_0(l)g\| + b \|R_0(l)g\|.$$

If l is purely imaginary, we have by (8) and (7)

$$\|VR_0(l)g\| \leq (a + b|l|^{-1}) \|g\|.$$

It means that $VR_0(l)$ is a bounded operator and that $\|VR_0(l)\| \leq a + b|l|^{-1}$. If $|x| < a^{-1}$, we can take $|l|$ so large that $\|xVR_0(l)\| \leq |x|(a + b|l|^{-1}) < 1$ holds. Then the series in (18) is absolutely convergent, proving that H_x is regular.

It should be noted that (30) is satisfied for any H_0 if V is bounded, for we can take $a=0, b=\|V\|$.

In the same way, we can treat more general case

$$H_x = H_0 + xV_1 + x^2V_2 + \dots \quad (31)$$

and show that H_x is regular if

$$\|V_n f\| \leq c^{n-1}(a \|H_0 f\| + b \|f\|), \quad (n=1, 2, \dots) \quad (32)$$

for some constants a, b, c .

The condition (30) or (32) is satisfied in most cases in the classical eigenvalue problems. For instance, take the case of Sturm-Liouville equation in the interval $a \leq x \leq \beta$

$$\frac{d}{dx}(\rho_0 + x\rho_1) \frac{d}{dx}u + \{\lambda(\rho_0 + x\rho_1) - (q_0 + xq_1)\}u = 0, \quad (33)$$

where we assume $\rho_0 > 0, \rho_1 > 0$. (33) can be transformed into the following form

$$H_x v = \lambda v, \quad v = (\rho_0 + x\rho_1)^{1/2}u,$$

$$H_x = -(\rho_0 + x\rho_1)^{-1/2} \frac{d}{dx}(\rho_0 + x\rho_1) \frac{d}{dx}(\rho_0 + x\rho_1)^{-1/2} + (q_0 + xq_1)(\rho_0 + x\rho_1)^{-1}$$

Then we can verify without difficulty that the condition (32) is satisfied provided that the coefficients $\rho_0, \rho_1, q_0, q_1, \rho_0, \rho_1$ are continuous in $a \leq x \leq \beta$. If some of them are discontinuous in the interior or at the boundary of the interval, special investigation is necessary.

In the problems of quantum mechanics, (30) or (32) is not always satisfied, for it is usual that V has singular points. Nevertheless the condition (30) has important application in quantum mechanics of atoms and molecules. Consider a system composed of finite number of particles interacting with each other through a potential energy of Coulomb type. Let $H=T+W$ be the Hamiltonian of the system, where T is the kinetic energy and W is the potential energy. Let W be divided into two parts: $W=V_0+V$, where V is regarded as a perturbation to the unperturbed Hamiltonian $H_0=T+V_0$. Then, though V is not bounded, it can be shown¹³⁾ that (30) is satisfied and, moreover, that a can be taken as small as we like. It follows that $H_x=H_0+xV$ is regular in x for $-\infty < x < +\infty$ and hence by § 2 that the energy levels of the system are analytic functions of x even if they cross each other when x grows from 0 to 1. It should be noted that this offers a theoretical basis for the one-electron model of atomic systems¹⁴⁾.

§ 5. Convergence Radii. Application to Mathieu Functions.

In this section we shall show that our method of contour integral allows us to estimate the convergence radii of the series representing eigen-values and eigen-spaces (or eigen-vectors). For the sake of simplicity, however, we restrict ourselves to the case where V in (17) is a bounded operator.

Let the *distance of isolation* of the eigen-value λ_0 (i.e. the distance of λ_0 from the rest of the spectrum of H_0) be d . Take as the curve C the circle $|L-\lambda_0|=d/2$. Then we have $\|R_0(L)\|=2/d$ on C as we see easily from (6). Hence we have

$$\|VR_0(L)\| \leq 2\|V\|/d$$

on C , and the series on the right-hand side of (18) is absolutely convergent if

$$|x| < d/2\|V\| \quad \text{or} \quad \|xV\| < d/2. \quad (34)$$

It follows immediately that the series representing E_x is convergent for (34). The same is true with the series of the eigen-value λ_x if splitting does not take place as e.g. in the non-degenerate case. In other words, *the series are certainly convergent if the "magnitude of the perturbation" $\|xV\|$ is less than $1/2^*$ of the distance of isolation of the eigen-value λ_0* . In many cases this estimation must be very crude, but it will be seen that in general convergence is good if x/d is small.

By the way it will be noted that the figure $1/2$ above cannot in general be replaced by a large one. This is shown by the following simple example in two-dimensional Euclidian space:

$$H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \|V\|=1,$$

$$\lambda_0 = 1, \quad d = 2, \quad \lambda_x = (1+x^2)^{1/2}.$$

* The figure obtained from the formula of Rellich is $1/16$ and much worse than ours. His method can be improved so as to give $1/4$ as the result, but seems to be unable to go further.

We can also estimate the error committed when we cut the series with finite terms. But the question is more conveniently treated in the general theory of perturbation as an asymptotic expansion, and will be deferred to a subsequent paper.

As an application of (34), consider the Mathieu equation

$$\frac{d^2 u}{dx^2} + (\lambda + 16x \cos 2x)u = 0, \quad (-\pi \leq x \leq \pi)$$

under the periodic boundary condition. If we put

$$H_0 = -d^2/dx^2, \quad V = -16 \cos 2x,$$

H_0 has the following eigen-functions and eigen-values

$$\begin{aligned} \varphi_0 &= (2\pi)^{-1/2}, & \lambda_0 &= 0, \\ \varphi_n &= (\pi)^{-1/2} \cos nx, & \lambda_n^+ &= n^2 \\ \varphi_n^- &= (\pi)^{-1/2} \sin nx, & \lambda_n^- &= n^2 \end{aligned} \quad \left. \vphantom{\begin{aligned} \varphi_n \\ \varphi_n^- \end{aligned}} \right\} (n=1, 2, \dots).$$

Since V is bounded and $\|V\|=16$, $H_n = H_0 + xV$ is regular by § 4. To obtain the respective convergence radii r_0, r_n^\pm for the eigen-values and eigen-functions of H_n , it is advantageous to observe the fact that the Hilbert space \mathfrak{H} is in this case divided into the following four orthogonal subspaces:

$$\begin{aligned} [\varphi_0, \varphi_2^-, \varphi_4^-, \dots], & \quad [\varphi_1^+, \varphi_3^+, \varphi_5^+, \dots], \\ [\varphi_2^+, \varphi_4^+, \varphi_6^+, \dots], & \quad [\varphi_1^-, \varphi_3^-, \varphi_5^-, \dots], \end{aligned}$$

all reducing V as well as H , and that the eigen-values are correspondingly divided into four groups. We can therefore consider the problem in each of these subspaces. Then all eigen-values of H_0 are non-degenerate and we have, denoting by d_0, d_n^\pm the isolation distances of λ_0, λ_n^\pm respectively,

$$\begin{aligned} d_0 &= 4, & d_1^+ &= d_1^- = 8, & d_2^+ &= 4, & d_2^- &= 12, \\ d_n^\pm &= n^2 - (n-2)^2 = 4(n-1), & (n &\geq 3). \end{aligned}$$

By (34) and $\|V\|=16$, we obtain the following estimation of the convergence radii:

$$\begin{aligned} r_0 &\geq 1/8, & r_1^+ &\geq 1/4, & r_1^- &\geq 1/4, \\ r_2^- &\geq 1/8, & r_2^+ &\geq 3/8, & r_n^\pm &\geq (n-1)/8 \quad (n \geq 3). \end{aligned}$$

This result may be compared with that of Watson¹⁵⁾. He showed by an elaborate calculation that $r_0 \geq \sqrt{2}/8$, but could not estimate other r_n^\pm . Our method gives us all r_n^\pm at once. Indeed our r_0 is slightly worse than his, but it must be noted that our formula (34) is the crudest estimate without considering special properties of V^* . If these are taken into account, we can easily obtain better result than Watson's.

* In fact, our result is immediately extended to more general Hill equations.

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On the Decay of Heavy Mesons. I.

—On Pauli's Regulator in the Calculation of Meson Decay Problems.—

S. OZAKI, S. ÔNEDA and S. SASAKI

Physics Institute, Tohoku University.

(Received August 15, 1949)

§ 1. Introduction.

Recently Tomonaga¹⁾ and Schwinger²⁾ have independently developed a relativistically covariant formulation of the quantum theory of field and have succeeded to the explanation of the level shift in the hydrogen atom and the anomalous magnetic moment of the electron. Since, however, the present field theory cannot be formulated in a relativistically and gauge covariant way without introduction of the singular delta-function of Jordan and Pauli, the present theory gives the divergent results to such field reaction problems and the remaining finite part can be not free from any ambiguity arising from the singular nature of delta function.

One typical example of the appearance of such an ambiguity is the photon self energy. As first pointed out by Schwinger²⁾, the photon self energy should be zero from the gauge covariant point of view, while, Wentzel³⁾ showed the photon self energy finite and non gauge covariant. Recently Pauli and Villars⁴⁾ succeeded to give unique result for the problem of the photon self energy by using "regulators" called by them. As the non gauge covariant and ambiguous terms appear also in the problem of meson decay⁵⁾, we used Pauli's regulator in order to obtain the unique and gauge covariant form. In the first place, in the γ -decay of the scalar neutretto firstly calculated by Fekuda and Miyamoto⁶⁾ there appears the convergent but non gauge covariant term, which further requires the necessity of condition $\int \sqrt{x} \rho(x) dx = 0$, $\sum c_i m_i = 0$ in addition to Pauli's conditions on their regulator in order to obtain the gauge covariant formulation. Secondly, there also appears the convergent but non gauge covariant term in the pseudovector coupling matrix element of the pseudoscalar neutretto, which vanishes by using Pauli's regulator. This non gauge covariant term is able to separate into non gauge covariant term not contributing to real transition process and gauge covariant one, but should be dropped off, on which we discussed in the previous paper⁷⁾. Further, in spite of being possible to prove formally the equivalence between pseudoscalar and pseudovector couplings of the pseudoscalar meson field before using the explicit expressions of Δ -functions, the identity relation does not hold only for the first non gauge covariant term of the pseudovector coupling and

the first gauge covariant one of the pseudoscalar coupling except the remainings. This fact seems to suggest the ambiguity of the first integral of the pseudoscalar coupling terms. Really this integral gives unique result if we use Schwinger's expressions of \mathcal{A} -functions, but gives another different result when another expressions of \mathcal{A} -functions are used. This situation is clearly seen in the fact that the value of $\bar{\mathcal{J}}$ at the origin is infinite in the Fourier representation but becomes zero in the Schwinger one. From this point of view, the first gauge covariant term of the pseudoscalar coupling should be dropped out, which obliges Pauli's regulator to have another condition, i.e., $\int \rho(\mathbf{x})/\sqrt{\mathbf{x}} d\mathbf{x}=0$, $\sum c_i/m_i=0$. Further, this condition compels the second term of the scalar coupling of the scalar neutretto to be zero.

But the use of this condition unrestrictedly gives rise the failure of the equivalence theorem in another case of meson decay problem. Namely, in the decay of a pseudoscalar neutral τ -meson into a vector neutral π meson with tensor coupling and a photon, the pseudoscalar coupling term is equivalent to the pseudovector coupling one without using $\int \rho(\mathbf{x})/\sqrt{\mathbf{x}} d\mathbf{x}=0$, except the first diverging and the second finite terms of the pseudoscalar coupling which vanish by the charge renormalization regulator $\int \rho(\mathbf{x}) \log |\mathbf{x}| d\mathbf{x}=0$ and $\int \rho(\mathbf{x}) d\mathbf{x}=0$ and the first finite but non gauge covariant term of the pseudovector coupling which disappears by the condition $\int \rho(\mathbf{x}) \sqrt{\mathbf{x}} d\mathbf{x}=0$. On the contrary, the use of $\int \rho(\mathbf{x})/\sqrt{\mathbf{x}} d\mathbf{x}=0$ destroys the equivalence theorem in this case. This situation tells us the following fact, that the condition $\int \rho(\mathbf{x})/\sqrt{\mathbf{x}} d\mathbf{x}=0$ should be used only after the relation $f_1=(2\mathbf{x}_0/\mathbf{x})f_2$ was set up, where f_1 and f_2 denote the pseudoscalar and the pseudovector coupling constants and \mathbf{x}_0 and \mathbf{x} means the reciprocal Compton wave length of proton and meson respectively. But the use of the condition $\int \rho(\mathbf{x})/\sqrt{\mathbf{x}} d\mathbf{x}=0$ after the relation $f_1=(2\mathbf{x}_0/\mathbf{x})f_2$ or $f_2=(\mathbf{x}/2\mathbf{x}_0)f_1$ was substituted removes sometimes even the term which satisfies the identity relation and gauge covariancy. As this result seems undesired, we shall adopt the following procedure.

Now we shall consider this situation more profoundly. When we prove the equivalence theorem between the pseudoscalar and the pseudovector couplings, the following procedure is used:

$$H_{ps} = -\frac{\partial}{\partial x} \iint \dots dx' dx'' + 2x_0 H_{ps} + \mathcal{A},$$

where H_m and H_{ps} denote the Hamiltonian density for the real transition process of the pseudovector and pseudoscalar coupling respectively, and \mathcal{A} expresses the term multiplies by $\delta(x)$, which is obtained by transforming the pseudovector coupling term as above using the Dirac equation of $\bar{\mathcal{J}}$ -function $\gamma_\mu \partial_\mu \bar{\mathcal{J}}(x) - x_\mu \bar{\mathcal{J}}(x) = -\delta(x)$. So that when we will regulate the matrix elements to preserve the equivalence theorem, both sides should be regulated as the same order of magnitude with respect to x_0 . If such procedure for regulating method is used, the

removal of divergency, gauge covariancy and identity relation are always preserved by using only Pauli's regulators and $\int \rho(x) \sqrt{x} dx = 0$. But there exists another alternative; namely when we regulate the both sides, the method which regulates the expression divided by x_0 in the formula cited above is also allowed. But in this case, the another condition $\int \rho(x) / \sqrt{x} dx = 0$ for regulator is necessary besides the above conditions in order to satisfy all requirements mentioned above. It is also noted that those alternatives yield the same results for removing the undesired terms. Further, it is interesting that the conditions $\int \sqrt{x} \rho(x) dx = 0$ or $\int \rho(x) / \sqrt{x} \times dx = 0$, which were necessary to preserve the gauge covariancy and the identity relation for the former and the latter alternative respectively makes the divergent term vanish, appearing in the decay of a heavy meson into lighter mesons, for example, $\tau^\pm \rightarrow \pi^\pm + \pi^0$ or $\tau^\pm \rightarrow \pi^\pm + \gamma$. From the considerations mentioned above, it may be concluded that careful treatment for regulating method gives almost satisfactory result.

§ 2. Details of the above considerations.

Fukuda and Miyamoto⁶⁾ first pointed out that the non gauge covariant and finite terms appears in the first term of the matrix element for the scalar coupling in the γ -decay of scalar neutretto. But as the gauge covariancy for the matrix element is formally proved⁷⁾, such a non gauge covariant term must be dropped out. Or, such a term should be removed out by the following reason. Generalized Schrödinger equation for such a term is

$$i \frac{\partial \Psi}{\partial \sigma(x)} = G A_\nu^2 \varphi \Psi, \quad G = \frac{x f}{8\pi^2} \left(\frac{e}{\hbar c} \right)^2 \int_{-1}^1 d\tau \int_1^\infty \frac{du}{u^3}, \quad (1)$$

where A_ν and φ denote electromagnetic potential and wave function of scalar neutretto respectively and Ψ means state vector and $x = mc/\hbar$, m proton mass. This equation is gauge invariant for the following gauge transformation $A_\mu \rightarrow A_\mu - \frac{\partial A}{\partial x_\mu}$, if we take

$$\Psi = e^{-iI} \Psi', \quad I = G \int_\sigma \varphi(x') \left\{ A(x') \frac{\partial A(x')}{\partial x'_\mu} - 2A_\mu(x') A(x') \right\} dx'_\mu, \quad (2)$$

but $\tilde{\Psi}' F_{\mu\nu} \Psi'$ is not gauge covariant, for $F_{\mu\nu}$ does not commute with I .

As, however, in this case there appears the odd power of mass m , Pauli's regulator method for even power of mass m can be not used as it stands. Therefore, if the following integral is utilized, which holds for positive and negative x , where $x = m^2 c^2 / \hbar^2$,

$$\sqrt{\frac{\pi i}{x}} = \int_{-\infty}^{\infty} e^{i x x'^2} dx', \quad (3)$$

we can regulate as the term of even power of m . The non gauge covariant term for the scalar neutretto with scalar coupling is produced from the following integral:

$$\begin{aligned} & \times \sqrt{x} \int d\xi d\eta \int (dk) (dk') (dk'') \int da db dc \left(\frac{a}{|a|} + \frac{b}{|b|} \right) \left(\frac{c}{|c|} + \frac{a+b}{|a+b|} \right) \exp i(k_v \hat{\xi}_v + k'_v \eta_v) \\ & \times \exp i(a+b+c) (k''_v + x) \exp \left[-i \frac{(ak_v + bk'_v)^2}{a+b+c} \right] \text{Sp}(\gamma_\mu \gamma_\nu) A_\mu(x+\xi) A_\nu(x-\eta) \varphi(x) \end{aligned} \quad (4)$$

(4) is regulated by using the relation (3) and

$$\begin{aligned} & \int x \sqrt{x} \rho(x) \exp i(a+b+c) x dx = -\frac{1}{\sqrt{\pi i}} \int_{-\infty}^{\infty} R''(a+b+c+x^2) dx \\ (4) = & -\frac{4}{\sqrt{\pi i}} \int d\xi d\eta \int_{-\infty}^{\infty} dx \int_1^{\infty} \frac{du}{u} \int_{-1}^1 d\tau \int_{-\infty}^{\infty} d\tau' \sigma(u^2 \tau' v) R''(u^2 \tau' v + x^2) \exp i(k_v \hat{\xi}_v + k'_v \eta_v) \\ & \times \exp \left[i \frac{\tau v}{4} (1-\tau^2) (k_v - k'_v)^2 \right] A_\mu(x+\hat{\xi}) A_\nu(x-\eta) \varphi(x) \\ = & -\frac{4}{\sqrt{\pi i}} \int_1^{\infty} \frac{du}{u^3} \int_{-1}^1 d\tau \int_{-\infty}^{\infty} dy \sigma(y-x^2) R''(y) \exp \left[\frac{i}{4u^2} (y-x^2) (\tau^2-1) \square \right] A_\mu^2(x) \varphi(x) \\ = & \frac{4}{\sqrt{\pi i}} \int_1^{\infty} \frac{du}{u^3} \int_{-1}^1 d\tau \left\{ \int_0^{\infty} R(y) / y^{3/2} dy - \frac{i}{u^2} (\tau^2-1) \square \int_0^{\infty} R(y) / \sqrt{y} dy + \dots \right\} A_\mu^2(x) \varphi(x) \end{aligned} \quad (5)$$

As the first term of the formula (5) is non gauge covariant, it must be dropped off, which requires the following condition for regulator:

$$\int_0^{\infty} R(y) / y^{3/2} dy = 0, \quad \text{or} \quad \int_{-\infty}^{\infty} \sqrt{x} \rho(x) dx = 0, \quad \text{or} \quad \sum c_i m_i = 0. \quad (6)$$

Next, also in the case of the pseudoscalar neutretto with pseudovector coupling there appears the non gauge covariant term in the first of matrix element, which is able to separate into non gauge covariant term not contributing to real transition process and gauge covariant one, but should be dropped off, on which we already discussed in the previous paper⁷⁾. This first convergent but non gauge covariant term disappears by Pauli's regulator $\int \rho(x) dx = 0$ or $R(0) = 0$ or $\sum c_i = 0$. Further, the matrix element due to the pseudovector coupling is transformed into the four dimensional divergent part not contributing to real transition process and the pseudoscalar coupling term multiplied by $2x$ (reciprocal Compton wave length of proton).

$$H_{pv} = \frac{\partial}{\partial x_\lambda} \iint dx' dx'' \frac{f_v}{2x_\pi} \left(\frac{c}{\hbar c} \right)^2 \text{Sp} \{ S^{(1)}(x''-x) \gamma_\lambda \bar{S}(x-x') \gamma_\mu \bar{S}(x'-x'') \gamma_\nu + \dots \}$$

$$\times A_\mu(x') A_\nu(x'') \varphi(x) + \frac{2x f_2}{x_\pi} \frac{1}{2} \left(\frac{e}{\hbar c} \right)^2 \iint dx' dx'' \\ \times \text{Sp} \{ S^{(1)}(x''-x) \gamma_5 \bar{S}(x-x') \gamma_\mu \bar{S}(x'-x'') \gamma_\nu + \dots \} A_\mu(x') A_\nu(x'') \varphi(x), \quad (7)$$

where x_π means reciprocal Compton wave length of π -meson. The first part vanishes by the law of conservation of energy and momentum for real transition process. Therefore, if we put $f_1 = (2x/x_\pi) f_2$, Nelson and Dyson's argument is established, which hereafter we call the equivalence theorem, while the first non gauge covariant term of $H_{\nu\mu}$ is not equal to the first gauge covariant one of H_{ps} and the equivalence theorem exactly holds for remaining terms of $H_{\nu\mu}$ and H_{ps} . Namely the matrix elements for pseudoscalar and pseudovector coupling are the following :

$$H_{\nu\mu} = \frac{f_1}{4\pi^2 x} \left(\frac{e}{\hbar c} \right)^2 \int_{-1}^1 d\epsilon \int_1^\infty \frac{du}{u} \left\{ \frac{1}{u^2} + \frac{1-v^2}{4x^2} \square - \frac{1}{u^2 \left(u^2 - \frac{1-v^2}{4x^2} \square \right)} \right\} \\ \times (F_{12} F_{34} + F_{23} F_{14} + F_{24} F_{31}) \varphi \quad (8)$$

$$H_{\nu\mu} = \frac{f_2}{8\pi^2 x_\pi} \left(\frac{e}{\hbar c} \right)^2 \int_{-1}^1 d\epsilon \int_1^\infty \frac{du}{u} \sum_\mu \left\{ \frac{1}{u^3} \sum_{cyc l} A_\nu F_{\sigma\tau} + \frac{1}{x^2} \frac{1-v^2}{u^2} \right. \\ \left. \times \frac{1}{u^2 - \frac{1-v^2}{4x^2} \square} \sum_\lambda \sum_{cyc l} F_{\lambda\nu} \partial_\lambda F_{\sigma\tau} \right\} \partial_\mu \varphi \quad (9)$$

Therefore, it seems appropriate to us that the first gauge covariant term of the pseudoscalar coupling (8) may be removed out. Really this integral gives unique result if we use Schwinger's expressions of Δ -functions, but gives another different result when another expressions of Δ -functions are used⁷⁾. The removal of such first term requires further the condition $\int_0^\infty R(y)/\sqrt{y} dy = 0$, or $\int_0^\infty \rho(x)/\sqrt{x} dx = 0$, or $\sum c_i m_i = 0$ for regulators, which is obtained by the calculation as before. Further, this condition compels the second term of the scalar coupling of the scalar neutretto to be zero (see (5)).

But the use of this condition unrestrictedly yields the failure of the equivalence theorem in another case of meson decay problem, Namely, in the decay of a pseudoscalar neutral τ -meson into a vector neutral π -meson with tensor coupling (interaction constant g_2) and a photon, the pseudoscalar coupling term (interaction constant G_1) is equivalent to the pseudovector coupling one (interaction constant G_2), except the terms mentioned below. The matrix element for pseudoscalar coupling of τ^0 becomes as follows :

$$H_{\nu\mu} = \frac{1}{4^2} \frac{G_1 g_2 e}{(2\pi)^2 (\hbar c)^2} \text{Sp}(\gamma_5 \gamma_\nu \gamma_\sigma \gamma_\alpha \gamma_\mu) \varphi \chi_{\nu\sigma} F_{\alpha\mu} \left\{ \int_0^\infty \frac{d\tau w}{\tau w} \cos w + \frac{1}{6} \right.$$

$$\begin{aligned}
& + \frac{1}{4} \frac{x_\pi^2}{x^2} \int_1^\infty \frac{d\eta}{\eta^6} \frac{2-\eta}{1 + \frac{1-\eta}{\eta^2} \frac{x_\pi^2}{x^2}} - \frac{1}{8} \int_{-1}^1 d\zeta \int_1^\infty \frac{d\eta}{\eta^6} \\
& \times \frac{(1-\eta)(4+8\eta-6\eta^2-\eta^3+8\eta\tau^2) \frac{x_\pi^2}{x^2} - (2-2\eta+2\eta^2+4\tau^2-\tau^3-2\eta\tau^2) \frac{x_\tau^2}{x^2}}{1 + \frac{(1+\tau)(1-\eta)}{2\eta^2} \frac{x_\pi^2}{x^2} - \frac{1-\tau^2}{4\eta^2} \frac{x_\tau^2}{x^2}}, \quad (10)
\end{aligned}$$

where φ , $\chi_{\rho\sigma}$ and $F_{\alpha\mu}$ denote the wave functions of scalar τ_0 , vector π_0 mesons and electromagnetic field respectively. The first diverging and the second finite terms are removed out by Pauli's regulators $\int \rho(\mathbf{x}) \log |\mathbf{x}| d\mathbf{x} = 0$ and $\int \rho(\mathbf{x}) d\mathbf{x} = 0$. The matrix element for pseudovector coupling becomes

$$\begin{aligned}
H_m = & - \frac{x_\tau}{8(2\pi)^2} \frac{G_2 g_2^{\rho\sigma}}{x_\tau (\hbar c)^2} \text{Sp}(\gamma_5 \tilde{\gamma}_\nu \tilde{\gamma}_\sigma \tilde{\gamma}_\alpha \tilde{\gamma}_\mu) \left\{ 2A_\mu \chi_{\nu\sigma} \partial_\alpha \varphi + \frac{1}{2} \varphi \chi_{\nu\sigma} F_{\alpha\mu} \int_{-1}^1 d\zeta \int_1^\infty \frac{d\eta}{\eta^5} \right. \\
& \times \left[(1-\tau^2+\eta-2\eta^2) \frac{x_\pi^2}{x^2} - (1-\tau^2+\eta) \frac{x_\tau^2}{x^2} \right] - \frac{1}{4} \varphi \chi_{\nu\sigma} F_{\alpha\mu} \int_{-1}^1 d\zeta \int_1^\infty \frac{d\eta}{\eta^7} \\
& \times \left[\frac{(1+\tau)(1-\eta) \frac{x_\pi^2}{x^2} - \frac{1-\tau^2}{2} \frac{x_\tau^2}{x^2}}{1 + \frac{(1+\tau)(1-\eta)}{2\eta^2} \frac{x_\pi^2}{x^2} - \frac{1-\tau^2}{4\eta^2} \frac{x_\tau^2}{x^2}} \right. \\
& \left. \left. \times \left[\frac{(1-\tau^2+\eta-\eta\tau-2\eta^2) \frac{x_\pi^2}{x^2} - (1-\tau^2+\eta-\eta\tau) \frac{x_\tau^2}{x^2}}{1 + \frac{(1+\tau)(1-\eta)}{2\eta^2} \frac{x_\pi^2}{x^2} - \frac{1-\tau^2}{4\eta^2} \frac{x_\tau^2}{x^2}} \right] \right\}, \quad (11)
\end{aligned}$$

the first non gauge covariant term of which is dropped by the condition $\int \sqrt{x} \times \rho(\mathbf{x}) d\mathbf{x} = 0$. And if we put $G_1 = (2\mathbf{x}/x_\tau) G_2$, the equivalence theorem holds exactly between the remaining terms of (10) and (11) except the terms dropped out by regulator. If we, further, use the condition $\int \rho(\mathbf{x})/\sqrt{x} d\mathbf{x} = 0$, the second term of (11) is dropped out, which destroys the equivalence theorem or requires the more restricted condition $\int \rho(\mathbf{x})/x d\mathbf{x} = 0$ or $\sum c_i m_i^2 = 0$ for regulator in order to bring about the equivalence theorem. As the new condition removes the unique integral, we must consider whether the equivalence theorem does not hold or the condition $\int \rho(\mathbf{x})/\sqrt{x} d\mathbf{x} = 0$ is inadequate. This situation tells us that $\int \rho(\mathbf{x})/\sqrt{x} \times d\mathbf{x} = 0$ should not be used unrestrictedly. Now we may consider the reason why this result yielded. As the equivalence theorem is proved by the procedure mentioned above as the formula (7), H_{pm} and H_{ps} should be the same order of magnitude referring to \mathbf{x} , whereas we regulated independently pseudoscalar and pseudovector matrix elements respectively. This is the reason why the equivalence theorem was destroyed. Consequently, we must regulate both terms of pseudoscalar and pseudovector coupling matrix elements as preserved as the same order of magnitude with respect to \mathbf{x} . For example, the equivalence between (8) and (9) is established by regulating after the relation $f_1 = (2\mathbf{x}/x_\pi) f_2$ was sub-

stituted. In this case it is sufficient for establishment of the theorem to use only Pauli's condition $\int \rho(x) dx = 0$. In the case of (10) and (11), if we put $G_1 = (2x/x_2)G_2$ the condition $\int \sqrt{x} \rho(x) dx = 0$ guarantees the theorem. Therefore, when we regulate by putting $H_{ps} = 2x/H_{ps}$, the requirement of the removal of divergency, gauge covariancy and equivalency theorem are accomplished sufficiently by Pauli's regulator and the condition $\int \sqrt{x} \rho(x) dx = 0$.

For example, in the decay problem of τ^\pm into $\pi^\pm + \pi^0$, there appear diverging integrals proportional to even and odd power of x , the former of which vanishes by Pauli's condition $\int \rho(x) \log |x| dx = 0$. Although in the latter diverging integral there appears apparently the term proportional to $\int \rho(x) / \sqrt{x} dx$ or $\int_0^\infty R(y) / \sqrt{y} dy$, this term exactly cancels out, which guarantees the removal of divergency by using only the condition $\int \sqrt{x} \rho(x) dx = 0$ or $\int_0^\infty R(y) / y^{3/2} dy = 0$. For example, this integral is written as follows:

$$\begin{aligned}
 & \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \sigma(u^2 \omega) R'(u^2 \omega + x^2) \exp(iu^2 \omega) A = \\
 & \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{y - x^2} \sigma(y - x^2) R'(y) \exp i(y - x^2) A \\
 & = -2iA \int_{-\infty}^{\infty} dx R(x^2) + \dots + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{(y - x^2)^2} \sigma(y - x^2) R'(y) \exp i(y - x^2) A \\
 & \quad - iA \int_{-\infty}^{\infty} dx \frac{dy}{y - x^2} \sigma(y - x^2) R(y) \exp i(y - x^2) A \\
 & = -2iA \int_{-\infty}^{\infty} R(x^2) dx + \dots \\
 & \quad + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{\sigma(y - x^2)}{(y - x^2)^2} R(y) + iA \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{\sigma(y - x^2)}{y - x^2} R(y) + \dots \\
 & \quad - iA \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{\sigma(y - x^2)}{y - x^2} R(y) + A^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \sigma(y - x^2) R(y) \dots \quad (12)
 \end{aligned}$$

In this formula the third and the fourth term cancels out. The second term is transformed into $\int_0^\infty R(y) / y^{3/2} dy$, the vanishing of which is necessary to obtain the convergency. The first term is transformed to $\int_0^\infty R(y) / \sqrt{y} dy$, the removal of which is unnecessary in this case. Although the condition $\int_0^\infty R(y) / y^{3/2} dy = 0$ was settled to preserve the gauge covariancy, it is interesting that it is also useful to remove the diverging integral.

Now there exists another alternative for regulating method, which regulates the term after the relation $H_{ps} = 1/2x H_{ps}$ was set up. But in this case, the another condition $\int \rho(x) / \sqrt{x} dx = 0$ is necessary besides the above conditions in order to satisfy all requirement mentioned above. However those alternatives yield the same results for removing the undesired terms. For example, in the γ -decay problem of pseudoscalar neutretto if we adopt the latter alternative, the

first terms of (8) and (9) are dropped by the condition $\int \rho(x)/\sqrt{x} dx = 0$. If the former alternative is adopted, the same first terms of (8) and (9) are removed by the condition $\int \rho(x)\sqrt{x} dx = 0$. And in the decay problem of τ_0 meson into π_0 meson and a photon, according to the former alternative, the first diverging and the second finite terms of (10) and the first non gauge covariant one of (11) are dropped out by the condition $\int \sqrt{x} \rho(x) dx = 0$. Regulating according to the latter alternative, the first diverging term of (10) is dropped out by the condition $\int \rho(x) \log|x| dx = 0$ and the second term of (10) and the first one of (11) are removed by $\int \rho(x) dx = 0$. It is evident that those alternatives yield the same result. In the decay of $\tau^\pm \rightarrow \pi^\pm + \pi^0$ (τ scalar, π^\pm, π^0 pseudoscalar) the diverging integral appears, which according to the latter alternative vanishes by the condition $\int \rho(x)/\sqrt{x} dx = 0$.

According to such an idea, as the same situation exists in the scalar and vector coupling of scalar meson, i.e.,

$$H_s = 2xH_s' - 2xH_s,$$

it seems appropriate to regulate the terms of scalar coupling multiplied by x , which removes the first and second terms by the conditions $\int x\rho(x) dx = 0$ and $\int \rho(x) dx = 0$ in the case of the former alternative. Even according to the latter alternative i.e., $H_s = (1/2x)H_s'$, the first and second terms of scalar coupling are dropped by the conditions $\int \sqrt{x} \rho(x) dx = 0$ and $\int \rho(x)/\sqrt{x} dx = 0$. Thus, the both alternatives yield the same result. It may be concluded that all requirements mentioned above are satisfied whenever we use the regulator by considering such a relation.

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On the Magnetic Moment of Nucleon*

Katurō SAWADA

Department of physics, Kyoto University

(Received August 24, 1949)

Recent formulation of relativistic field theory by Schwinger¹⁾ Tomonaga²⁾ and Feynman³⁾ enables one to calculate magnetic moment of nucleon as a finite re-active correction of the matrix element of nucleon transition in external electro-magnetic field. In the course of calculations, however, many ambiguous points resulting from the peculiar singularities of Pauli's D - and D_1 - functions appear, if one applies straightforwardly the method used by Schwinger in the calculation of intrinsic magnetic moment of electron. And, occasionally, one obtains non-vanishing charge renormalization for neutron. These inconsistent results seem to be removed by regularizing D - and D_1 - functions as proposed by Pauli and Villars⁴⁾, but without any caution on the selection of integral variables, one should impose very complicated condition on the regulator used. This difficulty can be removed to large extent by the suitable choice of the integral variable as will be done in this paper, and the charge renormalization for neutron calculated becomes independent of the masses of nucleon or meson, which can simply dropped by imposing the condition on the regulator which is weakest among appeared so far, $R(0)=0$ (in Pauli's notation⁴⁾).

The main results obtained are the same for that of Case's⁵⁾ or Luttinger's⁶⁾. We discuss scalar and pseudo-scalar meson theory in some detail and vector and pseudo-vector meson field theory are only referred to their results.

§ 1. Fundamental Formulas (charged theory)

The magnetic moment for nucleon is lead by the expectation value of the interaction Hamilton density between charge current density of meson and nucleon and external electromagnetic field. For scalar and pseudo-scalar meson field theory, we have for this quantity the following expression;

$$j_\mu \dot{A}_\mu = ie \left(\phi \frac{\partial \phi^*}{\partial x_\mu} - \frac{\partial \phi}{\partial x_\mu} \phi^* \right) \dot{A}_\mu + ie \bar{\Psi}^\dagger \gamma^\mu \tau_\mu \Psi \cdot \dot{A}_\mu \quad (1)$$

where ϕ and ϕ^* are meson variables, \dot{A}_μ is the external electromagnetic four

* Preliminary report of the results obtained appeared in Prog. Theor. Phys. 4 (1949), 383.

potential. The magnetic moment of nucleon is contained in the following expectation value of this quantity in the second order in interaction constant of meson with nucleon:

$$\langle j_\mu \hat{A}_\mu \rangle = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [H(x'') [H(x'), j_\mu(x) \hat{A}(x)]_-]_- d\omega' d\omega'' \quad (2)$$

where $H(x)$ is the interaction Hamilton density between meson and nucleon, and is given by:

Scalar theory: scalar coupling:

$$H(x) = f_1 \bar{\Psi}^\dagger(x) (\phi^*(x) \tau_{NP} + \phi(x) \tau_{PN}) \Psi(x) \quad (3)$$

The vector coupling can be removed by a "gauge" transformation proposed by Dyson⁷⁾ leaving only terms which contributed to the scattering of meson with nucleon;

$$\frac{1}{2} \left(\frac{f_2}{\mu} \right)^2 \bar{\Psi}^\dagger (\tau_N - \tau_P) i^\sigma \Psi \left(\phi \frac{\partial \phi^*}{\partial x_\sigma} - \phi^* \frac{\partial \phi}{\partial x_\sigma} \right) \quad (4)$$

in charged meson theory. The symmetrical theory also give (4), and the interaction Hamilton density is thoroughly equivalent to (3) with neutral interaction term.

Pseudo-scalar theory: pseudo-scalar coupling:

$$H(x) = if_1 \bar{\Psi}^\dagger(x) i^5 (\phi^*(x) \tau_{NP} + \phi(x) \tau_{PN}) \Psi(x) \quad (5)$$

The pseudo-vector coupling can also be removed, transforming into pseudo-scalar type interaction with interaction constant $2 \frac{m}{\mu}$ by the following "gauge" transformation (μ : meson mass, m : nucleon mass)

$$\Phi[\sigma] = U[\sigma] \Phi_0[\sigma];$$

$$U[\sigma] = \exp(-if_2 \int \bar{\Psi}^\dagger(x') i^5 i^\mu (\phi^*(x') \tau_{NP} + \phi(x') \tau_{PN}) \cdot F(x') \cdot dF_\mu) \quad (6)$$

leaving also the same term as (4) in charged theory, and in symmetrical theory.

From these circumstances, it is sufficient to calculate (2) with (3) and (5), since vector and pseudo-vector coupling do not give rise to any change of results, apart from the change in coupling constant.

The terms contained in (2) are separated into two parts, one of which is due to the interaction of meson charge current with external electromagnetic field (first term of (1)), and the other due to the recoil of the virtual meson on nucleon (second of (1)).

§ 2. Scalar theory

The first part of the expectation value (2) which is due to meson charge-current density becomes as follows, following the calculation of Schwinger¹⁾

$$\begin{aligned}
& -\frac{icf^2}{2} A_\mu \int_{-\infty}^{\infty} \Psi''' \tau_P \left[\bar{S}(x''-x') \left(\frac{\partial}{\partial x_\mu} A^{(1)}(x-x'') \cdot \bar{J}(x-x') - A^{(1)}(x-x'') \cdot \frac{\partial}{\partial x_\mu} \bar{J}(x-x') \right) \right. \\
& + \bar{S}(x''-x') \left(\frac{\partial}{\partial x_\mu} \bar{J}(x-x'') \cdot A^{(1)}(x-x') - \bar{J}(x-x'') \cdot \frac{\partial}{\partial x_\mu} A^{(1)}(x-x') \right) \\
& \left. + S^{(1)}(x''-x') \left(\frac{\partial}{\partial x_\mu} \bar{J}(x-x'') \cdot \bar{J}(x-x') - \bar{J}(x-x'') \cdot \frac{\partial}{\partial x_\mu} \bar{J}(x-x') \right) \right] \Psi' \cdot d\omega' d\omega'' \\
& + (\text{sign changed and } \tau_P \text{ replaced by } \tau_N), \tag{7}
\end{aligned}$$

where \bar{S} , $S^{(1)}$ are due to Schwinger with nucleon mass, \bar{A} , $A^{(1)}$ with meson mass.

The second part which is due to meson reaction on nucleon becomes

$$\begin{aligned}
& \frac{icf^2}{2} A_\mu \int_{-\infty}^{\infty} \Psi''' (S^{(1)}(x''-x') \bar{S}(x'-x) \bar{A}(x'-x'')) \\
& + \bar{S}(x''-x') \bar{S}(x'-x) A^{(1)}(x'-x'') \gamma^\mu \tau_P \Psi d\omega' d\omega'' \\
& + \frac{icf^2}{2} A_\mu \int_{-\infty}^{\infty} \Psi' \gamma^\mu (\bar{S}(x''-x') S^{(1)}(x'-x'') \bar{A}(x'-x'')) \\
& + \bar{S}(x-x') \bar{S}(x'-x'') A^{(1)}(x'-x'') \tau_P \Psi'' d\omega' d\omega'' \\
& + \frac{icf^2}{2} A_\mu \int_{-\infty}^{\infty} \Psi' \gamma^\mu (\bar{S}(x''-x) \gamma^\mu S^{(1)}(x-x') \bar{A}(x'-x'')) \\
& + S^{(1)}(x''-x) \gamma_\mu \bar{S}^{(1)}(x-x') \bar{A}(x'-x'') \\
& + \bar{S}(x''-x) \gamma_\mu \bar{S}(x-x') A^{(1)}(x'-x'') \tau_P \Psi' d\omega' d\omega'' \tag{8}
\end{aligned}$$

In the expression (8), the first two terms do vanish, which can be proved as follows; one firstly integrates over X'' , then this becomes of the form:

$$\int e^{-iq_\mu X_\mu'} \bar{S}(x'-x) d\omega' \propto \int e^{-iq_\mu X_\mu'} \frac{1}{(2\pi)^4} P \int \frac{e^{il_\lambda(X_\lambda' - X_\lambda)}}{l_\lambda^2 + m^2} d^{(4)}l \cdot d\omega'$$

which is zero, because by X' integration only $q_\lambda = l_\lambda$ remains and the condition $q_\lambda^2 + m^2 = 0$ implies that only value at $l_\lambda^2 + m^2 = 0$ remains which should vanish since we should take in l integration its principal value. So that the reaction of meson induces only effect on neutron, which is because the fact that the proton goes freely in external field once it emitted the positive meson.

There appears no vacuum polarization type term in the charged theory for this process, since if it appears, it comes from the second term of (1), but which should contain spur of the products of τ_{NP} or τ_{PN} with τ_P and vanishes.

Now, to evaluate (7) and (8), firstly write it in Schwinger's integral representation with attaching vanishing expression with combination of $S^{(1)}$, $A^{(1)}$, $A^{(2)}$ and $S^{(2)}$, $S^{(1)}$, $A^{(1)}$ respectively as done by Schwinger.

For (7): due to meson charge:

$$\begin{aligned}
 & -\frac{icf^2}{2 \cdot 64\pi^2} \hat{A}_\mu \cdot U_1(q) \cdot e^{-ip_\lambda X_\lambda} \cdot \tau_N \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{1}{a+b+c} - 2m^2 i \frac{c(2a+2b+c)}{(a+b+c)^2} \right] r^\mu - \\
 & - (p_\lambda - q_\lambda) \cdot \frac{1}{2} [r^\lambda r^\mu]_- \cdot m \frac{c(2a+2b+c)}{(a+b+c)^2} - \frac{(a-b)(2a+2b+c)}{(a+b+c)^2} m(p_\mu - q_\mu) \Big] \\
 & \cdot \frac{\sigma(a+b+c)}{(a+b+c)^2} \cdot (\sigma(a) + \sigma(b))(\sigma(c) + \sigma(a+b)) \cdot \\
 & \cdot \exp \left(i \frac{c^2}{a+b+c} m^2 + i(a+b) \mu^2 + i \frac{ab}{a+b+c} (p_\lambda - q_\lambda)^2 \right) \cdot da db dc \cdot e^{ip_\lambda X_\lambda} \cdot u(p) \\
 & + (\text{expression with sign changed and } \tau_N \text{ replaced by } \tau_N) \quad (9)
 \end{aligned}$$

where $u(q)$'s are Dirac amplitude of nucleon in state q_λ , $\sigma(a) = \begin{cases} 1, & a > 0 \\ -1 & a < 0. \end{cases}$

For the last term of (8): due to meson reaction:

$$\begin{aligned}
 & \frac{ief^2}{2 \cdot 64\pi^2} \hat{A}_\mu \cdot u^1(q) \cdot e^{ip_\lambda X_\lambda} \cdot \tau_N \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{1}{a+b+c} - m^2 i \frac{(a+b+2c)^2}{(a+b+c)^3} + \right. \\
 & + i \frac{ab}{(a+b+c)^2} (p_\lambda - q_\lambda)^2 \Big] r^\mu + (p_\lambda - q_\lambda) \frac{1}{2} [r^\lambda r^\mu]_- \cdot m \cdot \frac{(a+b)(a+b+2c)}{(a+b+c)^2} - \\
 & - \frac{(a-b)(a+b+2c)}{(a+b+c)^2} m(p_\mu - q_\mu) \Big] \cdot \frac{\sigma(a+b+c)}{(a+b+c)^2} \cdot (\sigma(a) + \sigma(b))(\sigma(c) + \sigma(a+b)) \cdot \\
 & \cdot \exp \left(i \frac{(a+b)^2}{a+b+c} m^2 + ic\mu^2 + i \frac{ab}{a+b+c} (p_\lambda - q_\lambda)^2 \right) \cdot da db dc \cdot e^{ip_\lambda X_\lambda} \cdot u(p) \quad (10)
 \end{aligned}$$

where, in (9) a, b refers to \bar{A} or $A^{(1)}$, c to \bar{S} or $S^{(1)}$; in (10) a, b to \bar{S} or $S^{(1)}$, c to \bar{A} or $A^{(1)}$.

Here, if one makes variable transformation used by Schwinger¹⁾ for both (9) and (10), the charge renormalization term of neutron cannot be removed by a simple regulator though it is finite; so we take instead the following transformation to bring the exponential factor of (9) and (10) coincide for $p_\lambda = q_\lambda$:

$$\text{For (9):} \quad a = \frac{1}{2} u(u-1)w(1-v)$$

$$b = \frac{1}{2} u(u-1)w(1+v)$$

$$c = uvw$$

$$da db dc = \frac{1}{2} (u^4 - u^2) w^2 du dv d\tau w$$

$$w: -\infty \sim \infty; \quad u: 1 \sim \infty; \quad v: -1 \sim 1 \quad (11)$$

For (10):

$$a = \frac{1}{2} u \tau v (1 - v)$$

$$b = \frac{1}{2} u \tau v (1 + v)$$

$$c = u (u - 1) \tau v$$

$$da db dc = \frac{1}{2} u^3 w^2 du dv d\tau w \quad (12)$$

the range of integration is the same as in (11).

Then (9) and (10) become as follows;

$$- \frac{ief}{64\pi^2} \hat{A}_\mu u(q) e^{-i\gamma_\mu \lambda \cdot p} \cdot \tau_N \int_{-\infty}^{\infty} \int_{-1}^1 \left\{ \left(\frac{1}{u^2} - \frac{1}{u^3} \right) \frac{1}{\tau v} - 2m^2 i \left(\frac{2}{u} - \frac{3}{u^2} + \frac{1}{u^3} \right) \right\} \tau^2 -$$

$$- (p_\lambda - q_\lambda) \frac{1}{2} [\gamma^\lambda \gamma^\mu]_- \cdot m \left(\frac{2}{u} - \frac{3}{u^2} + \frac{1}{u^3} \right) \Big] \sigma(\tau v)$$

$$\exp \left\{ i \left(m^2 + u(u-1) \mu^2 + \frac{(u-1)^2 (1-\tau^2)}{4} (p_\lambda - q_\lambda)^2 \right) \tau v \right\} \cdot du dv d\tau v \cdot e^{i\gamma_\lambda X_\lambda} \cdot u(p)$$

$$+ (\text{sign reversed and } \tau_N \text{ replaced by } \tau_P); \quad (13)$$

for meson charge effect, and for meson reaction effect;

$$\frac{ief^2}{64\pi^2} \hat{A}_\mu u(q) e^{-i\gamma_\lambda X_\lambda} \cdot \tau_N \int_{-\infty}^{\infty} \int_{-1}^1 \left\{ \frac{1}{u^3 \tau v} - im^2 \left(\frac{4}{u} - \frac{4}{u^2} + \frac{1}{u^3} \right) + \right.$$

$$\left. + i \frac{1-\tau^2}{4u^3} (p_\sigma - q_\sigma)^2 \right\} \gamma^\mu + (p_\lambda - q_\mu) \frac{1}{2} [\gamma^\lambda \gamma^\mu]_- \cdot m \left(\frac{2}{u^2} - \frac{1}{u^3} \right) \Big] \sigma(\tau v) \cdot$$

$$\exp \left\{ i \left(m^2 + u(u-1) \mu^2 + \frac{1-\tau^2}{4} (p_\lambda - q_\lambda)^2 \right) \tau v \right\} \cdot du dv d\tau v \cdot e^{i\gamma_\lambda X_\lambda} u(p). \quad (14)$$

By the integration by parts these goes to the following:

Due to meson charge;

$$- \frac{ief^2}{16\pi^2} \hat{A}_\mu u(q) e^{-i\gamma_\lambda X_\lambda} \tau_N \left[\int_0^\infty \frac{\cos m^2 \tau v}{2\tau v} d\tau v \cdot \gamma^\mu + \right.$$

$$+ \int_{-1}^1 2m^2 \left(\frac{2}{u} - \frac{3}{u^2} + \frac{1}{u^3} \right) - \left(\frac{1}{u} - \frac{1}{2u^2} \right) (2u-1) \mu^2 - \left(\frac{1}{u} - \frac{1}{2u^2} \right) \frac{(u-1)(1-\tau^2)}{2} (p_\sigma - q_\sigma)^2$$

$$\times \frac{m^2 + u(u-1) \mu^2 + \frac{(u-1)^2 (1-\tau^2)}{4} (p_\lambda - q_\lambda)^2}{4} \times$$

$$\left. \times du dv \cdot \gamma^\mu \right]$$

$$+ (\not{p}_\lambda - \not{q}_\lambda \cdot \sigma_{\lambda\mu}) m \int_0^1 \int_0^1 \frac{\frac{2}{u} - \frac{3}{u^2} + \frac{1}{u^3}}{m^2 + u(u-1)\mu^2 + \frac{(u-1)^2(1-v^2)}{4}(\not{p}_\lambda - \not{q}_\lambda)^2} du dv \Big] e^{i\not{p}_\lambda X_\lambda} \cdot u(p) \\ + (\text{sign reversed and } \tau_N \text{ replaced by } \tau_p); \quad (15)$$

Due to meson reaction ;

$$\frac{ief^2}{16\pi^2} \not{A}_\mu u^\dagger(q) e^{-iq_\lambda X_\lambda} \tau_N \left[\int_0^\infty \int_0^1 \frac{\cos \left\{ \left(m^2 + \frac{1-v^2}{4} (\not{p}_\lambda - \not{q}_\lambda)^2 \right) w \right\}}{2w} dw dv \cdot \gamma^\mu \right. \\ \left. + \int_1^\infty \int_0^1 \frac{m^2 \left(\frac{4}{u} - \frac{4}{u^2} + \frac{1}{u^3} \right) - \left(\frac{1}{u} - \frac{1}{2u^2} \right) \mu^2 - \frac{1-v^2}{4u^2} (\not{p}_\lambda - \not{q}_\lambda)^2}{m^2 + u(u-1)\mu^2 + \frac{1-v^2}{4} (\not{p}_\lambda - \not{q}_\lambda)^2} du dv \cdot \gamma^\mu \right. \\ \left. - (\not{p}_\lambda - \not{q}_\lambda \cdot \sigma_{\lambda\mu}) m \int_1^\infty \int_0^1 \frac{\frac{2}{u^2} - \frac{1}{u^3}}{m^2 + u(u-1)\mu^2 + \frac{1-v^2}{4} (\not{p}_\lambda - \not{q}_\lambda)^2} du dv \right] e^{i\not{p}_\lambda X_\lambda} \cdot u(p) \quad (16)$$

$$\text{where } \sigma_{\lambda\mu} = \frac{1}{2i} [\gamma^\lambda \gamma^\mu]_-.$$

The charge renormalization term can be obtained from the first two terms which is proportional to γ^μ as the first term in the expansion with respect to $(\not{p}_\lambda - \not{q}_\lambda)^2$. Especially, for neutron, this should vanish physically, but by calculating this from (15) and (16) and adding gives ;

$$u^\dagger(q) \cdot \not{A}_\mu \gamma_\mu \tau_N \cdot \frac{ief^2}{16\pi^2} \int_1^\infty \frac{2u-1}{u^3} du \cdot u(q) \quad (17)$$

which is finite but not zero. But a glance to this result shows that this is independent of the mass of meson or nucleon, so that the regularization with an regulator of either meson or nucleon part imposing weakest condition $R(0)=0$ can remove this term, and leads to the consistent result.

The magnetic moment of nucleon is obtained as the first expansion term of the last term of (15) and (16) in $(\not{p}_\lambda - \not{q}_\lambda)^2$; by using the equivalency of the expression :

$$-iu^\dagger(q) (\not{p}_\lambda - \not{q}_\lambda \cdot \sigma_{\lambda\mu}) \not{A}_\mu u(p) \equiv u^\dagger(q) \frac{\partial \not{A}_\mu}{\partial x_\lambda} \sigma_{\lambda\mu} u(p) \quad (18)$$

for any transition of nucleon from state p to q , one obtains ;

$$u^\dagger(q) e^{-iq_\lambda X_\lambda} \left(\frac{f^2/4\pi}{2\pi} (2F_1^{(1)} - 3F_2^{(1)} + F_3^{(1)}) \cdot \frac{e}{2m} \frac{\partial \not{A}_\mu}{\partial x_\lambda} \sigma_{\lambda\mu} \cdot \tau_N \cdot e^{i\not{p}_\lambda X_\lambda} \cdot u(p) \right)$$

$$+ (\text{sign reversed and } \tau_N \text{ replaced by } \tau_P) \quad (19)$$

for the meson charge effect, and for the meson reaction effect;

$$u^\dagger(q) e^{-iq_\lambda X_\lambda} \frac{(f^2/4\pi)}{2\pi} (2F_2^{(1)} - F_3^{(1)}) \cdot \frac{c}{2m} \frac{\partial \dot{A}_\mu}{\partial x_\lambda} \sigma_{\lambda\mu} \cdot \tau_N \cdot e^{iq_\lambda X_\lambda} u(p) \quad (20)$$

where $F_n^{(m)}$ is defined by;

$$F_n^{(m)} = \int_1^\infty \frac{du}{u^n (1+u(u-1)(\mu/m)^2)^m} \quad (21)$$

Integration gives the following by writing μ/m as δ ; Magnetic moment of proton (adding proper moment)

$$1 - \frac{(f^2/4\pi)}{2\pi} \left[-\frac{5}{2} + \delta^2 + (2 - 4\delta^2 + \delta^4) \log \frac{1}{\delta} + \frac{\delta(8 - 6\delta^2 + \delta^3)}{(4 - \delta^2)^{3/2}} \cos^{-1} \frac{\delta}{2} \right] \quad (22)$$

Magnetic moment of neutron;

$$\frac{(f^2/4\pi)}{2\pi} \left[-1 + (2 - \delta^2) \log \frac{1}{\delta} + \frac{\delta(4 - \delta^2)}{(4 - \delta^2)^{3/2}} \cos^{-1} \frac{\delta}{2} \right] \quad (23)$$

in unit of nuclear Bohr-magneton.

The extended charge distribution can be obtained from the higher term in expansion of the first two terms of (15) and (16) with respect to $(p_\lambda - q_\lambda)^2$, which becomes to contain $F^{(2)}$ or higher terms.

§ 3. Pseudo-scalar Theory

We obtain for this case the corresponding expression to (7) and (8);

Due to charge of meson;

$$\begin{aligned} & \frac{ief^2}{2} \dot{A}_\mu \int_{-\infty}^{\infty} \Psi^{\dagger\prime} \tau_P \gamma^5 \left[\bar{S}(x'' - x') \left(\frac{\partial}{\partial x_\mu} A^{(1)}(x - x'') \cdot \bar{A}(x - x') - A^{(1)}(x - x'') \cdot \frac{\partial}{\partial x_\mu} \bar{A}(x - x') \right) \right. \\ & \quad \left. + \bar{S}(x'' - x') \left(\frac{\partial}{\partial x_\mu} \bar{A}(x - x'') \cdot A^{(1)}(x - x') - \bar{A}(x - x'') \cdot \frac{\partial}{\partial x_\mu} A^{(1)}(x - x') \right) \right. \\ & \quad \left. + S^{(1)}(x'' - x') \left(\frac{\partial}{\partial x_\mu} \bar{J}^{(1)}(x - x'') \cdot \bar{J}(x - x') - \bar{J}(x - x'') \cdot \frac{\partial}{\partial x_\mu} \bar{A}(x - x') \right) \right] \gamma^5 \Psi' d\omega' d\omega'' \\ & \quad + (\text{sign reversed and } \tau_P \text{ replaced by } \tau_N) \end{aligned} \quad (24)$$

Due to meson reaction;

$$\begin{aligned} & -\frac{ief^2}{2} \dot{A}_\mu \int_{-\infty}^{\infty} \Psi^{\dagger\prime} (\gamma^5 S^{(1)}(x'' - x') \gamma^5 \bar{S}(x' - x) \gamma^\mu \bar{A}(x' - x'')) \\ & \quad + \gamma^5 \bar{S}(x'' - x') \gamma^5 \bar{S}(x' - x) \gamma^\mu A^{(1)}(x' - x'')) \tau_P \Psi d\omega' d\omega'' \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \frac{x_\pi^2}{x^2} \int_1^\infty \frac{d\eta}{\eta^6} \frac{2-\eta}{1 + \frac{1-\eta}{\eta^2} \frac{x_\pi^2}{x^2}} - \frac{1}{8} \int_{-1}^1 d\tau \int_1^\infty \frac{d\eta}{\eta^6} \\
& \times \frac{(1-\eta)(4+8\eta-6\tau-\tau^2+8\eta\tau) \frac{x_\pi^2}{x^2} - (2-2\tau+2\eta+4\tau^2-\tau^3-2\eta\tau^2) \frac{x_\tau^2}{x^2}}{1 + \frac{(1+\tau)(1-\eta)}{2\eta^2} \frac{x_\pi^2}{x^2} - \frac{1-\tau^2}{4\eta^2} \frac{x_\tau^2}{x^2}}, \quad (10)
\end{aligned}$$

where φ , $\chi_{\rho\sigma}$ and $F_{\alpha\mu}$ denote the wave functions of scalar τ_0 , vector π_0 mesons and electromagnetic field respectively. The first diverging and the second finite terms are removed out by Pauli's regulators $\int \rho(\mathbf{x}) \log |\mathbf{x}| d\mathbf{x} = 0$ and $\int \rho(\mathbf{x}) d\mathbf{x} = 0$. The matrix element for pseudovector coupling becomes

$$\begin{aligned}
H_{\rho\nu} = & - \frac{\mathbf{x}}{8(2\pi)^2} \frac{G_2 g_2'}{x_\tau (\hbar c)^2} \text{Sp} (\tilde{\psi} \tilde{\gamma}_\nu \tilde{\gamma}_\sigma \tilde{\gamma}_\alpha \tilde{\gamma}_\mu) \left\{ 2A_\mu \chi_{\rho\sigma} \partial_\alpha \varphi + \frac{1}{2} \varphi \chi_{\rho\sigma} F_{\alpha\mu} \int_{-1}^1 d\tau \int_1^\infty \frac{d\eta}{\eta^5} \right. \\
& \times \left[(1-\tau^2 + \eta - 2\eta^2) \frac{x_\pi^2}{x^2} - (1-\tau^2 + \eta) \frac{x_\tau^2}{x^2} \right] - \frac{1}{4} \varphi \chi_{\rho\sigma} F_{\alpha\mu} \int_{-1}^1 d\tau \int_1^\infty \frac{d\eta}{\eta^5} \\
& \times \frac{\left[(1+\tau)(1-\eta) \frac{x_\pi^2}{x^2} - \frac{1-\tau^2}{2} \frac{x_\tau^2}{x^2} \right]}{1 + \frac{(1+\tau)(1-\eta)}{2\eta^2} \frac{x_\pi^2}{x^2} - \frac{1-\tau^2}{4\eta^2} \frac{x_\tau^2}{x^2}} \left. \right\}, \quad (11)
\end{aligned}$$

the first non gauge covariant term of which is dropped by the condition $\int \sqrt{\mathbf{x}} \times \rho(\mathbf{x}) d\mathbf{x} = 0$. And if we put $G_1 = (2\mathbf{x}/x_\tau) G_2$, the equivalence theorem holds exactly between the remaining terms of (10) and (11) except the terms dropped out by regulator. If we, further, use the condition $\int \rho(\mathbf{x})/\sqrt{\mathbf{x}} d\mathbf{x} = 0$, the second term of (11) is dropped out, which destroys the equivalence theorem or requires the more restricted condition $\int \rho(\mathbf{x})/\mathbf{x} d\mathbf{x} = 0$ or $\sum c_i m_i^2 = 0$ for regulator in order to bring about the equivalence theorem. As the new condition removes the unique integral, we must consider whether the equivalence theorem does not hold or the condition $\int \rho(\mathbf{x})/\sqrt{\mathbf{x}} d\mathbf{x} = 0$ is inadequate. This situation tells us that $\int \rho(\mathbf{x})/\sqrt{\mathbf{x}} \times d\mathbf{x} = 0$ should not be used unrestrictedly. Now we may consider the reason why this result yielded. As the equivalence theorem is proved by the procedure mentioned above as the formula (7), $H_{\rho\nu}$ and $H_{\rho\sigma}$ should be the same order of magnitude referring to \mathbf{x} , whereas we regulated independently pseudoscalar and pseudovector matrix elements respectively. This is the reason why the equivalence theorem was destroyed. Consequently, we must regulate both terms of pseudoscalar and pseudovector coupling matrix elements as preserved as the same order of magnitude with respect to \mathbf{x} . For example, the equivalence between (8) and (9) is established by regulating after the relation $f_1 = (2\mathbf{x}/x_\pi) f_2$ was sub-

stituted. In this case it is sufficient for establishment of the theorem to use only Pauli's condition $\int \rho(x) dx = 0$. In the case of (10) and (11), if we put $G_1 = (2x/x_2)G_2$ the condition $\int \sqrt{x} \rho(x) dx = 0$ guarantees the theorem. Therefore, when we regulate by putting $H_{ps} = 2x/I_{ps}$, the requirement of the removal of divergency, gauge covariancy and equivalency theorem are accomplished sufficiently by Pauli's regulator and the condition $\int \sqrt{x} \rho(x) dx = 0$.

For example, in the decay problem of τ^\pm into $\pi^\pm + \pi^0$, there appear diverging integrals proportional to even and odd power of x , the former of which vanishes by Pauli's condition $\int \rho(x) \log |x| dx = 0$. Although in the latter diverging integral there appears apparently the term proportional to $\int \rho(x)/\sqrt{x} dx$ or $\int_0^\infty R(y)/\sqrt{y} dy$, this term exactly cancels out, which guarantees the removal of divergency by using only the condition $\int \sqrt{x} \rho(x) dx = 0$ or $\int_0^\infty R(y)/y^{3/2} dy = 0$. For example, this integral is written as follows:

$$\begin{aligned}
 & \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{d\tau w}{\tau w} \sigma(u^2 \tau w) R'(u^2 \tau w + x^2) \exp(iu^2 \tau w) A = \\
 & \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{y - x^2} \sigma(y - x^2) R'(y) \exp i(y - x^2) A \\
 & = -2iA \int_{-\infty}^{\infty} dx R(x^2) + \dots + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{(y - x^2)^2} \sigma(y - x^2) R'(y) \exp i(y - x^2) A \\
 & \quad - iA \int_{-\infty}^{\infty} dx \frac{dy}{y - x^2} \sigma(y - x^2) R(y) \exp i(y - x^2) A \\
 & = -2iA \int_{-\infty}^{\infty} R(x^2) dx + \dots \\
 & \quad + \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{(y - x^2)^2} \frac{\sigma(y - x^2)}{(y - x^2)^2} R(y) + iA \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{y - x^2} \frac{\sigma(y - x^2)}{y - x^2} R(y) + \dots \\
 & \quad - iA \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dy}{y - x^2} \frac{\sigma(y - x^2)}{y - x^2} R(y) + A^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \sigma(y - x^2) R(y) \dots \quad (12)
 \end{aligned}$$

In this formula the third and the fourth term cancels out. The second term is transformed into $\int_0^\infty R(y)/y^{3/2} dy$, the vanishing of which is necessary to obtain the convergency. The first term is transformed to $\int_0^\infty R(y)/\sqrt{y} dy$, the removal of which is unnecessary in this case. Although the condition $\int_0^\infty R(y)/y^{3/2} dy = 0$ was settled to preserve the gauge covariancy, it is interesting that it is also useful to remove the diverging integral.

Now there exists another alternative for regulating method, which regulates the term after the relation $H_{ps} = 1/2x H_{pm}$ was set up. But in this case, the another condition $\int \rho(x)/\sqrt{x} dx = 0$ is necessary besides the above conditions in order to satisfy all requirement mentioned above. However those alternatives yield the same results for removing the undesired terms. For example, in the γ -decay problem of pseudoscalar neutretto if we adopt the latter alternative, the

§ 4. Discussions.

By adding the above results for charged and neutral theory, we obtain the results in symmetrical theory;

Scalar theory;

For proton;

$$1 + \frac{(f^2/4\pi)}{2\pi} \left(4 - 2\delta^2 - (2 - 7\delta^2 + 2\delta^4) \log \frac{1}{\delta} - \frac{\delta(12 - 11\delta^2 + 2\delta^4)}{(4 - \delta^2)^{1/2}} \cos^{-1} \frac{\delta}{2} \right) \quad (35)$$

For neutron;

$$\frac{(f^2/4\pi)}{2\pi} \left(-1 + (2 - \delta^2) \log \frac{1}{\delta} + \frac{\delta(4 - \delta^2)}{(4 - \delta^2)^{1/2}} \cos^{-1} \frac{\delta}{2} \right) \quad (36)$$

Pseudo-scalar theory;

For proton;

$$1 - \frac{(f^2/4\pi)}{2\pi} \left(2\delta^2 - \delta^2(3 - 2\delta^2) \log \frac{1}{\delta} + \frac{\delta(2 - 7\delta^2 + 2\delta^4)}{(4 - \delta^2)^{1/2}} \cos^{-1} \frac{\delta}{2} \right) \quad (37)$$

For neutron;

$$- \frac{(f^2/4\pi)}{2\pi} \left(1 + \delta^2 \log \frac{1}{\delta} - \frac{\delta(2 - \delta^2)}{(4 - \delta^2)^{1/2}} \cos^{-1} \frac{\delta}{2} \right) \quad (38)$$

Comparing these with Luttinger's results⁶⁾, by changing the interaction constants as mentioned in the previous letter, our results coincide with his results apart from a slight modification.** But the constant use of regulator is necessary to bring the theory consistent in the Schwinger method as is revealed in the charge renormalization of neutron. The condition imposed on the regulator used is dependent on the choice of the integral variable and the above choice of integral variable leads to the simplest condition on the regulator.

Finally, we refer our results listed in the previous letter for vector and pseudo-vector theory. This result was obtained as the expectation value of the following interaction Hamilton density of meson and nucleon charge-current density with external electromagnetic field;

$$ie \left(A_\lambda(x) \frac{\partial A_\lambda^*(x)}{\partial x_\mu} - \frac{\partial A_\lambda(x)}{\partial x_\mu} A_\lambda^*(x) + B(x) \frac{\partial B^*(x)}{\partial x_\mu} - \frac{\partial B(x)}{\partial x_\mu} B^*(x) \right) \dot{A}_\mu(x) + ie \Psi^\dagger(x) \gamma^\mu \tau_P \Psi(x) \cdot \dot{A}_\mu(x) \quad (39)$$

where A and B are the meson variable in Stückelberg formulation of the meson field theory, and the notations are due to Miyamoto⁹⁾. The interaction Hamilton density of meson with nucleon is given by;

** See last note.

For vector meson theory ;

$$ig_1 \Psi^\dagger \gamma^\mu \left[\left(A_\mu^* + \frac{\partial B^*}{\partial x_\mu} \right) \tau_{NP} + \left(A_\mu + \frac{\partial B}{\partial x_\mu} \right) \tau_{PN} \right] \Psi \quad (40)$$

For pseudo vector meson theory ;

$$ig_1 \Psi^\dagger \gamma^\mu \gamma^5 \left[\left(A_\mu^* + \frac{\partial B^*}{\partial x_\mu} \right) \tau_{NP} + \left(A_\mu + \frac{\partial B}{\partial x_\mu} \right) \tau_{PN} \right] \Psi \quad (41)$$

To simplify the calculation, we take the "gauge" transformation, which is suggested from the analogous form of the B -field interaction with vector and pseudo-vector coupling in scalar and pseudo-scalar meson theory ; this removes B -field from (40) and transformed B -interaction to the pseudo-scalar type in (41). And calculate the magnetic moment without referring to the supplementary condition imposed on the A and B -field in this Stückelberg theory.

This procedure is not legitimated strictly, but for vector meson case, this gave the correct result coinciding with Luttinger's⁶⁾ result for neutral theory, so we gave our results in the previous letter.

In conclusion, I say much thanks to Prof. M. Kobayasi for his kind interest throughout this calculations.

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Note added in proof

After furnishing this calculation, we received Phys. Rev. **76** No. 1, on where Case's calculation in Feynman theory appeared. Since he states there that his results coincide with Luttinger's, and ours coincide with Case's, the letter of Luttinger appeared in Phys. Rev. (6) with which we compared our results in the previous letter seems to contain some misprint.

And we can now conclude that there can be found exact coincidence between these two independent methods.

On the Photodisintegration of the Deuteron.

(Pseudoscalar Meson Theory.)

HIROSHI ENATSU and YOSHIRO TAKANO

Institute of Physics, Kyoto University.

(Received August 25, 1949)

Introduction and Summary.

There are many difficulties in the meson theory at the present stage of its development. The singularities of nuclear potentials are the most serious ones in the deuteron problem. In order to overcome these points many attempts have been made repeatedly.

Recently van Hove¹⁾ and Araki²⁾ have shown that if the relativistic effects are taken into account the singularities are reduced to $1/r$ for the pseudoscalar meson theory. Thus we can find the solution of the deuteron problem, and, for example, the scattering cross-section of the neutron by the proton can be evaluated unambiguously. The photodisintegration of the deuteron is the same problem as the scattering. Its cross-section depends on the nuclear potential as much as the scattering cross-section does.

Although many authors³⁾ treated this problem to test the meson theory, it is necessary to consider it from the new theoretical standpoint on account of the recent discovery of π meson and the precise experimental results.

The purpose of this paper is to work out the total cross-section of the photodisintegration of the deuteron for the pseudoscalar meson theory by making use of the above mentioned new nuclear potentials.

In §1 we discuss the general theory of the cross-section of the photodisintegration of the deuteron. In §2 we consider the photoelectric effect which corresponds to the transition from the initial 3S state to the final 3P state. As stated above, we should decide exactly the solution of the Schrödinger equation of the deuteron for the new potential, but for simplicity we neglect the tensor forces and obtain the initial state function by the variational method and the final one by replacing the new potential with the square well potential. In §3 the photomagnetic effect which comes from $^3S \rightarrow ^1S$ transition is computed. In this case the initial and final states are also derived by the same method as before. In §4 numerical results are shown for various values of 3P state potential and compared with the recent experimental results.

§ 1. General theory

The outline of this calculation is more or less identical with that which have been discussed by Fröhlich, Heitler and Kahn¹⁾, and Ma²⁾ in the vector meson theory. We shall here quote briefly the derivation of matrix elements.

The processes in which a deuteron disintegrates into a neutron and a proton by absorbing a photon are as follows,

$$\left. \begin{aligned} N+P &\rightarrow N+N+Y_k^+ \xrightarrow{\gamma} N+N+Y_{k'}^+ \longrightarrow P+N, &<Y^-> \\ N+P &\xrightarrow{\gamma} N+P+Y^++Y^- \longrightarrow P+P+Y^- \longrightarrow P+N, &<Y^+, Y^-> \\ N+P &\longrightarrow N+N+Y^+ \longrightarrow P+N+Y^++Y^- \xrightarrow{\gamma} P+N. &<Y^+, Y^-> \end{aligned} \right\} \quad (1)$$

In addition to (1) there is another direct process with which the meson does not concern. The interaction energy is

$$H = -\frac{1}{c} \int \mathbf{A} \cdot \mathbf{S} dv, \quad (\text{div } \mathbf{S} + \dot{\rho} = 0), \quad (2)$$

where ρ , \mathbf{S} are the charge and current densities of the nucleon system including the exchange current, and \mathbf{A} is the vector potential of electromagnetic field

$$\mathbf{A} = \sqrt{4\pi c^2} \sqrt{\frac{\hbar}{2\nu}} e^{i\mathbf{k}_0 \cdot \mathbf{R}} = c\hbar \sqrt{\frac{2\pi}{\hbar\nu}} e(1 + i\mathbf{k}_0 \cdot \mathbf{R} + \dots). \quad (3)$$

From (2) we obtain

$$H = H_0 + H_1 + \dots \quad (4)$$

$$H_0 = -\sqrt{\frac{2\pi}{\hbar\nu}} \hbar \int \mathbf{e} \cdot \mathbf{R} \dot{\rho} dv, \quad (5)$$

$$H_1 = \frac{i\hbar}{2} \sqrt{\frac{2\pi}{\hbar\nu}} \int [\mathbf{e} \times \mathbf{k}_0] [\mathbf{R} \times \mathbf{S}] dv, \quad (6)$$

where H_0 and H_1 are the electric dipole and the magnetic dipole parts respectively. The electric quadrupole part and others are neglected. ρ and its time derivative are defined by

$$\rho = e [\Pi_1 \Pi_1^* \delta(\mathbf{R}_1) + \Pi_2 \Pi_2^* \delta(\mathbf{R}_2)] \quad (7)$$

$$\dot{\rho} = \frac{i}{\hbar} [H_0, \rho], \quad (8)$$

where
$$H_0 = -\frac{\hbar^2}{M} \nabla^2 + (\Pi_1^* \Pi_2 + \Pi_2^* \Pi_1) J(\sigma, \mathbf{r}) \quad (9)$$

Π^* : Proton \rightarrow Neutron, Π : Neutron \rightarrow Proton.

Introducing the relative coordinate $\mathbf{r} = \mathbf{R}_1 - \mathbf{R}_2$ and relative momentum

$\dot{\mathbf{r}} = \frac{2k\hbar}{M}$, we get the matrix elements^{(7) (8)}

$$\begin{aligned} H_0 &= -ic\sqrt{\frac{2\pi}{\hbar\nu}} \int \Psi_f^* (\mathbf{e} \cdot \mathbf{r}) (\Pi_1^* \Pi_2 - \Pi_2^* \Pi_1) J(\sigma, \mathbf{r}) \Psi_i d\mathbf{r} \\ &\quad + \hbar^2 \frac{c}{M} \sqrt{\frac{2\pi}{\hbar\nu}} \int \Psi_f^* (\Pi_1 \Pi_1^* - \Pi_2 \Pi_2^*) (\mathbf{e} \cdot \mathbf{k}) \Psi_i d\mathbf{r} \\ &= H_e^{(2)} + H_e^{(1)} \end{aligned} \quad (10)$$

$$H_1 = - \int \Psi_f^* \mathbf{H}_0 \cdot \boldsymbol{\mu} \Psi_i d\mathbf{r} = H_m$$

$$\boldsymbol{\mu} = \mu_P (\Pi_1 \Pi_1^* \boldsymbol{\sigma}_1 + \Pi_2 \Pi_2^* \boldsymbol{\sigma}_2) + \mu_N (\Pi_1^* \Pi_1 \boldsymbol{\sigma}_1 + \Pi_2^* \Pi_2 \boldsymbol{\sigma}_2) \quad (11)$$

$$+ \frac{ic}{2\hbar c} \frac{f_2^2}{\mathbf{x}^2} (\Pi_1^* \Pi_2 - \Pi_2^* \Pi_1) \left[\frac{(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \mathbf{r}}{r} \frac{\mathbf{r}}{r} (x+1) - x\mathbf{r} (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \right] \frac{e^{-x\mathbf{r}}}{r} \quad (12)$$

$$\mathbf{H}_0 = i [\mathbf{k}_0 \times \mathbf{e}] \sqrt{\frac{2\pi}{\hbar\nu}} c\hbar = ik_0 c\hbar \sqrt{\frac{2\pi}{\hbar\nu}} (-e_y, e_x, 0), \quad |\mathbf{k}_0| = k_0.$$

§ 2. Photoelectric effect.

The main contribution arises from the transition $^3S \rightarrow ^3P$ in the photoelectric effect. For simplicity, we neglect the 3D state of the deuteron and so the tensor force is omitted. According to Rarita and Schwinger, and Rose and Goertzel,⁽⁹⁾ the effect of tensor forces on the total cross-section is 4% or less. In this approximation the Hamiltonian of the deuteron is

$$H_0 = -\frac{\hbar^2}{M} \nabla^2 - B \frac{e^{-x\mathbf{r}}}{r} \quad \left(x = \frac{M_m c}{\hbar} r \right), \quad (13)$$

where B is a constant, and M_m is the meson mass. To solve the equation $H\psi = E\psi$ we insert in it a trial function $\psi \cong e^{-\frac{1}{2}ax\mathbf{r}}$ for the variational method. Using the binding energy of the deuteron, we get the normalized 3S state wave function

$$\Psi_i = \frac{1}{\sqrt{4\pi}} \sqrt{\frac{a^3 x^3}{2}} e^{-\frac{1}{2}ax\mathbf{r}} \quad (a = 1.618). \quad (14)$$

For the final 3P state it is reasonable for us to solve the Schrödinger equation exactly, but we shall use the same simplification as done in the initial 3S state equation.

It has shown by the recent experiments of the neutron-proton scattering that there is a weak repulsive force between the neutron and the proton in the 3P state. Therefore, we replace the potential by the square well one (height V_0 , radius R). The final state is given by

$$\Psi_f \sim 3ie^{-i\theta} F(kr) \cos \theta \quad (\text{cf. Fig. 1})$$

$$F(kr) = (\cos \delta f_1(kR) + \sin \delta g_1(kR)) \frac{f_1(k, r)}{f_1(k, R)} = f(\delta) f_1(k, r) \quad (r < R) \quad (15)$$

$$= \cos \delta f_1(kr) + \sin \delta g_1(kr), \quad (r > R) \quad (16)$$

$$f_1(kr) = \left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} J_{3/2}(kr) = \frac{1}{kr} \left(\frac{\sin kr}{kr} - \cos kr\right), \quad (17)$$

$$g_1(kr) = -\left(\frac{\pi}{2kr}\right)^{\frac{1}{2}} J_{-3/2}(kr) = \frac{1}{kr} \left(\frac{\cos kr}{kr} + \sin kr\right). \quad (18)$$

Normalizing this function for the coordinates and energy, it is found to be

$$\Psi_f = \left[\left(\frac{3}{4\pi}\right) \left(\frac{Mk}{\pi \hbar^2}\right) \right]^{\frac{1}{2}} F(kr) \cos \theta. \quad (19)$$

Now from (10), (14), and (19) the matrix element for the transition in which the spin state does not change is

$$H_c^{(1)} = \left(\frac{e \hbar^2}{M}\right) \left[\left(\frac{2\pi}{\hbar^2}\right) \left(\frac{Mk}{\pi \hbar^2}\right) \left(\frac{3}{4\pi}\right) \left(\frac{1}{4\pi}\right) \left(\frac{a^3 x^3}{2}\right) \right]^{\frac{1}{2}} \int F(kr) (\mathbf{e} \cdot \mathbf{k}) e^{-\frac{1}{2} \alpha x r} \cos \theta \, d\mathbf{r}. \quad (20)$$

By the help of the relation

$$\int \Psi_f^* (\mathbf{e} \cdot \mathbf{k}) \Psi_i \, d\mathbf{r} = -i \int \Psi_f^* (\mathbf{e} \cdot \text{grad}_r) \Psi_i \, d\mathbf{r}, \quad (21)$$

and introducing the polar coordinates $\mathbf{r} : (r, \theta, \Phi)$, $\mathbf{e} : (1, \varphi, \phi)$, $\mathbf{k}_0 : (k_0, \theta, 0)$ the equation (20) is written after the integrations of θ and Φ ,

$$H_c^{(1)} = \left(\frac{e \hbar^2}{3M}\right) \left[\left(\frac{2\pi}{\hbar^2}\right) \left(\frac{Mk}{\pi \hbar^2}\right) \left(\frac{3}{4\pi}\right) \left(\frac{1}{4\pi}\right) \left(\frac{a^3 x^3}{2}\right) \right]^{\frac{1}{2}} (i M_m c a) \sin \theta \sin \varphi_0 I(r) \quad (22)$$

$$I(r) = f(\delta) I_1(r) + \cos \delta I_2(r) + \sin \delta I_3(r) \quad (23)$$

$$I_1(r) = \frac{4\pi}{3} \int_0^R f_1(kr) r^2 e^{-\frac{1}{2} \alpha x r} \, dr \quad (r < R) \quad (24)$$

$$I_2(r) = \frac{4\pi}{3} \int_R^\infty f_1(kr) r^2 e^{-\frac{1}{2} \alpha x r} \, dr \quad (r > R) \quad (25)$$

$$I_3(r) = \frac{4\pi}{3} \int_R^\infty g_1(kr) r^2 e^{-\frac{1}{2} \alpha x r} \, dr \quad (r > R) \quad (26)$$

where

$$k = \sqrt{M(E + V_0)} / \hbar, \quad \varphi_0 : \text{cf. Fig. 2.}$$

In order to carry through the second matrix element $H_c^{(2)}$ the following relation must be regarded. If we consider the operator

$$S = \left\{ \frac{1}{3} (\sigma_1 \cdot \sigma_2) + \left(\frac{1}{3} + \frac{1}{x} + \frac{1}{x^2 r^2} \right) \left(\frac{3(\sigma_1 \cdot \mathbf{r})(\sigma_2 \cdot \mathbf{r})}{r^2} - (\sigma_1 \cdot \sigma_2) \right) \right\}, \quad (27)$$

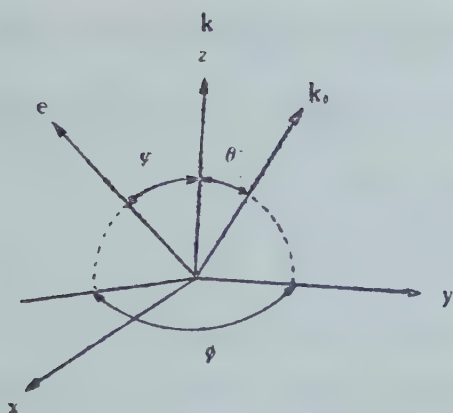


Fig. 1. Relative coordinates system.
 θ : Angle between r and k .

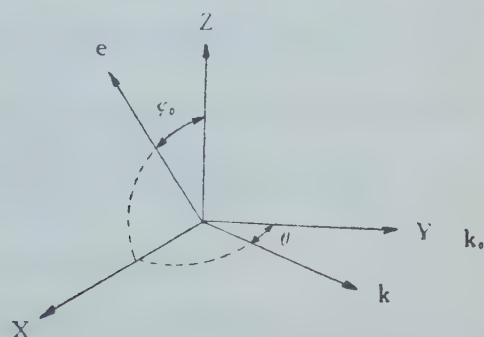


Fig. 2. Laboratory system:

and the spin functions

$$L = a(1)a(2), \quad M = \frac{1}{\sqrt{2}}(a(1)\beta(2) + a(2)\beta(1)), \quad N = \beta(1)\beta(2). \quad (28)$$

Among the matrix elements of the operator (27) corresponding to the transition ${}^3S \rightarrow {}^3P$ (we finished the angular integrations) there remain the following three elements by neglecting the second term of (27) as mentioned at the beginning of this section.

$$\left. \begin{aligned} S_{L \rightarrow L} &= \left[\frac{1}{3} \cdot \frac{4\pi}{3} \right] \\ S_{M \rightarrow M} &= \left[\frac{1}{3} \cdot \frac{4\pi}{3} \right] \\ S_{N \rightarrow N} &= \left[\frac{1}{3} \cdot \frac{4\pi}{3} \right] \end{aligned} \right\} \quad (29)$$

Now the nuclear potential by the pseudoscalar meson theory given by Araki is as follows:

$$J(\sigma, r) = \frac{1}{(1 + |\phi|^2)(1 + 2|\phi|^2)} f_2^2 \frac{e^{-\kappa r}}{r} \left\{ \frac{1}{3} (\sigma_1 \cdot \sigma_2) + \left(\frac{1}{3} + \frac{1}{\kappa r} + \frac{1}{\kappa^2 r^2} \right) A \right\}, \quad (30)$$

where

$$\phi = a \left(\frac{M}{M_m} \right) \left(\frac{f_2^2}{\hbar c} \right) \frac{e^{-\kappa r}}{\kappa r} = P \frac{e^{-\kappa r}}{r}$$

$$A = \left[\frac{3(\sigma_1 \cdot r)(\sigma_2 \cdot r)}{r^2} - (\sigma_1 \cdot \sigma_2) \right].$$

As can be seen in the later section, the potential $J(\sigma, r)$ does not contribute so much, then we are able to make use of the expression

$$J(\sigma, r) = \frac{(xr)^2}{2P^2} f_2^2 \frac{e^{-\alpha r}}{r} \left\{ \frac{1}{3} (\sigma_1 \cdot \sigma_2) + \left(\frac{1}{3} + \frac{1}{\alpha r} + \frac{1}{\alpha^2 r^2} \right) A \right\} \quad (r < R) \quad (31)$$

$$= f_2^2 \frac{e^{-\alpha r}}{r} \left\{ \frac{1}{3} (\sigma_1 \cdot \sigma_2) + \left(\frac{1}{3} + \frac{1}{\alpha r} + \frac{1}{\alpha^2 r^2} \right) A \right\} \quad (r > R) \quad (32)$$

Accordingly, from (10) the matrix element is given by

$$H_e^{(2)} = (ie) f_2^2 \left[\left(\frac{2\pi}{\hbar\nu} \right) \left(\frac{3}{4\pi} \right) \left(\frac{Mk}{\pi\hbar^2} \right) \left(\frac{1}{4\pi} \right) \left(\frac{\alpha^3 x^3}{2} \right) \right]^{\frac{1}{2}} \\ \times \int S_f F(kr) (\mathbf{e} \cdot \mathbf{r}) J(\sigma, r) e^{-\frac{1}{2}\alpha r} S_i \cos \theta \, d\mathbf{r},$$

where S_f and S_i are the spin functions given in (28) and the second terms in $J(\sigma, r)$ are neglected. As we are interested in the low energy photon, $f_1(kr)$ is able to be expanded into the power series,

$$f_1(kr) \cong \left(\frac{kr}{3} - \frac{k^3 r^3}{30} \right) \quad (kr \ll 2). \quad (34)$$

Then (33) is found to be

$$H_e^{(2)} = (ie f_2^2) \left[\left(\frac{2\pi}{\hbar\nu} \right) \left(\frac{3}{4\pi} \right) \left(\frac{Mk}{\pi\hbar^2} \right) \left(\frac{1}{4\pi} \right) \left(\frac{\alpha^3 x^3}{2} \right) \right]^{\frac{1}{2}} \sin \theta \sin \varphi_0 L(r), \quad (35)$$

where $L(r) = L_{1,1}(r < R) + L_{1,2}(r < R) + L_{2,1}(r > R) + L_{2,2}(r > R)$

$$L_{1,1}(r < R) = \frac{2\pi}{27} \frac{1}{P^2} f(\delta) k_p L(5)$$

$$L_{1,2}(r < R) = -\frac{\pi}{135} \frac{1}{P^2} f(\delta) k_p^3 L(7)$$

$$L(m) = x^2 \int_0^R r^m e^{-\bar{\alpha} r} \, dr, \quad \bar{\alpha} = \left(1 + \frac{a}{2} \right) x$$

$$L_{2,1}(r > R) = \frac{4\pi}{9} \cos \delta \int_R^\infty \frac{1}{kr} \left(\frac{\sin kr}{kr} - \cos kr \right) e^{-\bar{\alpha} r} r^2 \, dr$$

$$L_{2,2}(r > R) = \frac{4\pi}{9} \sin \delta \int_R^\infty \frac{1}{kr} \left(\frac{\cos kr}{kr} + \sin kr \right) e^{-\bar{\alpha} r} r^2 \, dr$$

Finally, H_e is given as follows (for $L \rightarrow L$, $M \rightarrow M$, and $N \rightarrow N$.)

$$H_e = H_e^{(1)} + H_e^{(2)} = G \sin \theta \sin \varphi_0 \left[\left(\frac{M_m a}{2M} \right) I(r) + \left(\frac{f_2^2}{\hbar c} \right) L(r) \right], \quad (36)$$

where G is given by

$$G = i \left(\frac{c}{\sqrt{\hbar c}} \right) \left[\left(\frac{3\alpha^3}{4^2 \pi^2} \right) \left(\frac{Mc^2}{\hbar\nu} \right) \left(\frac{k M_m^3 c^4}{\hbar^2} \right) \right]^{\frac{1}{2}}. \quad (37)$$

The total cross-section for the photoelectric disintegration is

$$\Phi_e = \frac{2\pi}{\hbar c} |H_e^{(1)} + H_e^{(2)}|^2_{av} \quad (38)$$

where *av.* means the average taken over the initial spin states and the polarization directions of the photons.

§ 3. Photomagnetic effect.

The interaction H_1 leads to the photomagnetic disintegration which corresponds to the transition from 3S state to 1S state.

The final state functions are, however, given by

$$\psi_f = \frac{1}{\sqrt{4\pi}} \frac{1}{kr} \sqrt{\frac{M\hbar}{\pi\hbar^2}} \sin kr \quad (39)$$

$$\text{and} \quad K = \frac{1}{\sqrt{2}} (a(1)\beta(2) - a(2)\beta(1)), \quad (40)$$

for the radial and spin parts respectively. It is convenient to separate the magnetic moment μ into three parts,

$$\mu = \mu_1 + \mu_2 + \mu_3, \quad (41)$$

$$\mu_1 = \frac{1}{2} (\mu_p - \mu_n) (\sigma_1 - \sigma_2)$$

$$\text{where} \quad \mu_2 = -\frac{ie}{2\hbar c} \frac{f_2^2}{x^2} \mathbf{x}r (\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \frac{e^{-\mathbf{x}r}}{r}$$

$$\mu_3 = \frac{ie}{2\hbar c} \frac{f_2^2}{x^2} \left[\frac{(\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2) \cdot \mathbf{r}}{r} \cdot \frac{\mathbf{r}}{r} (xr + 1) \right] \frac{e^{-\mathbf{x}r}}{r}. \quad (42)$$

These parts lead to the expressions for the spin transitions from L , M , and N to K ,

$$\left(K \begin{smallmatrix} \mu_1 + \mu_2 \\ N \\ M \end{smallmatrix} \right) = \left[\frac{1}{\sqrt{2}} (\mu_p - \mu_n) + \frac{e}{\sqrt{2}\hbar c} \frac{f_2^2}{x} e^{-\mathbf{x}r} \right] \cdot \begin{bmatrix} (-1, -i, 0) \\ (1, -i, 0) \\ \sqrt{2}(0, 0, 1) \end{bmatrix} \quad (43)$$

and

$$\left(K \begin{smallmatrix} \mu_3 \\ N \\ M \end{smallmatrix} \right) = \frac{ie}{2\hbar c} \frac{f_2^2}{x^2} (xr + 1) \frac{e^{-\mathbf{x}r}}{r} (\sin \theta_1 \cos \varphi_1, \sin \theta_1 \sin \varphi_1, \cos \theta_1) \\ \times \begin{bmatrix} \sqrt{2}(-i \sin \theta_1 \cos \varphi_1 + \sin \theta_1 \sin \varphi_1) \\ \sqrt{2}(i \sin \theta_1 \cos \varphi_1 + \sin \theta_1 \sin \varphi_1) \\ 2(i \cos \theta_1) \end{bmatrix}, \quad (44)$$

where $\mathbf{r} = (r, \theta_1, \varphi_1)$.

Finally, the matrix elements are found to be

$$H_1 = k_0 c \hbar \sqrt{\frac{2\pi}{\hbar \nu}} \frac{1}{\sqrt{2}} \left[-(\mu_P - \mu_N) I_0 - \left(\frac{2c}{3\hbar c} \cdot \frac{f_2^2}{x} \right) I_1 + \left(\frac{c}{3\hbar c} \cdot \frac{f_2^2}{x} \right) I_2 \right] \cdot \begin{bmatrix} e^{i\phi_0} \\ e^{-i\phi_0} \\ 0 \end{bmatrix}, \quad (45)$$

where

$$\left. \begin{aligned} I_0 &= 4\pi \int_0^\infty \Psi_f^* \Psi_i r^2 dr \\ I_1 &= 4\pi \int_0^\infty \Psi_f^* \Psi_i e^{-\kappa r} r^2 dr \\ I_2 &= 4\pi \int_0^\infty \Psi_f^* \Psi_i e^{-\kappa r} \frac{r}{x} dr \end{aligned} \right\} \quad (46)$$

and ϕ_0 is the angle between \mathbf{e} and \mathbf{X} -axis.

The total cross-section for the photomagnetic effect is

$$\phi_m = \frac{2\pi}{\hbar c} |H_m|_{Av.}^2, \quad (47)$$

where av. means the same average as before. This result can be reduced to

$$\phi_m = \frac{4}{3} \pi^2 \left(\frac{1}{\lambda} \right) \left[-(\mu_P - \mu_N) I_0 + \frac{cf_2^2}{3\hbar c x} (I_2 - 2I_1) \right]^2. \quad (48)$$

§ 4. Numerical Results.

The numerical values for various constants are

$$M_m = 286 m$$

$$R = 2.8 \times 10^{-13} \text{ cm}$$

$$\frac{f_2^2}{\hbar c} = 0.05 \quad \text{Binding energy: } \frac{|E|}{Mc^2} = 0.0023$$

$$P = 1 \quad \mu_P = 2.789 \left(-\frac{c\hbar}{2Mc} \right), \quad \mu_N = -1.910 \left(\frac{c\hbar}{2Mc} \right). \quad (49)$$

With respect to the height of the potential for P state three values (0 MeV, 5 MeV, 10 MeV) are inserted, the one for 1S state is taken, however, only 0 MeV by taking into account of the experimental results.⁶⁾

Further we introduce the notation

$$F(\phi) = \frac{1}{(1+|\phi|)(1+2|\phi|)} \quad (50)$$

then $J(\sigma, r)$ in the (33) reduces to the ordinary potential, when $F(\phi)$ is equal to 1. We carried out also the calculation in this case.

The results are shown in the Table I.

		$F(\phi) = \frac{1}{(1+\phi)(1+2\phi)}$			$F(\phi) = 1$
V_0	$h\nu$	ϕ_e	ϕ_m	$\phi_{tot.}$	ϕ_{tot}
0	2.6	2.59	0.07	2.66	2.97
	6.2	12.19	0.17	12.36	12.36
	17.1	5.13	0.18	5.31	5.31
5	2.6	3.16	0.07	3.23	3.23
	6.2	12.95	0.17	13.12	13.12
	17.1	5.53	0.18	5.71	5.71
10	2.6	2.78	0.07	2.85	2.85
	6.2	12.95	0.17	13.12	13.12
	17.1	6.43	0.18	6.61	6.61

V_0 : Height of the 3P state potential.

$h\nu$: Incident γ -ray energy in MeV.

ϕ_e : Photoelectric cross-section (10^{-28} cm 2).

ϕ_m : Photomagnetic cross-section (10^{-28} cm 2).

$\phi_{tot} = \phi_e + \phi_m$: Total cross-section (10^{-28} cm 2).

The recent experiment by Wilson, Collie and Halban³⁾ shows that for the 2.6 MeV γ -ray the total cross-section is $(16.2 \pm 1.0) \times 10^{-28}$ cm 2 . It is hardly possible to obtain the agreement with experimental data in all cases. Our results are, rather, in conformity with the Bethe-Peierls' curve.

The authors express cordial thanks to Professor G. Araki who showed his work about new potentials, and to Professor M. Kabayasi for valuable discussions.

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Erratum. The sentence "The case (ii).....reult (2)." in the letters to the editor (Prog. Theor. Phys. **4** (1949), 376) should be omitted.

On the mu-Meson Capture^{*)}

Mituo TAKETANI, *Tokyo*; MUNEO SASAKI, *Tokyo Bunrika University*; and
Seitaro NAKAMURA, *Tokyo University*

(Received August 28, 1949)

§ I. Introduction

On the features of mu-meson capture, various investigations, qualitative as well as quantitative, have been given by many authors.^{1),-10)} The capture process consists of two parts: at first step the interaction is purely electrical one, while in the second step the specific interaction between mu-meson and nucleon plays the essential role. The problems here treated are concerned with the second step and deal with 1) Z -dependence of the excitation energy due to the mu meson capture and 2) the establishment of the interaction scheme between elementary particles.

In the previous paper¹⁾ we have shown the formula for the mu-capture probability as follows:

$$\frac{1}{\tau_{\mu\text{-cap}}} = \left(\frac{M_1^2}{T_{01}} + \frac{M_2^2}{T_{02}} \right) \frac{4\pi (uZ)^{2s+1}}{l'(2s+1)} \times \frac{(2m_\mu c^2 r_0)^{2s-2}}{\hbar} (\Delta\bar{W} + s)^2,$$

$$s = \sqrt{1 - u^2 Z^2}, \quad \frac{1}{T_{01}} = \frac{8\theta^2}{\pi} \frac{m_\pi c^2}{\hbar} \left(\frac{g_1 r_1}{\hbar c} \right)^2, \quad \frac{1}{T_{02}} = \frac{32\theta^2}{3\pi} \frac{m_\pi c^2}{\hbar} \left(\frac{g_2 r_2}{\hbar c} \right)^2,$$

$$M_1 = \int \Psi^* \varphi dV, \quad M_2 = \int \Psi^* \omega \varphi dV.$$

where all the notations are the same given in 1); here $\Delta\bar{W}$ is the exciting energy shared to the nucleus by absorbing the mu meson in the K -shell, the reason why we have assumed its value as $\Delta\bar{W} = \frac{20}{216}$ will be shown in this paper (see Section II). In this formula, the Z -dependence of $1/\tau_{\mu\text{-cap}}$ seems to be proportional to the 3rd power of Z . However, if we take into account the Z -dependence of the unknown nuclear matrix elements, it will be shown, in the Section II of this paper, that the capture probability is found to be dependent, at least, on the 4th power of Z .

On the lights of recent experiments, it has been almost safely established that the pi-meson is a Boson and the mu meson a Fermion, with the obvious information that the pi-meson interacts strongly with nucleons but the mu meson much weaker. Nevertheless, the interaction schemes are supposed to be rather uncertain, there seems many alternatives for that, such as Fig. 1, 2, and 3. etc.¹¹⁾

The possibility given in Fig 4 may be more attractive, since the processes between two Fermions are coupled through virtual emission and absorption of pi-mesons as originally put forward by Yukawa. However, if we determine the pi-lepton coupling by the beta decay of nuclei in this model, the computed life time of the beta decay of pi mesons becomes $6 \sim 2 \times 10^{-9}$ sec, which is faster than the life time of pi mesons for pi-mu decay. To remove this difficulty some separation of couplings is suggested, by assuming pi mesons as pseudoscalar field and imposing essential role to the pseudoscalar coupling for nuclear beta decay, to the pseudovector coupling for the beta-decay of pi-mesons.¹²⁾ In spite of this advantage, the nuclear beta-decay through pseudoscalar pi-mesons¹³⁾ gives the seventh power law for the Sargent law, which is apparently at variances with the experiments. Moreover, in this substantialistic arguments, the elaborate discussions, such as parastic to the accidental nature of the present field theory, should essentially distinguished from the phenomenological point of view. The model 4 is, thus, ruled out in this stage. For the choice between that given in the Fig. 1 and Fig. 2 will be discussed in the Section III.

§ II. The Z-Dependence of μ -Meson Capture Processes.

As will be shown in the following, the interaction between nucleons and μ -mesons are supposed, with good reasons, to be caused through the virtual emissions and reabsorptions of π -mesons. However, in the phenomenological treatment of the problem, these interactions may be reduced to the direct ones similar to Fermi interactions of β -decay, i. e., proton P captures μ -meson and transmutes itself into neutron N , by successively emitting neutral meson ν :

$$\mu^- + P \rightarrow N + \nu. \quad (1)$$

To obtain somewhat quantitative results from the process (1), we assume scalar coupling between particles, whose interaction energy is then given by:

$$H = g \int \Psi_i^* \Psi_f \varphi_i^* \psi_f d\tau \quad (2)$$

where

$$\varphi = (a_0^3 \pi)^{-1/2} \exp(-r/a_0)$$

denotes the waves functihn for μ -meson in K -orbit, a_0 being Bohr radius and equal to $\hbar^2/Zm_\pi e^2$ and $\psi_f = \exp(i\mathbf{p}\mathbf{r})$ is that of the emitted neutral meson. As the detailed forms of the wave functions of nucleons in the intial and final states, Ψ_i, Ψ_f are not yet known, we take the approximate model given by Majorana, i. e., Ψ 's are the plane waves:

$$\Psi = R^{-2/3} (4\pi/3)^{-1/2} \exp(i\mathbf{k}\mathbf{r})$$

inside the nucleus, $|\mathbf{r}| < R$, and are zero outside, $|\mathbf{r}| > R$, where R is the nuclear radius:

$$R = \frac{e^2}{2m_\pi c^2} A^{1/3}$$

The interaction probability p that the nucleon acquires the momentum between $(\mathbf{k}, \mathbf{k} + d\mathbf{k})$ and the neutral meson is emitted with the momentum between $(\mathbf{p}, \mathbf{p} + d\mathbf{p})$ is then calculated to be

$$p \sim Z^3 \cdot f(y) p^2 d\Omega_p \cdot \rho_k, \quad (3)$$

with

$$\rho_k = \frac{2\Gamma k^2 dk d\Omega_k}{(2\pi)^3},$$

$$y = |\mathbf{k} + \mathbf{p} - \mathbf{k}_0| \cdot R,$$

where \mathbf{k}_0 denotes the average momentum of initial protons in the nucleus. From (3) we can roughly investigate the Z -dependence of capture processes. Firstly the function

$$f(y) = (y^{-2/3} J_{3/2}(y))^2 \approx 0.707 \exp(-y^2/4)$$

in the probability p has appreciable values only in the small region of $0 \leq y^2 \leq 4$. Further, the average initial momentum of protons and that of the emitted neutral meson depend scarcely on the charge Z , because of the nuclear model we have taken and of the fact that the latter particle carries away the greater part of energy due to the mass annihilation and thus we may put the effective force range $m_\mu c/\hbar$. Therefore the region which gives the appreciable contribution to the probability depends only on \mathbf{k} and is proportional to R^{-3} . As the final level density ρ_k is proportional to R^3 , the integration with respect to the final level is thus almost independent on Z and only factors which give the charge dependence are the volume concentration of μ -meson in K -orbit, Z^3 , and the number of protons in the nucleus, Z , which is to be multiplied to obtain the total probability. From these discussions we can conclude that the total probability would behave as $\sim Z^4$, which agrees with the result of the consideration made by Wheeler.

Now we shall proceed to the next step. After the integration with respect to angles, we have the probability curve that the nucleon remains with the momentum between k and $k + dk$. The curve shows only one peak at $x = x_0$ and its shape around this point is of Gauss-function type;

$$\exp\left(-\frac{R^2 x'^2}{4} (x - x_0)^2\right) \quad \text{with } x'x = k \quad (4)$$

where x' is the average value of the vector $(\mathbf{p} + \mathbf{k})$ with respect to the angle of \mathbf{p}

$$x' = \sqrt{(\mathbf{p} + \mathbf{k})^2} \approx 2x$$

and the maximum point x_0 which the curve shows is given by the relation:

$$x_0 = \left(\tanh \frac{R^2 x'^2}{2} x_0 \right)^{-1} \quad (5)$$

For large Z , x_0 is nearly equal to unity and increases slightly as Z decreases. Because of the sharpness of the curve we may identify this x_0 with the average value of momentum \bar{x} of the excited nucleon. Really the difference between x_0^2 and \bar{x}^2 is calculated to be less than several percents. The average energy, when measured in the unit of $m_\mu c^2 = 100 \text{ MeV}$, is given by

$$\frac{\hbar^2 \bar{x}^2}{2Mx} - \bar{E} = \frac{x_0^2}{5} - E \quad (6)$$

where \bar{E} is the average kinetic energy of the initial nucleon and is equal to ~ 0.14 in the present unit. The variation of the excitation energies with the charge number Z is considerably small. From (5) and (6) we have the excitation energy for large Z , $\sim 6 \text{ MeV}$, and for small Z (say, Be ($Z=4$)), $\sim 14 \text{ MeV}$.

§ III. The Interaction Scheme.

The value of the coupling constant, g , in (2) has been estimated by Wheeler

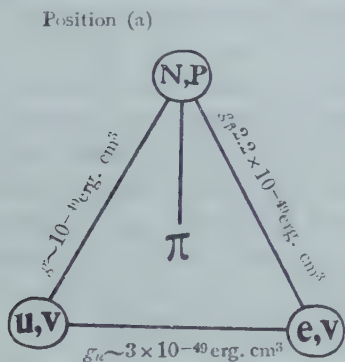


Fig. 1. π occupies the position (a). In this scheme, π - μ decay could only occur through virtual nucleon pairs.

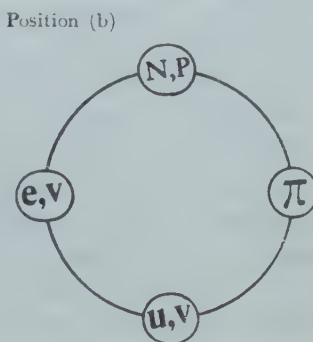


Fig. 2. π occupies the position (b). In this scheme, π - μ decay can occur through direct process, while μ -meson capture through virtual π -meson.

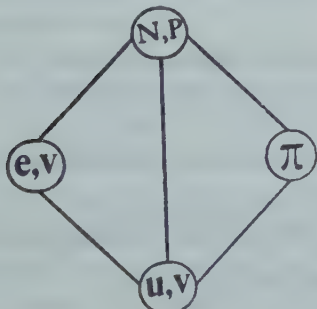


Fig. 3. Scheme (c). μ -capture, π - μ decay, nuclear beta decay and the beta decay of μ mesons are all given through direct processes.

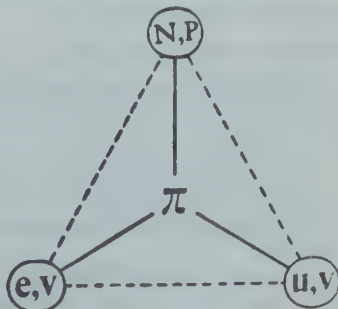


Fig. 4. The processes between two Fermions, such as (N, P) (μ, ν) and (e, ν) are intermitted through π -mesons. The direct processes may eventually be supposed.

and others to be $\sim 10^{-49}$ erg cm³, which is the same order of magnitude for the coupling constants g_β of β -decay and g_μ of μ -meson decay. From these facts Wheeler has drawn a interaction triangle. To complete, however, the actual interaction scheme, we must add one more line thereon, which indicate the rôle of π -meson. The position of π -meson is considered to be put either in (a) or in (b). In the position (a) the $(\pi-\mu)$ -decay is given by the process:

$$\pi \rightarrow \underline{P} + N \begin{matrix} \nearrow [N] + N + \mu^+ + \nu \\ \searrow \underline{P} + [\underline{P}] + \mu^+ + \nu \end{matrix} \mu^+ + \nu \quad (7)$$

where [] denotes anti-particle, while the position (b) gives the simple process

$$\pi \rightarrow \mu + \nu \quad (8)$$

Both these two processes, (7) and (8), can give the right order of lifetime for $(\pi-\mu)$ -decay theoretically, if one wish, and thus one can not decide which of these two is better suited. However, it is indicated that we can give the decision, if we take into consideration the experimental fact on the conflict between $(\pi-\beta)$ and $(\pi-\mu)$ -decays. According to the experiments π -mesons decay more rapidly into μ -meson than into leptons. This fact cannot be explained by (7), for, here the π -mesons would decay into leptons rather than into μ -mesons, unless unusual asymmetrical interactions for both decay processes are assumed. This argument will be understood from the fact that as the coupling constants of $(\pi-\mu)$ and $(\pi-\beta)$ -decays and the path from π to μ and from π to the leptons are essentially the same in the process (7), the only difference in the probabilities for both decay processes may be the density factors of the final momentum space which has the larger magnitude for the lighter particle. In fact, the ratio of the probabilities for both decay is calculated to be the order of

$$\tau_{\pi\mu} / \tau_{\pi e} \sim \frac{g_\mu^2}{g_\beta^2} \frac{\int P_\mu^2 d p_\mu}{\int P_e^2 d p_e} \cong 0.05.$$

Therefore, we can conclude that the position (b), with the all over phenomenological model (c), will be preferable for the interaction scheme including μ -mesons, before the position (a).

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On the Production of π -Mesons in Nucleon-Nucleon Collisions

Shūji TAKAGI

Department of Physics, Kyoto University.

(Received August 31, 1949)

§ 1. Introduction.

Recent development of nuclear physics confirms experimentally the theoretical prediction that at least one type of mesons is produced by the nuclear interaction. On the other hand, π -mesons in cosmic radiation are found to interact strongly with nucleons in the experiments of photographic emulsions, and it is believed that π -mesons are produced by the interaction between primary nucleon components and air nucleus. It is reasonable, therefore, to identify the artificially produced mesons with cosmic π -mesons. According to the original idea of Yukawa, these mesons are responsible for at least a part of the nuclear forces.

Fields with various transformation characters and their mixed fields have hitherto been proposed to describe the nuclear mesons. As far as the single field is concerned, pseudoscalar type seems to be most plausible. Mixed fields, on the other hand, were introduced to remove the so-called $1/r^3$ singularity which appeared in the nuclear force operator. It has been found, however, that this singularity has been the result of careless treatment of intermediate states and has been able to be removed by taking account of the recoil of nucleons.¹⁾ Furthermore in the case of pseudoscalar field the so-called static nuclear force is almost zero, while in the case of vector field it has the different order of singularity. Hence, to take a mixed field to reduce the singularity is not only unnecessary, but also almost meaningless.

Data of cosmic ray experiments do not make it difficult to represent π -mesons in terms of pseudoscalar fields. (The possibility of taking vector field can, of course, not be excluded decisively.)

We assume, therefore, that the nucleon interacts with pseudoscalar meson field and thus consider the production of mesons in nucleonic collisions. In particular, we first treat the case of nucleonic collision with the energies just above the threshold energy for meson creation to compare with the experiment of synchro-cyclotron, and then we consider the collision of high energy nucleons.

§ 2. Method of Calculation.

We perform the calculation throughout standing on the interaction between

nucleons and pseudoscalar meson field without assuming special form for the nuclear forces phenomenologically. For this purpose we utilize the formalism of relativistic quantum electrodynamics developed by Tomonaga, Schwinger and Feynman. This method is not only elegant formally but also has advantages to clarify the ambiguities inherent in the theories which have been currently used up to present and to put forward the computation relativistically at least but a few practical final stages. By this we can avoid the errors caused from the indiscriminate application of ambiguous non-relativistic approximation in the starting point.

Adopting the unit system in which $\hbar = c = 1$, the generalized Schrödinger equation is

$$i \frac{\partial \Psi}{\partial \sigma} = H(x) \Psi(\sigma). \quad (2.1)$$

(2.1) can be solved by the use of generalized transformation function $U(\sigma, \sigma_0)$ as

$$\Psi(\sigma) = U(\sigma, \sigma_0) \Psi(\sigma_0),$$

where $U(\sigma, \sigma_0)$ can be written in the form of series expansion

$$\begin{aligned} U(\sigma, \sigma_0) = & 1 - i \int_{\sigma_0}^{\sigma} H(x_1) (dx_1) - \int_{\sigma_0}^{\sigma} (dx_1) \int_{\sigma_0}^{\sigma(x_1)} (dx_2) H(x_1) H(x_2) \\ & + i \int_{\sigma_0}^{\sigma} (dx_1) \int_{\sigma_0}^{\sigma(x_1)} (dx_2) \int_{\sigma_0}^{\sigma(x_2)} (dx_3) H(x_1) H(x_2) H(x_3) + \dots \end{aligned} \quad (2.2)$$

The transition probability can be deduced from this U as $|U|_{if}^2$. For the sake of practice we push σ and σ_0 into $+\infty$ and $-\infty$ respectively.

Now the interaction Hamiltonian is usually written as follows:

$$\begin{aligned} H_I(x) = & -(\zeta(x)\phi(x) + \zeta^*(x)\phi^*(x) + \zeta^0(x)\phi^0(x)) \\ & - \left(\gamma_\mu \frac{\partial}{\partial x_\mu} \phi(x) + \gamma_\mu^* \frac{\partial}{\partial x_\mu} \phi^*(x) + \gamma_\mu^0 \frac{\partial}{\partial x_\mu} \phi^0(x) \right) \\ & + \frac{1}{2} \left\{ (\gamma_\mu, N_\mu) (\gamma_\mu^*(x), N_\mu) + (\gamma_\mu^*(x), N_\mu) (\gamma_\mu(x), N_\mu) \right. \\ & \left. + (\gamma_\mu^0(x), N_\mu) (\gamma_\mu^0(x), N_\mu) \right\}, \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} \zeta(x) &= if_1 \phi^\dagger(x) \gamma_5 \tau \psi(x), & \zeta^*(x) &= if_1 \psi^\dagger(x) \gamma_5 \tau^* \phi(x), \\ \zeta^0(x) &= if_1^{(0)} \phi^\dagger(x) \gamma_5 \tau^0 \psi(x), & \gamma_\mu &= i \frac{f_2}{x} \phi^\dagger(x) \gamma_5 \gamma_\mu \tau \psi(x), \\ \gamma_\mu^* &= i \frac{f_2}{x} \psi^\dagger(x) \gamma_5 \gamma_\mu \tau^* \phi(x), & \gamma_\mu^0 &= i \frac{f_2^{(0)}}{x} \phi^\dagger(x) \gamma_5 \gamma_\mu \tau^0 \psi(x). \end{aligned}$$

ϕ , ϕ^* and ϕ^0 denote the variables of meson field, which contain the emission operator of negative, positive and neutral mesons and the absorption operator of positive, negative and neutral mesons respectively, ψ that of nucleon field and ψ^\dagger is the adjoint operator to ψ , among which the following commutation rules hold:

$$[\psi(x), \phi^*(x')] = i\mathbf{J}_\pi(x-x') = [\phi^0(x), \phi^0(x')]$$

$$\{\psi_\alpha(x), \psi_\beta^\dagger(x')\} = \frac{1}{i} \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - M \right)_{\alpha\beta} \delta_M(x-x') = \frac{1}{i} S(x-x')$$

while the other pairs are commutable or anticommutable, the curly bracket meaning the anticommutator.

In (2.3) τ , τ^* are the usual isotopic spin operators which change the neutron and proton state to the proton and neutron state respectively, and τ^0 is given as $\tau^0 = \tau\tau^* - \tau^*\tau$, f_1 , $f_1^{(0)}$, f_2 and $f_2^{(0)}$ are the coupling constants and κ and M are the mass of meson and nucleon respectively. The last term of (2.3), the so-called direct interaction term, is added to secure the integrability condition for (2.1). Thus (2.3) satisfy the integrability condition up to the order of f^2 . We believe that this direct term is physically meaningless and that it should not make any effect to physical processes. So we perform a canonical transformation which has the nature of gauge transformation

$$\Psi = e^{iS} \Phi(\sigma)$$

$$S = \int_\sigma (\eta_\mu(x') \phi(x') + \eta_\mu^*(x') \phi^*(x') + \eta_\mu^0 \phi^0(x')) d\sigma'_\mu.$$

The transformed Hamilton density H is

$$H = H_1 + H_2,$$

$$\begin{aligned} H_1 = & -if_3(\psi^\dagger(x)\gamma_3\tau\psi(x)\phi(x) + \psi^\dagger(x)\gamma_3\tau^*\psi(x)\phi^*(x)) \\ & -if_3^{(0)}\psi^\dagger(x)\gamma_3\tau^0\psi(x)\phi^0(x), \\ H_2 = & -\frac{2f_2}{\kappa}f_4\psi^\dagger(x)\psi(x)\phi(x)\phi^*(x) - \frac{2f_2^{(0)}}{\kappa}f_4^{(0)}\psi^\dagger(x)\psi(x)\phi^0(x)\phi^0(x) \\ & -\frac{1}{2}\left(\frac{f_2}{\kappa}\right)^2\psi^\dagger(x)\gamma_\mu(\tau_N - \tau_P)\psi(x)\left(\phi^*(x)\frac{\partial\phi(x)}{\partial x_\mu} - \phi(x)\frac{\partial\phi^*(x)}{\partial x_\mu}\right) \\ & -\frac{f_2f_2^{(0)}}{\kappa^2}\left\{\psi^\dagger(x)\gamma_\mu\tau\psi(x)\left(\phi^0(x)\frac{\partial\phi(x)}{\partial x_\mu} - \phi(x)\frac{\partial\phi^0(x)}{\partial x_\mu}\right) \right. \\ & \left. -\psi^\dagger(x)\gamma_\mu\tau^*\psi(x)\left(\phi^0(x)\frac{\partial\phi^*(x)}{\partial x_\mu} - \phi^*(x)\frac{\partial\phi^0(x)}{\partial x_\mu}\right)\right\}, \end{aligned} \quad (2.4)$$

$$f_3 = f_1 + \frac{2M}{\kappa}f_2, \quad f_3^{(0)} = f_1^{(0)} + \frac{2M}{\kappa}f_2^{(0)}; \quad f_1 = f_1 + \frac{M}{\kappa}f_2, \quad f_1^{(0)} = f_1^{(0)} + \frac{M}{\kappa}f_2^{(0)}.$$

(2.4) contains no direct terms. If we push forward the above transformation to

higher order in coupling constant, there appear the terms containing direct terms. If we take (2.4) as Hamilton density it satisfies the integrability condition up to the order of f^3 , and the terms of f^3 coming from the transformation other than those containing direct terms do not contribute to the meson emission. Therefore, it may be reasonable to take (2.4) as the effective interaction for the meson-creation in f^3 -approximation.

Inserting (2.4) into (2.2) we obtain the generalized transformation function

$$U = 1 - \int_{-\infty}^{\infty} (dx_1) \int_{-\infty}^{\sigma(x_1)} [H(x_2), H(x_1)] + i \int_{-\infty}^{\infty} (dx_1) \int_{-\infty}^{\sigma(x_1)} (dx_2) \int_{-\infty}^{\sigma(x_2)} [H(x_3), [H(x_2), H(x_1)]] \quad (2.5)$$

§ 3. The Transition Matrix Element.

To fix the consideration we consider the case of positively charged mesons in proton-neutron collisions. The other cases can be computed in similar way. First, we investigate $[H_1(x_3), [H_1(x_2), H_1(x_1)]]$.

$[H_1(x_2), H_1(x_1)]$ contains the nuclear force operator and the meson scattering (including the emission of two mesons) operators. The operators corresponding to the nuclear force are

$$if_3^2 \mathcal{J}_N(x_1 - x_2) \frac{1}{2} \left(\{ \phi^\dagger(x_2) \gamma_5 \tau^* \phi(x_2), \phi^\dagger(x_1) \gamma_5 \tau \phi(x_1) \} \right. \\ \left. + \{ \phi^\dagger(x_1) \gamma_5 \tau \phi(x_1), \phi^\dagger(x_2) \gamma_5 \tau^* \phi(x_2) \} \right) \\ if_3^{(0)2} \mathcal{J}_N(x_1 - x_2) \frac{1}{2} \{ \phi^\dagger(x_2) \gamma_5 \tau^0 \phi(x_2), \phi^\dagger(x_1) \gamma_5 \tau^0 \phi(x_1) \}$$

In $[H_1(x_3), [H_1(x_2), H_1(x_1)]]$ the terms of creation of positive mesons coming from this force operator are, apart from the terms of one particle transition operators which do not contribute to our processes:

$$-if_3^3 \mathcal{J}_N(x_1 - x_2) \left(\{ \phi^\dagger(x_3) \gamma_5 S(x_3 - x_1) \gamma_5 \tau^* \tau \phi(x_1) \phi^\dagger(x_2) \gamma_5 \tau^* \phi(x_2) \right. \\ \left. - \phi^\dagger(x_2) \gamma_5 \tau^* \phi(x_2) \phi^\dagger(x_1) \gamma_5 S(x_1 - x_3) \gamma_5 \tau \tau^* \phi(x_3) \} \right. \\ \left. + [x_1, x_2] \right) \phi^*(x_3) \quad (3.1) \\ -if_3^{(0)2} \mathcal{J}_N(x_1 - x_2) \left(\{ \phi^\dagger(x_3) \gamma_5 S(x_3 - x_1) \gamma_5 \tau^* \tau^0 \phi(x_1) \phi^\dagger(x_2) \gamma_5 \tau^0 \phi(x_2) \right. \\ \left. - \phi^\dagger(x_2) \gamma_5 \tau^0 \phi(x_2) \phi^\dagger(x_1) \gamma_5 S(x_1 - x_3) \gamma_5 \tau^0 \tau^* \phi(x_3) \} \right. \\ \left. + [x_1, x_2] \right) \phi^*(x_3),$$

where $[x_1, x_2]$ means the terms which are obtained exchanging x_1 and x_2 in the foregoing expressions. (3.1) corresponds to the process of Bremsstrahlung. The

remaining terms in $[H_1(x_3), [H_1(x_2), H_1(x_1)]]$ mainly come from the scattering processes. They are:

$$\begin{aligned}
 & -if_3^3(\{A_\pi(x_2-x_3)(\phi^\dagger(x_3)\gamma_5\tau^*\phi(x_3)\phi^\dagger(x_2)\gamma_5S(x_2-x_1)\gamma_5\tau^*\phi(x_1) \\
 & \quad -\phi^\dagger(x_3)\gamma_5\tau^*\phi(x_3)\phi^\dagger(x_1)\gamma_5S(x_1-x_2)\gamma_5\tau^*\phi(x_2))\phi^*(x_1)\} \\
 & \quad -[x_1, x_2]) \\
 & -if_3^{(0)2}f_3(\{A_\pi(x_2-x_3)(\phi^\dagger(x_3)\gamma_5\tau^0\phi(x_3)\phi^\dagger(x_2)\gamma_5S(x_2-x_3)\gamma_5\tau^*\tau^0\phi(x_1) \\
 & \quad -\phi^\dagger(x_1)\gamma_5S(x_1-x_2)\gamma_5\tau^0\tau^*\phi(x_2)\phi^\dagger(x_3)\gamma_5\tau^0\phi(x_3))\phi^*(x_2)\} \\
 & \quad -[x_1x_2]).
 \end{aligned} \tag{3.2}$$

Next we compute $[H_1(x_2), H_2(x_1)]$. Terms which contribute to the process of meson creation coming from this are:

$$\begin{aligned}
 & -f_3f_4\left(\frac{f_2}{x}\right)A_\pi(x_1-x_2)\{\phi^\dagger(x_2)\gamma_5\tau^*\phi(x_2), \phi^\dagger(x_1)\phi(x_1)\}\phi^*(x_1) \\
 & -\frac{1}{4}f_3\left(\frac{f_2}{x}\right)^2\left(\frac{\partial A_\pi(x_1-x_2)}{\partial x_{1\mu}}\phi^*(x_1)-A_\pi(x_1-x_2)\frac{\partial\phi^*(x_1)}{\partial x_{1\mu}}\right) \\
 & \quad \times\{\phi^\dagger(x_2)\gamma_5\tau^*\phi(x_2), \phi^\dagger(x_1)\gamma_\mu(\tau_N-\tau_P)\phi(x_1)\} \\
 & -\frac{1}{2}f_3^{(0)}\frac{f_2^{(0)}}{x}\frac{f_2}{x}\left(\frac{\partial A_\pi(x_1-x_2)}{\partial x_{1\mu}}\phi^*(x_1)-A_\pi(x_1-x_2)\frac{\partial\phi^*(x_1)}{\partial x_{1\mu}}\right) \\
 & \quad \times\{\phi^\dagger(x_2)\gamma_5\tau^0\phi(x_2), \phi^\dagger(x_1)\gamma_\mu\tau^*\phi(x_1)\}
 \end{aligned} \tag{3.3}$$

We must integrate these terms over dx_1 , dx_2 and dx_3 . For this, it is convenient to use the momentum representation and the step function $\epsilon(x)$ introduced by Schwinger.

First we expand ϕ , ϕ^\dagger and ϕ^* in momentum space:

$$\begin{aligned}
 \phi(x) &= \int \psi(P)e^{i(P \cdot X)}(dP), & \phi^\dagger(x) &= \int \psi^\dagger(P)e^{-i(P \cdot X)}(dP), \\
 \phi^*(x) &= \int (A^*(K)e^{-i(K \cdot X)} + B(K)e^{i(K \cdot X)})\frac{dK}{\epsilon(K)},
 \end{aligned}$$

where $\epsilon(x) = \sqrt{k^2 + x^2}$ and the block letter means three dimensional vector. Here A^* contains the emission operator of the positive meson. Inserting these expressions into (3.1), (3.2) and (3.3), the integration over space-time can be easily performed. For instance, from (3.3) we obtain

$$\begin{aligned}
 & \int_{-\infty}^{\infty}(dx_1)\int_{-\infty}^{\infty}(dx_2)[H_1(x_2), H_2(x_1)] = -\frac{1}{2}\int_{-\infty}^{\infty}(dx_1)\int_{-\infty}^{\infty}(dx_2)[H_1(x_2), H_2(x_1)]\epsilon(x_1-x_2) \\
 & = +\int_{-\infty}^{\infty}(dx_1)\int_{-\infty}^{\infty}[f_3f_4\frac{f_2}{x}A_\pi(x_1-x_2)\{\phi^\dagger(x_2)\gamma_5\tau^*\phi(x_2), \phi^\dagger(x_1)\phi(x_1)\}\phi^*(x_1) + \dots
 \end{aligned}$$

$$\begin{aligned}
&= \int \dots \int (dp) d\mathbf{k} (2\pi)^4 \delta(P_1 + P_2 - P_3 - P_4 - K) \left[\left(+2f_3 f_4 \frac{f_2}{x} \right) \psi^\dagger(P_3) \gamma_5 \tau^* \psi(P_1) \right. \\
&\quad \times \psi^\dagger(P_4) \psi(P_2) A^*(k) \frac{1}{\epsilon(k)(x^2 + (P_1 - P_3)^2)} \\
&\quad + \left(+\frac{i}{2} f_3 \left(\frac{f_2}{x} \right)^2 \right) \psi^\dagger(P_3) \gamma_5 \tau^* \psi(P_1) \psi^\dagger(P_4) \gamma_\mu \tau_N \psi(P_2) A^*(k) \frac{(P_1 - P_3 + K)_\mu}{\epsilon(k)(x^2 + (P_1 - P_3)^2)} \\
&\quad \left. + \left(-i f_3^{(0)} \frac{f_2^{(0)}}{x} \frac{f_2^{(0)}}{x} \right) \psi^\dagger(P_3) \gamma_\mu \tau^* \psi(P_1) \psi^\dagger(P_4) \gamma_5 \tau_N \psi(P_2) \times \frac{(P_2 - P_4 + K)_\mu}{\epsilon(k)(x^2 + (P_2 - P_4)^2)} \right]. \quad (3.4)
\end{aligned}$$

During this computation we have dropped the terms which do not contribute to our proton-neutron collision processes throughout.

Similarly, we can calculate $\{ \{ [H_1(x_3), [H_1(x_2), H_1(x_1)]] \}$.

The result is

$$\begin{aligned}
&\int \dots \int (dp) d\mathbf{k} (2\pi)^4 \delta(P_1 + P_2 - P_3 - P_4 - K) \\
&\quad \times \left[- (4i f_3^3) \psi^\dagger(P_3) \gamma_5 \tau^* \psi(P_1) \psi^\dagger(P_4) \left(\frac{i \gamma_\mu (P_4 + K)_\mu + M}{M^2 + (P_4 + K)^2} \right) \tau_N \psi(P_2) A_*(k) \right. \\
&\quad \times \frac{1}{\epsilon(k)(x^2 + (P_1 - P_3)^2)} - (4i f_3 (f_3^{(0)})^2 \psi^\dagger(P_3) \left\{ \frac{i \gamma_\mu (P_3 + K)_\mu + M}{M_2^2 + (P_3 + K)^2} \right. \\
&\quad \left. \left. - \frac{i \gamma_\mu (P_1 - K)_\mu + M}{M^2 + (P_1 - K)^2} \right\} \tau^* \psi(P_1) \psi^\dagger(P_4) \tau_N \psi(P_2) A^*(k) \times \frac{1}{\epsilon(k)(x^2 + (P_2 - P_4)^2)} \right] \quad (3.5)
\end{aligned}$$

It is remarkable that in (3.5) the contribution from (3.1) is equal to that from (3.2) so that (3.5) is essentially the terms of Bremsstrahlung. It should also be remembered that (3.4) comes into U twice.

Thus having got all terms in U , the estimation of transition probability is straightforward and is performed as usual.

§ 4. Results and Discussions.

The exact expression of the cross-section is very much complicated in general and practically it seems almost impossible to obtain it in terms of simple analytic functions. We content ourselves, therefore, to two extreme cases.

1) $p_0 \ll M$

First we consider the case where the collision occurs with the energy just above the threshold energy for meson creation. Suppose that a proton with momentum p_0 collides with a neutron of momentum $-p_0$ (i. e., we take the center of mass coordinate system). In this case the approximation of $p_0 \ll M$

holds. The region of applicability of this approximation is considerably restricted. Hence in cosmic ray phenomena this case does not bring appreciable contributions. Present experiments of artificial creation of mesons are, however, restricted almost in this case. Therefore it may not be useless to examine it. The differential cross-section becomes

$$d\sigma = \left[\left(\frac{f_3^2}{4\pi} \right)^3 + 6 \left(\frac{f_3^{(0)2}}{4\pi} \right) \left(\frac{f_2^2}{4\pi} \right) \right] \left(\frac{Mk}{M\epsilon(k)} \right)^2 \left(\frac{k\rho}{\rho_0} \right) \left(\frac{\rho_0}{M} \right)^2 \frac{1}{2\pi} \\ \times \left\{ \frac{1}{(\mathbf{x}^2 + \rho^2 + \rho_0^2) - 4\rho_0^2 \rho^2} - \frac{\tan^{-1} [2\rho_0 \rho / (\mathbf{x}^2 + \rho_0^2 + \rho^2)]}{2\rho_0 \rho (\mathbf{x}^2 + \rho_0^2 + \rho^2)} \right\} d\epsilon(k) d\Omega \quad (4.1)$$

where $\rho = \sqrt{p_0^2 - M\epsilon(k)}$ and $d\Omega$ denotes the solid angle element. It can easily be seen in the course of above estimation that here the contributions from (3.5) are predominant while that from (3.4) is smaller by the factor (ϵ/M) . We can conclude, therefore, that the creation of mesons in low energy nucleonic collision takes place mainly through the agency of the nuclear force operator. The result (4.1) is effectively the same as obtained by Foldy and Marshak²⁾ who argued the same process assuming a phenomenological form for the nuclear force.

2) $p_0 \gg M$

We consider two cases under this approximation.

a) The case where the momentum of the emitted meson is small compared with the momentum of the incident proton p_0 . This case resembles to that of 1). The result is

$$d\sigma \cong \left(\frac{f_3^2}{4\pi} \right)^3 \frac{2}{2\pi} \left(\frac{k}{\epsilon} \right) \frac{1}{p_0^2} \left(\log \frac{p_0}{x} + \log 2 \right) \frac{d\epsilon}{\epsilon} d\Omega \quad (4.2)$$

all the other contributions are smaller than this by the power of (ϵ/p_0) . It is remarkable here that the angular distribution is spherically symmetric as is the case 1).

b) The case where the momentum of the emitted meson has the order of magnitude comparable with that of incident proton.

In this case the contributions from (3.4) is predominant. For instance, the term of $(f_3^2/4\pi)(f_4^2/4\pi)(f_2^2/4\pi)$ is, apart from angular factor, of the order $\sim (1/x^2)d\epsilon/\epsilon$ while the contribution from the force operator is $\sim (f_3^2/4\pi)^3(1/p_0^2)\frac{d\epsilon}{\epsilon}$. Therefore, the former is greater than the latter by the factor $(p_0/M)^2$ taking account of equivalence of f_3 and $(M/x)f_2$. High energy mesons are created by the part of pseudovector coupling which is not translated into nuclear force operator. After some calculation, the differential cross-section in this case becomes

$$d\sigma \cong \left[\left(\frac{f_3^2}{4\pi} \right) \left(\frac{f_2^2}{4\pi} \right)^2 f(\theta, p_0, k) + 4 \left(\frac{f_3^{(0)2}}{4\pi} \right) \left(\frac{f_2^2}{4\pi} \right) \left(\frac{f_2^{(0)2}}{4\pi} \right) g(\theta, p_0, k) \right] \frac{1}{2(2\pi)} \\ \times \frac{k}{x^4 p_0^2} \frac{1}{(1 - \cos^2 \theta)} d\epsilon d\Omega, \quad (4.3)$$

where

$$f \sim \frac{15}{4} p_0^2 + 7 p_0^2 \cos \theta - 8 p_0^2 \cos^2 \theta - \frac{9}{4} k^2 \cos^3 \theta,$$

$$g \sim \frac{16}{4} p_0^2 + \frac{74}{4} p_0^2 \cos \theta + \frac{51}{4} p_0^2 \cos^2 \theta - p_0^2 \cos^3 \theta,$$

$$\cos \theta = (\mathbf{p}_0 \cdot \mathbf{k}) / p_0 k.$$

The other terms being smaller by the factor of $(M/p_0)^2$ or of its higher power. (Of course, (4.3) is not valid for $\cos^2 \theta = 1$.)

From the above investigation it can be seen that the emission of high energy nucleonic collisions originates mainly from the interaction between the nucleonic current and the current of meson field, and the other type of coupling contributes less than this. Furthermore, the angular distribution of the emitted mesons is not symmetric in this case but restricted into the direction of the incident proton, as it should be the case.

§ 5. Conclusion.

The emission of π -mesons in nucleonic collisions are investigated assuming the interaction between nucleonic field and the pseudoscalar field which we suppose to represent π -mesons. In low energy collisions the creation of mesons occurs mainly by the process of usual Bremsstrahlung and the cross-section is essentially the same when we assume a phenomenological form for nuclear force operator. In high energy collisions it is found that the low energy mesons are produced in the same process as in low energy collisions. On the other hand high energy mesons are created in the processes which originate from the pure pseudovector coupling. With respect to the angular distribution, low energy mesons are produced symmetrically in the center of mass system while the high energy mesons are emitted mainly in the direction of collision.

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Letters to the Editor

Remarks on the Ambiguity in Quantum Field Theory

Y. Katayama and S. Hori

Dept. of Physics, Kyoto University

August 20, 1949

The so-called ambiguities raised by the photon self-energy problem have been seemed to be removed for a time by the ingenious method of Pauli and Villars.¹⁾ However, if we want to rewrite this method in the customary form, many difficulties arise.

Here, we intend to examine an alternative method to remove the ambiguities. The first problem concerning above mentioned ambiguities is whether the quadratic divergences exist or not. As is well known, we get them by the usual momentum space integration, but do not by Schwinger's calculation method. This difference is due to the definition of $\mathcal{A}^{(1)}(x)$ at $x=0$. The latter calculation takes its principal value as follows:

$$\mathcal{A}^{(1)}(0) = \frac{1}{2} \left[\lim_{X\mu^2 \rightarrow 0+0} + \lim_{X\mu^2 > 0+0} \right] \mathcal{A}^{(1)}(x) \quad (1)$$

and then removes the first quadratic divergence contrary to the former. If we want to take the same definition in the former, we must use the time-like and space-like λ -process.²⁾

The second problem is how to treat the integral

$$\int q_\mu q_\nu e^{iaq\lambda^2} dq. \quad (1)$$

According to Miyazima,³⁾ it is evaluated as

$$-\delta_{\mu\nu} \frac{\varepsilon(a)}{2a^2} \left[\frac{1}{a} - c\varepsilon(a)\delta(a) \right] \quad (3)$$

and an ambiguity arises from the last c -term. If we determine this term uniquely by the suitable definition of the integral (2), a part of ambiguities will be removed. It is likely

to define it so as to reduce

$$\delta(x) \frac{\partial \mathcal{A}^{(1)}}{\partial x_\mu} = (-\square^2 + m^2) \bar{\mathcal{A}} \cdot \frac{\partial \mathcal{A}^{(1)}}{\partial x_\mu} \quad (4)$$

to zero. In fact, Miyazima has indicated that taking $c=2$, contrary to Wentzel,⁴⁾ the photon self-energy vanishes.

However, we notice that this result does not give unique answer. This arises from the fact that one can reduce

$$\bar{\mathcal{A}}(-\square^2 + m^2) \mathcal{A}^{(1)} \quad (5)$$

to zero by taking $c=1$, but not $c=2$. Therefore, if we put $(q_\lambda^2 + m^2)\delta(q_\lambda^2 + m^2) = 0$ before calculations, we get different result, the difference being the value of (4) with $c=2$. When we take $c=1$, the photo self-energies do not vanish, but become to the finite values $\frac{\alpha}{\pi} \frac{m_e^2}{2}$, $-\frac{\alpha}{\pi} \frac{\mu_s^2}{4}$, and $-\frac{\alpha}{\pi} \frac{3\mu_v^2}{4}$ for the spinor, scalar and vector fields respectively, contradicting to the conditions of gauge invariance. (Notice that the results are half of Wentzel's one). This discrepancy is due to the fact that the left-hand side of (4) vanishes conditionally while the right-hand side does not.

In order to remove this discrepancy, we are obliged to rely on the mixed theory,⁵⁾ but there arises a new difficulty which is caused by the vector field introduced in order to compensate the divergences of charge-renormalizations with the condition $4n+m-3l=0$, where n , m and l are the numbers of particles of spinor, scalar and vector fields. In spite of them, there remain other difficulties which are not removed by the mixed theory, for instance, the problems of Dyson theorem.

In these circumstances, we can hardly say that we have any consistent method of calculations and so, at the present stage, to get a temporary way of escape, we must utilise the following three methods: i) Pauli's regulator,

ii) the definition $c=2$ and iii) the definition $c=0$ after using (4) at first in the calculation (by which the p. s. e. vanishes). The last two methods, however, make us resign ourselves to prove the identity (5) by direct calculations.

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We calculated by Schwinger's calculation methods and so the conclusions are a few different with them, but essentially the same.

Reaction of Meson Field on Nucleon

H. Kita

Physics Institute, Kyoto University.

August 30, 1949

Recently, applying the "Self-Consistent" subtraction method to the interaction between scalar meson and nucleon field, Itô and Baba⁽¹⁾ have obtained the following results in the second order approximation: the terms containing the mesic self-energy of the nucleon can be unified in the nucleon mass. The vacuum polarization terms corresponding to the meson self-energy, on the contrary, consist of two types, one of which is $U\phi^*\phi$, and the other $V \text{ Grad } \phi^* \text{ Grad } \phi$. The former has the desired form but the latter does not.

But since there exists the relation:

$$\begin{aligned}\frac{\partial G_\mu}{\partial x_\mu} &\equiv \frac{1}{2} \frac{\partial}{\partial x} \left(\phi^* \frac{\partial \phi}{\partial x_\mu} + \frac{\partial \phi^*}{\partial x_\mu} \phi \right) \\ &= \frac{\partial \phi^*}{\partial x_\mu} \frac{\partial \phi}{\partial x_\mu} + \mu^2 \phi^* \phi,\end{aligned}$$

after the gauge transformation

$\exp[-iV \int G_\mu d\sigma_\mu]$ to the equation that Itô and Baba reached, we obtain $\delta H_{\text{pol}} = (U - \mu^2 V) \times \phi^* \phi$, which can be unified in the meson

mass term.

(A) Pseudo-scalar meson.⁽²⁾ In this case the situation is the same as the scalar meson. There are three different canonical transformations to eliminate the first order interaction: $\exp[i \int (H_{ps} + H_{pv}) dV_4]$, $\exp[i \int H_{ps} dV_4]$ $\exp[i \int H_{pv} dV_4]$ and $\exp[i \int H_{pv} dV_4]$ $\exp[i \int H_{ps} dV_4]$, and these lead to $U\phi^*\phi + V \frac{\partial \phi^*}{\partial x_\mu} \frac{\partial \phi}{\partial x_\mu}$ with different U and V . But after gauge transformations these lead to the same $2\mu\delta\mu\phi^*\phi$. The results are as follows:

$$\begin{aligned}\delta H &= \delta M_N \bar{\Psi}^\dagger \tau_N \Psi + \delta M_P \bar{\Psi}^\dagger \tau_P \Psi + 2\mu\delta\mu\phi^*\phi, \\ \delta M_N &= \frac{1}{\pi} \int_0^\infty \frac{k}{\sqrt{k^2 + \mu^2} \sqrt{k^2 + M_P^2}} \\ &\quad \times \frac{k}{\{M_N^2 - (\sqrt{k^2 + \mu^2} + \sqrt{k^2 + M_P^2})\}} \\ &\quad \times \left[f_1^2 \{M_N \sqrt{k^2 + M_P^2} - M_P (\sqrt{k^2 + \mu^2} \right. \\ &\quad \left. + \sqrt{k^2 + M_P^2})\} + \frac{f_2^2}{\mu^2} \{M_N (2k^2 \sqrt{k^2 + \mu^2} \right. \\ &\quad \left. + 2k^2 \sqrt{k^2 + M_P^2} + \mu^2 \sqrt{k^2 + M_P^2}) \right. \\ &\quad \left. - M_P \mu^2 (\sqrt{k^2 + \mu^2} + \sqrt{k^2 + M_P^2})\} + \frac{2f_1 f_2}{\mu} \right. \\ &\quad \left. \{2k^2 (\sqrt{k^2 + \mu^2} + \sqrt{k^2 + M_P^2}) + \mu^2 \sqrt{k^2 + M_P^2} \right. \\ &\quad \left. + M_P \sqrt{k^2 + \mu^2} - M_P M_N \sqrt{k^2 + \mu^2}\} \right] dk, \\ \delta M_P &= (\text{obtained by exchanging } M_N \\ &\quad \text{and } M_P \text{ in } \delta M_N). \\ \delta \mu &= \frac{1}{2\mu\pi^2} \int_0^\infty \frac{l}{\sqrt{l^2 + M_P^2} \sqrt{l^2 + M_N^2}} \\ &\quad \times \frac{l}{\{\mu^2 - (\sqrt{l^2 + M_P^2} + \sqrt{l^2 + M_N^2})\}} \\ &\quad \times \left[f_1^2 \{2l^2 (\sqrt{l^2 + M_P^2} + \sqrt{l^2 + M_N^2}) \right. \\ &\quad \left. + M_P^2 \sqrt{l^2 + M_N^2} + M_N^2 \sqrt{l^2 + M_P^2} \right. \\ &\quad \left. + M_P M_N (\sqrt{l^2 + M_P^2} + \sqrt{l^2 + M_N^2})\} \right]\end{aligned}$$

$$\begin{aligned}
& +f_1^2\{M_P^2\sqrt{l^2+M_N^2}+M_N^2\sqrt{l^2+M_P^2} \\
& +M_P M_N(\sqrt{l^2+M_P^2}+\sqrt{l^2+M_N^2})\} \\
& +2f_1 f_2\{M_N\sqrt{l^2+M_P^2}+M_P\sqrt{l^2+M_N^2}\}
\end{aligned}$$

where Ψ , Ψ^\dagger and ϕ , ϕ^* describe the nucleon and meson field respectively and M , μ are the rest mass of nucleon and meson.

($\hbar=c=1$).

(B) Vector and pseudovector meson.⁽²⁾ After eliminating the first order interactions by the canonical transformation the second order Hamiltonian is rearranged in the correct order. Then the term corresponding to the nucleon self-energy has directly the form $U_N\Psi^\dagger\tau_N\Psi+U_P\Psi^\dagger\tau_P\Psi$, while those corresponding to the meson self-energy consist of $\phi_i^*\phi_i$, $\chi_{jm}^*\frac{\partial^2\chi_{lm}}{\partial x_i\partial x_j}+\frac{\partial^2\chi_{lm}^*}{\partial x_i\partial x_j}\chi_{jm}$, $\chi_{lm}^*\chi_{lm}$ and $\phi_i^*\frac{\partial\chi_{lm}}{\partial x_m}+\frac{\partial\chi_{lm}^*}{\partial x_m}\phi_l$ associated each with the diverging factors. These terms, however, can be transformed into $A_i^*A_i+B^*B$ multiplied by 1, $-2\mu^4$, $-2\mu^2$, $-2\mu^2$, by an appropriate gauge transformation analogous to the case (A) making use of the auxiliary condition, where the first order corrected terms in the transformed auxiliary condition are neglected as the third order correction in the Schrödinger equation. Thus

$$\begin{aligned}
\delta H &= \delta M_N\Psi^\dagger\tau_N\Psi + \delta M_P\Psi^\dagger\tau_P\Psi \\
& + 2\mu\delta\mu(A_i^*A_i+B^*B) \\
\delta M_N &= \frac{g_1^2}{\pi}\left[\frac{2M_N}{\mu^2}\int_0^\infty\frac{k^4(\beta_P+\alpha)dk}{\beta_P\alpha E}\right. \\
& \mp 3M_P\int_0^\infty\frac{k^2(\beta_P+\alpha)dk}{\beta_P\alpha E}\left.+\frac{g_2^2}{\pi}\left[\frac{2M_N}{\mu^2}\right.\right. \\
& \times\int_0^\infty\frac{k^2dk}{\alpha E}\mp 3M_P\int_0^\infty\frac{k^2(\beta_P+\alpha)dk}{\beta_P\alpha E}\left.\right] \\
& -\frac{3g_1g_2}{\pi\mu}\left[2\int_0^\infty\frac{k^2(\beta_P+\alpha)dk}{E}\right. \\
& \left.+2\int_0^\infty\frac{k^4(\beta_P+\alpha)dk}{\beta_P\alpha E}+M_P M_N\int_0^\infty\frac{k^2dk}{\beta_P E}\right], \\
\delta M_P &= (\text{obtained by exchanging } M_N \text{ and}
\end{aligned}$$

M_P in δM_N),

$$\begin{aligned}
\delta\mu &= \frac{g_1^2}{2\mu\pi^2}\left[\int_0^\infty\frac{k^2(\beta_P+\beta_N)dk}{F}\right. \\
& + 3\int_0^\infty\frac{k^4(\beta_P+\beta_N)dk}{\beta_P\beta_N F}\pm M_P M_N \\
& \times\int_0^\infty\frac{k^2(\beta_P+\beta_N)dk}{\beta_P\beta_N F}\left.+\frac{g_2^2}{2\mu\pi^2}\right. \\
& \times\left[\int_0^\infty\frac{k^2(\beta_P+\beta_N)dk}{F}-3\int_0^\infty\frac{k^4(\beta_P+\beta_N)dk}{\beta_P\beta_N F}\right. \\
& \left.\pm M_P M_N\int_0^\infty\frac{k^2(\beta_P+\beta_N)dk}{\beta_P\beta_N F}\right]-\frac{g_1g_2}{\pi^2} \\
& \times\left[M_P\int_0^\infty\frac{k^2dk}{\beta_P F}+M_N\int_0^\infty\frac{k^2dk}{\beta_N F}\right], \\
\beta_P &= \sqrt{k^2+M_P^2}, \quad \beta_N= \sqrt{k^2+M_N^2}, \\
E &= M_N^2-(\beta_P+\alpha)^2, \quad F= \mu^2-(\beta_P+\beta_N)^2,
\end{aligned}$$

where the double signs correspond to the vector and pseudovector meson field, and especially for the pseudovector meson all (g_1g_2) terms vanish.

As to the diverging terms when an external meson field is present, the investigation is in course. Detailed account will appear in a later issue.

In conclusion the writer wishes to express his cordial thanks to Mr. T. Inoue for his kind interest through this work.

- 1) K. Baba and D. Itô, Prog. Theor. Phys. **4** (1949) 95.
- 2) Y. Miyamoto, Prog. Theor. Phys. **3** (1948), 128.

Note added in proof

One nucleon part of any contact interaction term, which was missed out in the above letter, each forms a quadratically diverging mass correction. It turns out that in the case of scalar and pseudo-scalar meson, as it should be, meson- and nucleon-mass correction are irrelevant to whether the Dyson's transformation is used or not.

A Note on the Self-Energy and Self-Stress

K. Sawada

Department of Physics, Kyoto University

September 2, 1949

Self-stress of an electron due to electromagnetic field is $\alpha m/2\pi$ (α : fine structure constant, m : electron mass) and this shows that we cannot give consistent particle aspect to the electron as Pais⁽¹⁾ has pointed out, and if the self-stress vanishes the electron can be pictured as particle and the additional quantity to $-T_{44}$ can be renormalized as mass correction. But this statement that the vanishment of self-stress ensures particle aspect only holds for Fermi-particle obeying Dirac equation. Since in this case, the canonical energy momentum tensor is given by the following, if one takes, for example, the system of electron and scalar meson (f -field);

$$T_{\mu\nu} = \frac{\partial\phi}{\partial x_\mu} \frac{\partial\phi}{\partial x_\nu} - \frac{1}{2} \delta_{\mu\nu} \left[\frac{\partial\phi}{\partial x_\sigma} \frac{\partial\phi}{\partial x_\sigma} + \mu^2 \phi\phi \right] + \frac{1}{2} \left[\Psi^\dagger \gamma^\mu \frac{\partial\Psi}{\partial x_\nu} - \frac{\partial\Psi^\dagger}{\partial x_\nu} \gamma^\mu \Psi \right] + f N_\mu N_\nu \Psi^\dagger \Psi \quad (1)$$

in interaction representation; where ϕ is meson variable, μ is meson mass and N_μ is normal unit vector of the space-like surface σ . To renormalize the mass correction, we transform (1) by the canonical transformation⁽²⁾;

$$\Psi_m[\sigma] = U[\sigma] \Psi_{m+\delta m}[\sigma],$$

$$i \frac{\delta U[\sigma]}{\delta \sigma(x)} = \delta m \Psi^\dagger \Psi(x) U[\sigma] \quad (2)$$

Then

$$U^{-1}[\sigma] T_{\mu\nu} U[\sigma] = T_{\mu\nu} - N_\mu N_\nu \delta m \Psi^\dagger \Psi \quad (3)$$

So that, the renormalization of electron mass can be done independently to self-stress, and if self-stress vanishes the electron moves as particle with mass $m + \delta m$.

On the other hand, if we want to renor-

malize additional meson mass which came from the pair creation and annihilation of electron, we should transform $T_{\mu\nu}$ by $U[\sigma]$ which satisfy;

$$i \frac{\delta U[\sigma]}{\delta \sigma(x)} = \mu \delta \mu \phi \phi U[\sigma] \quad (4)$$

this changes the equation of motion of ϕ to the following;

$$\left(\frac{\partial^2}{\partial x_\lambda^2} - (\mu^2 + 2\mu\delta\mu) \right) \phi = \left(\frac{\partial^2}{\partial x_\lambda^2} - (\mu + \delta\mu)^2 \right) \times \phi = 0 \quad (5)$$

in the order considered. Then one has

$$U^{-1}[\sigma] T'_{\mu\nu} U[\sigma] = T_{\mu\nu} + \delta_{\mu\nu} \cdot \mu \delta \mu \phi \phi \quad (6)$$

This just cancels the mass correction appearing in $T'_{\mu\nu}$, but moreover modifies the self-stress. So that, even if one obtains finite self-energy and vanishing self-stress which ensures the correct relation between energy and momentum for meson, one cannot renormalize $-\int T_{44} dx$ identifying it as additional energy due to mass change, since it modifies the self-stress by an amount $\langle \mu \delta \mu \phi \phi \rangle = \delta \mu$, though $-\int T_{44} dx$ has always of the form $E_0 + \int_{E_0}^{\mu} \delta \mu^{(3)}$, and the relation of energy and momentum becomes to be wrong; we cannot picture meson with mass $\mu + \delta \mu$ consistently. Only if both self-energy and self-stress vanishes, one can give word particle to meson.

The actual situation for the case of scalar meson interacting with electron is as follows; (resting meson)

$$\delta \langle \int T_{44} dx \rangle = \frac{2f^2}{\pi^2} \int \frac{l^2 dl}{\mu E (4E^2 - \mu^2)} = -W_{self} \quad (7)$$

$$\delta \langle \int T_{41}, 42, 43 dx \rangle = 0 \quad (8)^*$$

$$\delta \langle \int T_{11}, 22, 33 dx \rangle = -\frac{4f^2}{3\pi^2} \int \frac{m^2 l^2 dl}{\mu E^3 (4E^2 - \mu^2)} + \frac{2f^2}{3\pi^2} \int \frac{l^4 (4E^2 + \mu^2) dl}{\mu E^3 (4E^2 - \mu^2)^2} - \frac{2f^2}{\pi^2} \int \frac{l^2 dl}{\mu E (4E^2 - \mu^2)} \quad (9)$$

where $N = \sqrt{m^2 + l^2}$. The quadratic divergences are reduced to logarithmic one if one uses another definition of $\Delta^{(1)}$ -function⁽¹⁾. Self-energy and self-stress contain quadratic and logarithmic divergences. Of course, if one applies regulator of electron satisfying the general condition;

$$\sum_i c_i = 0, \quad \sum_i c_i m_i^2 = 0, \text{ etc.} \quad (10)$$

one can reduce (7) (9) to finite value but cannot bring them to zero.

So that the so-called compensation method of f -field is not yet satisfactory as to the self-stress of f -particle itself, and due to the peculiar property of 4-4-component discussed above, although the 4-4-component can be made finite by the condition $\mu^2 = 6m^2$ ⁽³⁾.

Full accounts of the related subject will soon appear in this journal.

- 1) A. Pais: *Developments in the Theory of the Electron*, Princeton University press, (1948).
- 2) J. Schwinger: *Phys. Rev.* **75**, (1949), 651, Eq (3.100).
- 3) A. Pais: loc. cit, p. 31.
- 4) By using the Schwinger's representation of $\Delta^{(1)}$ -function, or by defining $\Delta^{(1)}(x)$ as

$$\lim_{\tau \rightarrow 0} \frac{1}{4} (\Delta^{(1)}(x + \tau_1) + \Delta^{(1)}(x - \tau_1) + \Delta^{(1)}(x + \tau_2) + \Delta^{(1)}(x - \tau_2))$$

where τ_1, τ_2 are four-vector satisfying the condition $\tau_1^\mu \tau_1^\mu = -\tau_2^\mu \tau_2^\mu$.

- 5) Y. Katayama and K. Sawada: *Prog. Theor. Phys.* **4** (1949), 377.

* The self-momentum of moving meson is also zero, since the intermediate states are such as they have same momentum as initial states.

[Note added] It was found that there exists general relation

$$\langle \bar{T}_{11} \rangle = -\frac{1}{3} \left\{ \langle \bar{T}_{44} \rangle - m \frac{\partial}{\partial m} \langle \bar{T}_{44} \rangle - \mu \frac{\partial}{\partial \mu} \langle \bar{T}_{44} \rangle \right\}$$

for self-stress and self-energy, where m and μ is Fermion and Boson mass respectively. The relation of Pais and Epstein (*Rev. Mod. Phys.*, **21**

(1949),) for the self-stress of electron due to photon is the special case of this with $\mu=0$. One can easily verify that (9) can be derived from this relation by using (7). Full discussions will soon appear.

On the Nuclear Stars (Addendum:)

Y. Yamaguchi

Physics Institute, Tokyo University.

September 5, 1949

In §3 of our paper "On the Nuclear Stars"⁽¹⁾, we pointed out that the fluctuation of p will be nearly of Poisson type. This fact is understandable as follows (although it is not a complete proof): Let k_n and k_p be $\langle \Gamma_n / (\Gamma_n + \Gamma_p) \rangle_{AV}$, $\langle \Gamma_p / (\Gamma_n + \Gamma_p) \rangle_{AV}$ ($k_n \sim 2/3$, $k_p \sim 1/3$), and $f(p)$ be the probability that just p protons are evaporated from the nucleus with the initial excitation energy x , then $f(p)$ is given by

$$f(p) \approx \sum_{n=0}^{\infty} \binom{n+p}{p} (k_n)^n (k_p)^p \{ P(X, n+p) - P(X, n+p+1) \}.$$

For $k_p \ll 1$ (i. e. $p \ll \langle a \rangle$) we obtain

$$f(p) \approx \frac{\langle p \rangle^p e^{-\langle p \rangle}}{p!},$$

and the average number of p is

$$\langle p \rangle = k_p \langle a \rangle,$$

where $\langle a \rangle$ is given in (11). Actually, $k_p \approx 1/3$ is considerably small, so that the above approximation by Poisson distribution low does not seem to be worse. Similarly, for the probability that just n neutrons are evaporated from the nucleus,

$$g(n) \approx \frac{e^{-(n-k_n \langle a \rangle) / 2\tau^2}}{\sqrt{2\pi} \tau}, \quad \tau = \sqrt{k_n^2 + 1} \cdot (\Delta a),$$

and the average value of n is

$$\langle n \rangle = k_n \langle a \rangle$$

- 1) Y. Fujimoto and Y. Yamaguchi, *Prog. Theor. Phys.* **4** (1949),

The another τ -Meson

S. Hayakawa and Y. Yamaguchi

*Department of Physics,
University of City Osaka*

September 24, 1949

In a previous note Fukuda, Hayakawa and Miyamoto¹ discussed the nature of the τ -meson which was capable to disintegrate into three π -mesons. Recent experiments make us incline to expect more sorts of mesons heavier than the π -meson.² Especially, three tracks of " τ -mesons" found in a Ilford C2 plate at a very high altitude, 90,000 to 100,000 feet³, seem to be a different kind of " τ -mesons" from that previously discussed. In the present note we discuss the nature of this "another τ -meson", tentatively called as τ' -meson.

The τ' -meson is distinguished from the τ -meson in the following points: (1) The mass, $725 \pm 40 m$, does not allow 3π -decay. (2) A star produced by the capture has only two prongs and the energy liberated in a star production may amount about only 50 MeV, whereas a capture of τ -meson, so far reported, show six prongs and the total energy given to the star exceeds 350 MeV⁴. (3) Relative abundance of such tracks seems to be rather large, though the insensitiveness for fast particles may bias the result.

The second argument suggests that the most part of the rest energy of a captured τ' -meson is carried away by a partner which may be a neutral or a light charged particle (of course, the number of such particles is not necessary one, but may be likely one). If the escaping particle has zero mass, the distribution of the transferred energy to a nucleon is calculated by the translation of the theory of μ -capture⁵. The distribution has a maximum at about 50 MeV and is ranged from 0 to 100 MeV. The excitation energy, 50 MeV, is capable to emit one proton on the average folloing the evapola-

tion theory⁶. Thus the observed star with the small number of prongs can be explained.

On the other hand, there remains a possibility that Leprince-Ringuet's τ -meson and the new τ' -meson are the same kind, since the large difference in size of the two stars might be caused by the fluctuation of the energy transfer and the meson production. But this difference and also the mass difference between 930 ± 40 seem to be so large to identify the τ with τ' -meson, that we are inclined to distinguish them.

Two other τ' -mesons reach the ends without producing any observable event. These may be disintegrations to emit a partner and an other nuclear meson, presumably π -meson since the π -meson directly couples with the nucleon and the strongest coupling meson lighter than τ' is the π -meson. The energy of a disintegrated π -meson is 70 MeV or 35 MeV, corresponding to the mass of the partner zero or π -meson mass. Neither energy lies within the observable limit of the Ilford C2 plate, 5.7 MeV. This is the reason why disintegration processes are not observed.

The life of τ' -meson is estimated as long as 10^{-12} sec. from their track lengths, 700 and 300 μ . Such a value of life time is not inconsistent with the magnitude of coupling constant $G^2 = 10^{-2} \sim 10^{-3}$, as inferred by the theory of meson disintegration⁷. This can explain the third argument.

Above discussions lead us to the conclusion that the τ' -meson seems to interact with a nucleon by a pair type. The partner of this pair may be neutral or light charged particle, or, of course, it may be neutrino. Each of the pair must be only Boson or Only Fermion. Following the customary manner, the τ' -meson may be spin 1/2 Fermion.

We wish express our gratitude to Messrs Fujimoto and Sasaki for their stimulating discussions.

- 1) H. Fukuda, S. Hayakawa and Y. Miyamoto :
Prog. Theor. Phys. in press.
- 2) A. I. Alichaniyan, A. I. Alichanow and A.

- Vaisenberg: Jour. Exp. Theor. Phys. USSR. **18**, No. 3 (1948). A. I. Alichanian and A. I. Alichanow: Nature **163** (1949), 761. J. G. Retallack and R. B. Brode: Phys. Rev. **75** (1949), 1716.
- 3) N. Wagner and D. Cooper: Phys. Rev. **76** (1949) 499.
- 4) L. Leprince-Ringuet: Rev. Mod. Phys. **21** (1949), 43.
- 5) J. Tiomno and J. A. Wheeler: Rev. Mod. Phys. **21** (1949), 153.
M. Taketani, M. Sasaki and S. Nakamura: Prog. Theor. Phys. in press.
- 6) Y. Fujimoto and Y. Yamaguchi: Prog. Theor. Phys. in press.
- 7) H. Fukuda and Y. Miyamoto: Prog. Theor. Phys. in press.
S. Oneda, S. Ozaki and Y. Sasaki: Prog. Theor. Phys. in press.
J. Steinberger: Phys. Rev. in press.

Directional Distribution of Extensive Air Showers

Y. Fujimoto, S. Hayakawa
and
Y. Yamaguchi

Department of Physics, Tokyo University
and University of City Osaka

September 24, 1949

In a previous paper¹ we reported the analytical calculation of the altitude variation of extensive air showers on the basis of proton primary hypothesis. Although our approximation becomes worse at higher altitude, its result is applicable to the directional distribution at a mountain elevation, since the more inclined showeres correspond to the vertical ones at the deeper region.

We use the altitude-frequency relation before integrating over the angle of falling directions:

$$H(A, l) \propto e^{-a' \pi l} \{ (l^* - a' l) l \}^{\pi l - 2}. \quad (1)$$

The notations are the same as used previously. From (1) we obtain the directional intensity

at the depth l_0

$$I(A, l_0, \theta) \sin \theta d\theta = H(A, l) l_0 dl / l^2, \quad (l \cos \theta = l_0). \quad (2)$$

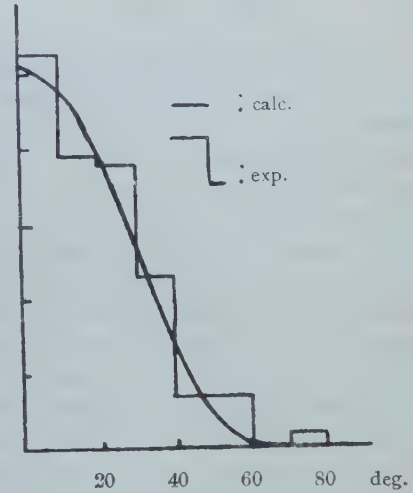


Fig. 1.

Directional distribution of extensive air showers.

The result is shown in Fig. 1, compared with an experiment at $l=20^2$. The calculated angular dependence at smaller angle $\cos^6 \theta$ is compared as experimental one $\cos^{3.6} \theta$. The agreement may be good, considering that our approximation gives a little steeper altitude variation.

- 1) Y. Fujimoto, S. Hayakawa and Y. Yamaguchi: Prog. Theor. Phys. in press.
- 2) W. W. Brown and A. S. McKay: Phys. Rev. in press.

The authors thank to Prof. Greisen who sent us this paper before publication.

On the Nuclear Forces.

H. Enatsu

Institute of Physics, Kyoto University.

October 11, 1949.

Using the newly developed method Case¹⁾ showed that the additional nuclear magnetic moment calculated by the pseudoscalar meson theory in the Tomonaga-Schwinger formalism could not fit the existing experimental data. This is serious difficulty for the present meson theory, because it has been believed that in order for agreements with various experimental results to be achieved the pseudoscalar meson theory should be adopted. Therefore, it is necessary to consider other models for the meson field. Taking into account the fact that the pseudoscalar and scalar meson fields do not give the divergences of the vacuum polarization in the anomalous magnetic moment¹⁾, we take the Schoenberg's²⁾ generalized interactions between nucleon and meson fields (this contains the so-called mixed theory in its special case), and assume the following interaction Lagrangians for the simplicity,

Scalar field (Neutral field)

$$L_a = -[f_1 u \phi + f_2 v_a \chi^2 + \frac{1}{24} f_1' S_{\alpha\beta\gamma\delta} \eta^{\alpha\beta\gamma\delta} \phi + \frac{1}{6} f_2' t_{\alpha\beta\gamma} \eta^{\alpha\beta\gamma\delta} \chi^2] \quad (1)$$

$$(\phi = U, \quad \chi_0 = -V_0, \quad \chi_i = \vec{V}) \quad (1)$$

Pseudoscalar field (Neutral field)

$$L_a = -\left[\frac{1}{24} f_1' S_{\alpha\beta\gamma\delta} \chi^{\alpha\beta\gamma\delta} - \frac{1}{6} f_2' t_{\alpha\beta\gamma} \phi^{\alpha\beta\gamma} - \frac{1}{24} f_1 u \eta^{\alpha\beta\gamma\delta} \chi_{\alpha\beta\gamma\delta} - \frac{1}{6} f_2 v_a \eta^{\alpha\beta\gamma\delta} \phi_{\beta\gamma\delta} \right] \\ (\chi^{0123} = U, \quad \phi^{123} = -V_0, \quad \phi^{ijk} = -\vec{V}) \quad (2)$$

where

$$u = (\psi^* \rho_3 \psi), \quad (v_0, v_i) = (-\psi^* \psi, \psi^* \vec{\alpha} \psi), \\ (t_{0ijk}, t_{ijkl}) = (\psi^* \vec{\sigma} \psi, -\psi^* \rho_1 \psi)$$

$$S_{0123} = \psi^* \rho_1 \psi, \quad \eta^{\alpha\beta\gamma\delta} = (-g)^{1/2} \epsilon^{\alpha\beta\gamma\delta}, \\ g = |g^{\alpha\beta}|, \quad \hbar = c = 1,$$

$$(\alpha, \beta, \gamma, \delta) = 0, 1, 2, 3 \quad (i, k, l) = 1, 2, 3$$

These two Lagrangians lead to the same form and are expressed by the usual notations as follows:

$$L_0 = -\left[f_1 \psi^* \rho_3 \psi U + \frac{f_2}{x} (\psi^* \psi \frac{\partial U}{\partial t} + \psi^* \vec{\alpha} \psi \text{grad} U) \right] - \left[f_1' \psi^* \rho_3 \psi U + \frac{f_2'}{x} (\psi^* \rho_1 \psi \frac{\partial U}{\partial t} + \psi^* \vec{\sigma} \psi \text{grad} U) \right] \quad (3)$$

Namely, two fields are unified to the spin 0 field. By the same method we are able to define the spin 1 field for the vector and pseudovector fields. Making use of the canonical transformations as adopted by van Hove³⁾, we obtain the transformed Hamiltonian H_T , and it is the remarkable fact that the contact terms drop out entirely in this result. Then, the second order nuclear force parts are reduced by the Dirac equation for the deuteron to the usual ones, and lead to the next nuclear potentials in the non-relativistic and static approximation,³⁾

$$W = -T \left[f_1^2 - \left(\frac{\mu}{2M} \right)^2 \left\{ (f_1' + \frac{2M}{\mu} f_1')^2 + f_1^2 \right\} \right. \\ \left. \times \left\{ \frac{1}{3} (\sigma^{(1)} \cdot \sigma^{(2)}) + A \cdot G \right\} \right] \phi_{r_1} \quad (4)$$

where μ : Meson mass, M : Proton mass

$$A = \frac{\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} (\sigma^{(2)} \cdot \vec{r})}{r^2} - (\sigma^{(1)} \cdot \sigma^{(2)})$$

$$G = \left(\frac{1}{3} + \frac{1}{\mu r} + \frac{1}{\mu^2 r^2} \right),$$

$$\phi_r = \frac{1}{4\pi} \frac{e^{-\mu r}}{r} \quad (5)$$

and T is added to consider the symmetrical and charged theory, and given by $(\tau^{(1)} \cdot \tau^{(2)})/2$ and $(\tau_1^{(1)} \tau_1^{(2)} + \tau_2^{(1)} \tau_2^{(2)})/2$ respectively.

Assuming the $f_1 = f_2 = f_1' = f_2' = f$ between the coupling constants, these give:

Symmetrical theory

$${}^1W_s \sim -f^2 \phi_r$$

$${}^3W_s \sim \frac{3}{2} \left[\frac{2}{3} f^2 - f^2 A \cdot G \right] \phi_r \quad (6)$$

Charged theory

$${}^1W_c \sim -2f^2 \phi_r$$

$${}^3W_c \sim \left[\frac{2}{3} f^2 - f^2 A \cdot G \right] \phi_r \quad (7)$$

where 1W and 3W correspond to the singlet state and the triplet state of the deuteron. Comparing these results with the Bethe's³⁾ potential by the neutral vector meson theory, i.e.

$${}^1W_N^R = -2g^2 \phi_r$$

$${}^3W_N^R = \left[\frac{2}{3} g^2 - g^2 A \cdot G \right] \phi_r \quad (8)$$

which is the most successful one in the deuteron problem, we find that the charged theory potential has just the same form as Bethe's, and the symmetrical theory one differs only from Bethe's in numerical factors provided that $f=g$. Thus, there is the possibility to obtain the reasonable results for the deuteron problem.

As mentioned above the ambiguity of contact terms are not involved in these potentials, but the cross term which exists in H_T vanishes in (4), therefore, the difference between the spin 0 field considered here and the usual scalar-pseudoscalar mixed meson theory could not be found in the nuclear potential in this approximation.

It will be seen in the anomalous magnetic moment of nucleon.

The detailed account will be soon published in this journal. (Just finished this letter Mr. S. Nakamura wrote to me that E. Yamada obtained the reasonable anomalous magnetic moment of nucleon by mixing the scalar and pseudoscalar meson field.)

3) L. van Hove, Phys. Rev. **75** (1949), 1819.

G. Araki, Prog. Theor. Phys. **4** (1949), 193.

4) H. A. Bethe, Phys. Rev. **67** (1940), 260, 390.

Note on the Vacuum Polarization.

G. Takeda

Department of Physics, Tokyo University.

October 14, 1949.

The problem of the vacuum polarization was discussed by many authors and the appearance of a non-gauge-invariant polarization term was inherent in this problem. The recent development¹⁾ of quantum electrodynamics has shown that these difficulties are due to singularities of D and D_1 functions, and can be removed by a skilful calculation method²⁾. But since, by nature, we are concerned with integrations of mathematically undefined expressions in the current theory, a fundamental settlement would be far from being achieved. So we examine this problem from another point of view similar to those of Dirac³⁾ and Heisenberg⁴⁾. According to Feynman-Dyson's theory the expression for the induced vacuum polarization current in e^2 -approximation is as follows:

$$J_\lambda^{(2)}(x) = (ie^2/4) \int_{-\infty}^{+\infty} dy^4 S_F[\gamma_\lambda S_F(x-y) \\ \times \gamma_\mu S_F(y-x)] A_\mu(y). \quad (1)$$



Fig. 1.



Fig. 2.

1) K. M. Case, Phys. Rev. **76** (1949), 1.

2) M. Schoenberg, Phys. Rev. **60** (1941), 468.

Now in place of it we take

$$J_{\lambda}^{(2)}(\xi, Z) = (ie^2/4) \int_{-\infty}^{\infty} dy^4 S_F[\gamma_{\lambda} S_F(x-y) \times \gamma_{\mu} S_F(y-x')] A_{\mu}(y) \quad (2)$$

where $X_{\mu} = \xi_{\mu} + (Z_{\mu}/2)$, $X'_{\mu} = \xi_{\mu} - (Z_{\mu}/2)$.

This just means to adopt an open loop having a mouth with small breadth Z_{μ} instead of a closed one in graphic representation. (Fig. 1, Fig. 2) Performing the same alteration for the vacuum polarization current of every order in e , the charge-current conservation law becomes as follows:

$$\partial J_{\lambda}(\xi, Z)/\partial \xi_{\lambda} - ie[A_{\lambda}(x') - A_{\lambda}(x)]J_{\lambda}(\xi, Z) = 0 \quad (3)$$

After the integration of the right-hand side of expression (2)

$$J_{\lambda}^{(2)}(\xi, Z) = (im^2 e^2/2\pi) \left[- \int_0^1 dv \cos(v(q \cdot Z)/2) H_2^{(1)}(p\lambda^{1/2}) L_{\lambda} + \left\{ \cos((q \cdot Z)/2) (H_1^{(1)}(m\lambda^{1/2})/m\lambda^{1/2}) - \int_0^1 dv \cos(v(q \cdot Z)/2) (p^2/m^2)(H_1^{(1)}(p\lambda^{1/2})/p\lambda^{1/2}) \right\} L_{\lambda} + (q^2/4m^2) \times \int_0^1 dv \cos(v(q \cdot Z)/2) H_0^{(1)}(p\lambda^{1/2}) \{ (1-v^2)M_{\lambda} + vL_{\lambda} \} \right]^* \quad (4)$$

where $L_{\lambda} \equiv A_{\lambda} - [Z_{\lambda}(q \cdot A) + q_{\lambda}(Z \cdot A)]/(q \cdot Z)$,

$$M_{\lambda} \equiv A_{\lambda} - q_{\lambda}(A \cdot q)/q^2,$$

$$N_{\lambda} \equiv Z_{\lambda}(A \cdot Z)/\lambda, \quad (5)$$

$$p^2 = m^2 + (q^2/4)(I - v^2),$$

$$\lambda = -Z^2$$

and q_{μ} is the energy-momentum four vector of potential A_{μ} . ($\partial(\lambda)$ -type singularities are omitted and * means complex conjugate.)

Neglecting terms of higher order in Z_{μ} which become zero in the limit $Z_{\mu} \rightarrow 0$,

$$J_{\lambda}^{(2)}(\xi, Z) = (2e^2/\pi^2) [(1/\lambda + m^2/4) - (1/24)\{(q \cdot Z)^2/\lambda - q^2\} N_{\lambda} + (e^2/12\pi^2)\{(q \cdot Z)^2/\lambda - q^2\} L_{\lambda} + (e^2 q^2/16\pi^2)[I' + q^2 \int_0^1 dv v^2(1-v^2/3) \times \{m^2 + (q^2/4)(1-v^2)\}^{-1}]^* M_{\lambda} \quad (6)$$

$$\text{where } I' \equiv (8/3)[\gamma + \log(m\lambda^{1/2}/2) + \pi/2i] \quad (7)$$

We require in the limit the usual conservation law $\partial J_{\lambda}/\partial \xi_{\lambda} = 0$ and zero photon mass. If we take Z_{μ} so as to satisfy $(A \cdot Z) = 0$, then follows $\partial J_{\lambda}^{(2)}/\partial \xi_{\lambda} = 0$ from (6) because of $(q \cdot L) = (q \cdot M) = N_{\lambda} = 0$ for finite breadth Z_{μ} and further in the limit $Z_{\mu} \rightarrow 0$ $\partial J_{\lambda}/\partial \xi_{\lambda} = 0$. This can be understood from $\lim_{Z_{\mu} \rightarrow 0} [A_{\lambda}(x') - A_{\lambda}(x)] J_{\lambda}(\xi, Z) = 0$ for every order in e . The procedure means that we ought to take Z_{μ} according to the given potential. Another method will be to take an average over various positions of Z_{μ} . The zero photon mass condition requires in addition $[(q \cdot Z)^2/\lambda - q^2] L_{\lambda} = 0$ for $q^2 = 0$ and $(q \cdot A) = 0$, which can be easily satisfied by reasonable prescriptions. At any rate we can show equivalence of our result with Schwinger-Feynman's one if terms of L and N vanishes. In our method ambiguity lies not in the integration but in the choice of Z_{μ} and this ought to be determined by physical conditions. Finally we take some remarks on the logarithmic diverging term. It seems to us that logarithmic divergences as the self-energy and the charge-renormalization term of an electron are not serious difficulties different from more stronger divergences, for these terms will give only small corrections to the mass and the charge respectively by taking account physically acceptable universal length which corresponds our Z_{μ} . Our method for finite Z_{μ} is an incomplete but a suggestive model for new theories where universal length would play a decisive role and the condition $(Z \cdot A) = 0$ together with the Lorentz condition $(P \cdot A) = 0$ make a set of reciprocal equations.

- 1) J. Schwinger; *Phys. Rev.* **74**, (1948) 1439.
T. Tati and S. Tomonaga; *Prog. Theor. Phys.* **3**, (1948) 391.
R. P. Feynman; *The Theory of Positrons*;
Space-Time Approach to Quantum Electrodynamics.
- 2) W. Pauli and P. Villars; *On the Invariant Regularization in Relativistic Quantum Theory.*
- 3) P. A. M. Diracs; *Proc. Camb. Phil. Sec.* **30**, (1934) 150.
- 4) W. Heisenberg; *ZS. f. Phys.* **90**, (1934) 209.

Remarks on the Nuclear Disintegration by Meson Capture.

Y. Fujimoto, S. Hayakawa and
Y. Yamaguchi

*Department of Physics, Tokyo University
and University of City Osaka.*

October 15, 1949

What happens by the capture of a meson, π or τ , is of great importance both to infer the nature of the captured meson and to get some informations about nuclear reactions. There have, however, been few theory about this problem except that by Fujimoto and Yamaguchi¹⁾, which can unfortunately not give the satisfactory explanation of stars by π -meson capture. Main defect of this theory exists in the neglect of fluctuation, for which some improvement is now carried out. Further, they did not consider the recoil of nucleons in a nucleus. This effect may not be negligible in higher energy region under consideration.

The capture of π -meson occurs by the following process:

$$\pi^- + P + P(N) \rightarrow N^* + P(N) \rightarrow$$

$$P + \pi^-(\pi^0) + P(N) \rightarrow N + P(N),$$

Thus two nucleons start with the same but opposite momenta, corresponding mean kinetic energies 83 MeV, and the energy transferred to the nucleus may somewhat increase. Taking into account this fact we have improved

the former theory in I and calculated the distribution of prong numbers for several models. The results for these models are practically in agreement with each other except for the number of prongs 1 and 0, and explain the experiments²⁾ considerably good as seen in the next letter.

As for the capture of a τ -meson, there has been so few data that we can not derive the definite conclusion. The capture process of the lowest order may be

$$\tau^- + P \rightarrow N^* \rightarrow P + \pi^-$$

or

$$\tau^- + P \begin{cases} \nearrow N + P + P \\ \searrow \tau^- + P + \pi^0 \end{cases} \rightarrow N + \pi^0$$

Thus a neutron (proton) and a π -meson start with the same and opposite momenta and their kinetic energies are 80 and 275 MeV, respectively. Although the excitation caused by the nucleon is low, the π -meson may suffer large angle scatterings and transfer its energy to nucleons effectively because of its large Compton wave length and lighter mass. Thus the outgoing meson may be decelerated to very low energy as seen in photographic plate by Leprince-Ringuet³⁾, and the energy lost is spent by nuclear excitation which leads to the star with several prongs. If this meson is produced near the edge of the nucleons and escapes without scattering, the excitation energy amounts only about 80 MeV. Therefore, the 725 m τ -meson producing two prong star¹⁾ does not necessarily rule out the ordinary type of interaction against the previous note⁵⁾. But the probability that the τ -meson is captured near the edge of the nucleus may be very small, because the K -orbit of the τ -meson lies in considerably inner part of the nucleus.

- 1) Y. Fujimoto and Y. Yamaguchi; *Prog. Theor. Phys.* in press, which will be cited as I.
- 2) A private letter from Dr. Gardner to Prof. Taketani.
- 3) L. Leprince-Ringuet; *Rev. Mod. Phys.* **21** (1949), 42.

- 4) N. Wagner and D. Cooper: *Phys. Rev.* **76** (1949), 449.
 5) S. Hayakawa and Y. Yamaguchi: *Prog. Theor. Phys.* in press.

Nuclear Disintegration by π -Meson Capture.

Y. Fujimoto, S. Hayakawa and
Y. Yamaguchi

*Department of Physics, Tokyo University
and University of City Osaka.*

October 15, 1949

Following the notion described in the above note¹⁾, we calculate the distribution of prong numbers in the stars caused by π^- -meson captures. This problem is solved by determining the excitation energy x and its distribution $f(x)$. The former quantity x determines the average number of evaporated protons $\langle p(x) \rangle$. The distribution of observed prong numbers may be given by Poisson distribution as was discussed in our previous paper²⁾:

$$\int f(x) \frac{\langle p(x) \rangle^p}{p!} e^{-\langle p(x) \rangle} dx. \quad (1)$$

To calculate the excitation energy x we must notice the place where the π^- -meson is captured by the nucleus. The probability that the proton at r captures a π^- -meson may be proportional to $|\psi(r)|^2$, where $\psi(r)$ means the wave function of the π^- -meson in $1s$ -state in the three dimensional parabolic potential³⁾.

Under such remarks we discuss this problem assuming the following models.

(a) The rest energy of captured π^- -meson is transferred into two nucleons as suggested in I. Each nucleon starts with mean kinetic energy:

$$\begin{aligned} &1/2 \text{ (rest energy of a } \pi^- \text{-meson)} - 1/2 \\ &\text{(binding energy of a } \pi^- \text{-meson in } 1s\text{-} \\ &\text{level)} + 3/5 \text{ (Fermi energy in a nucleus)} \end{aligned}$$

$$= 83 \text{ MeV.}$$

Our problem is the same as a nucleon with energy 83 MeV hits the nucleus. Using the method of II we find that $\langle p(x) \rangle$ and $f(x)$ are derived by the range-energy relation which will be published lately. Thus we get the distribution of prong numbers, which is represented in column A of Table I.

(b) The initial energy of starting nucleons 83 MeV is so high that we must not neglect the transparency of nuclear matter. But after one collision the nucleon loses the large part of its energy and falls into the "opaque" energy region. We may, therefore, assume that the nucleon stops in the nucleus after at least one collision and its energy excess (70 MeV, subtracting Fermi energy part) is used to heat the nucleus. Thus x is able to have one of three discrete values, 0, 70, and 140 MeV, and the values of $f(x)$ for these cases, that is, two, one and no nucleons escape out of the nucleus, are 0.113, 0.470 and 0.417, respectively. By introducing these figures in (1) we get the result shown in column B of Table I.

(c) There may be the other possible mode of capture than mentioned in (a) and (b), where a single nucleon plays a role in the capture process. π^- -meson may simultaneously interact with many protons in the nucleus and the interference among the captures by these protons may be considerably effective, since the K -radius of π^- -meson is only slightly smaller than the nuclear radius. In this case the all available energy of π^- -meson (140 MeV) is consumed to excite the whole nucleus, as emphasized by K. Ono,

Table I.

Number of prongs	Percentage frequencies emitting the corresponding number of prongs.			
	A	B	C	Experiment
0	14.1 %	28.5 %	10.0 %	27.0 %
1	31.9	30.4	23.1	23.4
2	28.8	20.0	26.5	24.0
3	14.6	11.5	20.3	14.8
4	6.4	5.9	11.7	8.7
5	2.1	2.4	5.4	1.9
6	0.6	0.9	2.1	0

and then ordinary evaporation process takes place. The result is given in column C of Table I.

As seen in Table I, the results for these models are in agreement roughly with each other and with experiment¹⁾. But there is a slight discrepancy between the theories and the experiment for the cases of 0- and 1-prong which may be due to the special character of Ag- and Br-nucleus.

Furthermore, the contribution from the disintegrations of light nuclei, which is taken in II without reasonable explanation, is to be considered, but they may not essentially change the above distribution.

- 1) Y. Fujimoto, S. Hayakawa, and Y. Yamaguchi: Prog. Theor. Phys. in press. quoted as I.
- 2) Y. Fujimoto, and Y. Yamaguchi: Prog. Theor. Phys. in press:
Y. Yamaguchi: Prog. Theor. Phys. in press, quoted as II.
- 3) J. A. Wheeler: Rev. Mod. Phys. **21** (1949), 133, 153.
- 4) A private letter from Dr. Gardner to Prof. Taketani.

Remark on the Energy Distribution of Disintegrated μ -Mesons.

S. Hayakawa and J. Nishimura

*Department of Physics, University of City
Osaka and Scientific Research Institute.*

October 15, 1949

In our previous papers¹⁾ we treated the energy distribution of μ -mesons disintegrated from π -meson as uniform from 0 to the energy of the agent π -meson E_0 . But this is erroneous, as was pointed out by Ogawa²⁾, because of the small mass difference of π - and μ -mesons, contrasting with the case of μ -electron decay.

If the π -meson with energy E_0 and momentum P_0 in the laboratory system disintegrates into a μ -meson and a neutrino,

the energy of the disintegrated μ -meson is represented by

$$E = (E_0/2)\{1 + (\mu/x)^2\} + (P_0 c/2) \{1 - (\mu/x)^2\} \cos \theta, \quad (1)$$

where x and μ mean the masses of π - and μ -mesons and θ the angle of emission of μ -meson in the center of mass system. In our approximation $E_0 \approx P_0 c$, the energy of μ -meson lies in

$$(\mu/x)^2 E_0 \leq E \leq E_0. \quad (2)$$

The energy spectrum of μ -mesons should, therefore, be taken as

$$\frac{1}{1 - (\mu/x)^2} \cdot \frac{dE}{E_0} \quad (3)$$

This results in the average energy $\{1 + (\mu/x)^2\} E_0/2 = 0.78 E_0$ of disintegrated μ -mesons.

Above modification affects our previous work in two points. One is that the integration over E_0 must be limited by (2); the upper limit should be replaced by $(x/\mu)^2 E$. The other is the factor before dE/E_0 in (3). These effects lead to give about two times larger intensity of μ -mesons than I. In order to compensate this change, the production of π -mesons must be reduced by the corresponding factor. In I the value of H may be reduced for this purpose. But then we can no longer relate H with v_0 . As was suggested in I, v_0 and H is not necessary related so closely that the whole energy lost by nucleons is transported into π -mesons. Thus we may once determine v_0 and H independently. This results in that about a half of energy lost by nucleons is transferred to others than π -mesons. They may be slow nucleons not capable to produce further mesons and photons directly produced. Furthermore, the reason to take $\gamma=2.0$ vanishes and we may put $\gamma=1.8$, being consistent with other phenomena. By such remarks the main results of I need not be changed. But it seems us to be a

wonder that unexpectedly few π -mesons are produced by a nuclear collision.

- 1) S. Hayakawa and J. Nishimura: *Prog. Theor. Phys.* **4** (1949), 232; *Jour. Sci. Res. Inst.* in press. These papers will be cited as I.
- 2) S. Ogawa: private communication. The present authors thank him for his kind suggestion.

Soft Component in Upper Atmosphere.

S. Hayakawa and J. Nishimura

*Department of Physics, University of City
Osaka and Scientific Research Institute.*

October 15, 1949

Following the previous analysis about the hard component in the atmosphere¹⁾, we treat the intensity of the soft component in the upper atmosphere, which will be of importance to settle the origins of electronic rays. Although in the current notion only the μ -meson has been the main origin, some authors have pointed out the necessity of the other origin to explain the height and the place of the maximum intensity of total cosmic rays^{2,3)}. But recent experiments assure the mode of the disintegration of μ -meson into one electron and two neutrinos and make us infer the too small intensity of soft rays unless we assume other origins besides μ -mesons. Among the possible origins the γ -decay of the neutral π -meson with the suitable life may be the most reliable one. The hypothetical β -meson is now disproved by Berkeley experiments and the charge exchange process may not be the case because of its small cross-section. Thus we concern only μ - and π^0 -mesons as the origins of electronic rays.

In order to perform the satisfactory calculation of this problem, it seems to be rather essential to take into consideration the angular divergence of produced radiations at the transmutations because of their low momenta. But here we confine ourselves to the one dimensional treatment, though regrettable, and estimate

only the order of maximum intensity.

For this simplification we may put the differential energy spectrum of soft rays produced between x and $x+dx$ with energy between E and $E+dE$

$$S(E, x) = S_0 \exp(-x/l) / (E+a)^{r+1}, \quad (1)$$

where S_0 , l and a are chosen in order to fit the result of I:

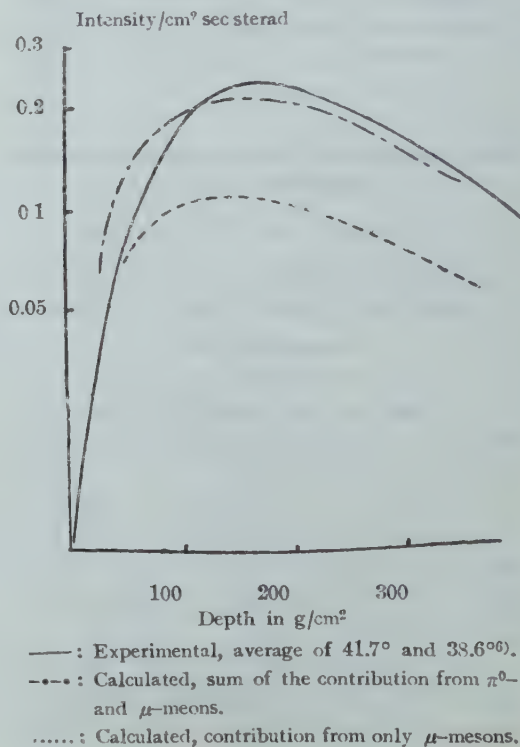
$$\left. \begin{aligned} l &= 125 \text{ g cm}^{-2} \text{ and } a = 0.5 \text{ Bev for } \pi\text{-meson,} \\ l &= 280 \text{ g cm}^{-2} \text{ and } a = 0.1 \text{ Bev for } \mu\text{-meson,} \end{aligned} \right\} \quad (2)$$

The intensity of electrons at the depth x is given by

$$I(x) = \int_0^x dx' \int_1^\infty dE S(E, x') \Pi(E, 0, x-x'). \quad (3)$$

Cascade function Π is transformed by Mellin transformation and the integration of (3) is carried out by saddle point method.

Fig. 1. Intensity-depth curve of soft component.



The result is represented in Fig. 1, in which one sees the decay of μ -meson can only explain about a half of whole soft rays. The reason why the calculated intensity in the most upper part is too high, may be due to the rough exponential approximation of (1) for μ -mesons. Adding the contribution from π^0 -mesons, we obtain the right order of the intensity. We may conclude that the neutral meson disintegrates into electronic rays with considerable short life as was previously discussed⁵⁾. Detailed account will be published in Jour. Sci. Res. Inst.

- 1) S. Hayakawa and J. Nishimura: Prog. Theor. Phys. **4** (1949), 232 and Jour. Sci. Res. Inst. in press, which will be as I.
- 2) M. Taketani: Prog. Theor. Phys. **3** (1948), 349.
- 3) W. Heitler and S. Power: Phys. Rev. **72** (1947), 266.
- 4) M. Sasaki, S. Nakamura and S. Hayakawa: Prog. Theor. Phys. **4** (1948), 454.
- 5) S. Hayakawa: Prog. Theor. Phys. in press.
- 6) A. J. Biehler, R. A. Montgomery, H. V. Neher, W. H. Pickering and W. C. Roesh: Rev. Mod. **20** (1948), 360. This curve is usually called as Pfozter curve, but we must not refer the Pfozter's result (ZS. f. Phys. **102** (1936), 23.) He overcorrected the counting rate in upper height, so that his intensity was about two times larger than those of others, because he took the deat time of G-M counter too large.

Neutron Component in Extensive Air Showers.

Y. Fujimoto, S. Hayakawa
and Y. Yamaguchi

Department of Physics, Tokyo University and
University of City Osaka.

October 15, 1949

Recent experiments¹⁾ have established the existence of the resonance level in high energy (γ, n) reactions, and theoretical explanation has been given by Goldhaber and Teller²⁾. This reaction will be the effective source of

neutron component in air showers. Another source is the nuclear stars produced by high energy nucleons. We estimate these two contributions.

(γ, n) reaction possesses the resonance level at 20 MeV and its cross-section is 1 MeV barn at this energy. The cross-section for a nuclear star production may be about a half of the geometrical cross-section of an air nucleus.

The relative intensity of photons with energy ~ 20 MeV can be estimated assuming the energy spectrum at the shower maximum³⁾. While the intensity of high energy nucleons was measured by Tongiorgi⁴⁾.

	Then we get the following values.
agent intensity ⁵⁾	cross-section
photon 2.8 %	10^{-24} cm ²
nucleon 2~3 %	1.8×10^{-26} cm ²
	1~2
	1.6 ⁶⁾

Multiplying these three factors, we see that the two contributions are the same order of magnitude.

- 1) G. C. Baldwin and G. S. Klaiber: Phys. Rev. **71** (1947), 649; **73** (1948), 1156.
- 2) M. Goldhaber and E. Teller: Phys. Rev. **74** (1948), 1046.
- 4) H. A. Bethe, cited in B. Rossi and K. Greisen: Rev. Mod. Phys. **13** (1941), 240.
- 4) Vanna Cocconi Tongiorgi: Phys. Rev. **75** (1949), 1532.
- 5) The total electron intensity is put to be 100 %.
- 6) Vanna Cocconi Tongiorgi: Phys. Rev. **75** (1949), 517. This value is taken from the measurement of neutron production in C not accompanying with air showers. The multiplicity of neutron production in air shower may be somewhat larger.

High Energy Electronic Component in Extensive Air Showers.

Y. Fujimoto and S. Hayakawa

Department of Physics, Tokyo University
and University of City Osaka.

October 15, 1949

The question whether the agent of an extensive air shower is an electron or a nu-

clean becomes gradually apparent by various arguments. A conclusive decision would be possible by the analysis of the spatial distribution of high energy electronic component, since its distribution is mainly due to initial emergent angle rather than multiple scattering. It is closely related with the decoherent curve of core selectors measured by Cocconi *et al.*¹⁾ In this note we will discuss this problem on the basis of electron primary hypothesis referring to their experiment.

The unit of length is Molière unit at Echo Lake, 114 m, and the unit of energy is the critical energy of air, 86 MeV. The distribution function with fixed energy E is approximately represented by

$$f(E, x) = 1.64(E/K)^2 x^{-2/3} \exp(-x) [1 - 0.82x \times \exp(-0.38x)], \quad (1)$$

$$x = 1.71Er,$$

for $x \gtrsim 1$, based on Molière's treatment.²⁾ $K = 21$ MeV means the characteristic energy of multiple scattering in air and r represents the distance from the shower axis. The density $\rho(r)$ can be obtained by integrating (1) over the differential energy spectrum of electronic component $n(E)$. $n(E)$ may be assumed as the same as the spectrum at the shower maximum:

$$n(E)dE = NdE/E^2. \quad (2)$$

N means the total number of electrons in a shower, and in this case it is 3.6×10^5 for a small shower and 5.8×10^5 for a large one. The integration over E ranges from $E_0 = 70$, which is the minimum energy capable to discharge the core selector, to ∞ . By the method of steepest descent we get

$$\rho(x) = 1.92 \times 10^5 N x_0^{-3/5} e^{-x_0} \times [1 - 0.594x_0 e^{-0.38x_0}], \quad (3)$$

where $x_0 = 1.71E_0 r = 1.05R$, R in meter. Neglecting the second term, the decoherent curve $I(d)$ is expressed as the function of separation d in meter,

$$I(d) \sim \exp(-1.05d)/d \quad (4)$$

for $d \gg R_c$, $R_c = 3.9m$ for a large shower and $6.3m$ for a small one. This exponential decreases drastically deviates from experimental one.

Above consideration concerns only the multiple scattering, but the single scattering may be predominantly effective in such greater distances measured by Cocconi *et al.* In this case, the distribution function has less steep decrease as³⁾

$$f(E, r) = (1.17 \times 10^{-2}/E^2)(dr'/r^4). \quad (5)$$

The distance, where (1) is equal to (5), is about $x = 14$ or $R = 13m$. We see the single scattering is more effective in farther regions than this distance. Following the similar method as in the case of multiple scattering, we obtain

$$\rho(r) = 3.9 \times 10^{-3} N/E_0^2 r^4 \quad (6)$$

and

$$I(d) \sim d^{-4}. \quad (7)$$

Even (7) shows steeper decrease than the experimental one, which finds approximately d^{-2} decrease.

We may conclude from the above analysis that the electron primary hypothesis is decisively ruled out. The spatial distribution is considered to be mainly due to the initial divergence of electronic rays in nuclear collisions caused by primaries.

We thank to Prof. Greisen who communicated us their paper before publication.

- 1) G. Cocconi, V. Tongiorgi Cocconi and K. Greisen: Phys. Rev. in press.
- 2) G. Molière: The Cosmic Radiation edited by W. Heisenberg.
- 3) L. Eugy: Phys. Rev. **74** (1948), 1801.

(Continued from p. 476)

Addendum: Just after we had finished this paper, the Proceedings of the Royal Society, Vol. 196, No. 1646 arrived, in which the papers of G. Thomson and of J. B. Harding, S. Lattimore and D. H. Perkins were found. Those authors reported that considerable part of star prongs were α -particles. Although this fact seems to contradict with our conclusion, we must take account of the following points.

i) They estimated the excitation energy of a

nucleus assuming that nearly the same number of neutrons were emitted as protons. We fear, it would be rather underestimated and should be nearly doubled.

ii) They observed particularly the large stars caused by cosmic radiation. As was pointed out by them, a large cosmic ray star can not be explained by the evaporation theory, and will need some new explanation.

We shall discuss in detail these large stars in another paper.

Errata (Vol. 4, No. 3)

p. 353 **On the Dirac's Indefinite Metric Space.** Naomi Shono and Nobuo Oda.

On the equation (4): *for* $\eta = \mathbf{I} - \mathbf{P}$, *read* $\eta = \mathbf{I} - 2\mathbf{P}$.

p. 372 **An Attempt to Pauli's Regulator.** Y. Katayama.

In the tenth line of right column on p. 373: *for* electron field, *read* electromagnetic field.

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